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# NULL CONTROLLABILITY OF FRACTIONAL INTEGRODIFFERENTIAL SYSTEMS IN BANACH SPACES WITH DISTRIBUTED DELAYS IN THE LIMITED CONTROL POWERS

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***Abstract***

***In this work, Fractional Integro-differential Systems in Banach Spaces with Distributed Delays in the Limited Control Power of the form:*** $\frac{d^n x(t)}{dt^n} = Ax(t) + \int_{-h}^{0} d_\theta H(t,\theta)u(t+\theta) + f\left(t, x(t), \int_0^t g\left(t,s,x(s)\right)ds\right)$ ***is presented for controllability analysis. Necessary and Sufficient Conditions for the system to be Null Controllable are established. The Set Functions upon which our results hinged were extracted. Uses were made of: Unsymmetric Fubini theorem, the Controllability Standard and the Concept of Fractional Calculus were used to establish results.***

***Keywords*****:** Null Controllability, Fractional Integro-differential Systems, Banach Spaces, Fractional Calculus, Unsymmetric Fubini Theorem, Positive Definite Limited Power Control

## 1. INTRODUCTION

According to **[1],** fractional differential equations emerged as a new branch of mathematics. Fractional differential equations have been used for many mathematical models in Sciences and Engineering. The equations are considered as an alternative model to nonlinear differential equations. The theory of fractional differential equations has been studied extensively by many authors(**[2, 3 and 4] ) .**While the problems of stability for fractional differential systems are discussed in **[5,6 , and 7].**
Apart from stability, another important qualitative behavior of a dynamical system is controllability. Systematic study of controllability started over years at the beginning of the sixties when the theory of controllability based on the description in the form of state space for both time-varying and time-invariant linear control systems are carried out. Roughly speaking, controllability generally means that, it is possible to steer a dynamical control system from an initial state x(o) of the system to any final state x(t) in some finite time using the set of admissible controls**[8] .**The concept of controllability plays a major role in both finite and infinite dynamical systems, that is systems represented by ordinary differential equations and partial differential equations respectively. So it is natural to extend this concept to dynamical systems represented by fractional differential equations. Many partial fractional differential equations and Integro-differential equations can be expressed as fractional differential equations and Integro-differential equations in Banach spaces **[9].**
There exist many works on finite dimensional controllability of linear systems **[10]** and infinite dimensional systems in abstract spaces **[11].** The controllability problems of nonlinear systems and Integro-differential systems with delays have been carried out by many researchers in both finite and infinite dimensional spaces **[12,13 and 14].** Controllability of fractional differential systems in finite dimensional space have been studied by **[15]** and **[16].**While in **[17]** Controllability of fractional Integro-differential systems in Banach spaces wasstudied. Not alone,Relative Controllability of fractional Integro-differential systems in Banach spaces with Distributed Delays in the Constrained Control was studied.
One of the celebrated triumphs of LaSalle as contained **[18]** was his solution of the null controllability problem of Linear Ordinary Differential Control System

$$\dot{x} = Ax + Bu \qquad (1)$$

Where A isnxn square constant matrix, B is nxm constant matrix, while the control powers small. i.e. Control powers are square integrable and lie in the unit cube,

$$C^m = \{\, u \in C^m : |u_i| \le 1\,;\, i = 1,2,\dots,n \,\} \qquad (2)$$

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Here, $u_i$ denotes the ith component of u. In his work,**[19]** "The Time Optimal Control Problem, in Theory of Nonlinear Oscillations", he showed that if system (1) is proper (and this holds if and only if

$$rank[B, AB, A^2B, \ldots, A^{n-1}B] = n \;) \quad (3)$$

And if the system

$$\dot{x} = Ax \quad (4)$$

Is stable ( i.e., all the eigen values of A have no positive real part ) , then system(1) is null controllable with constraints ( i.e. , limited control powers u lie in $\bar{B}_1(0)$ ).

The condition in system (3) is equivalent to the controllability of system (1) when the controls are unlimited "big" in the sense that they are only assumed to be square integrable. This is equivalent to null controllability with square integrable controls. We call such controls unrestrained or unlimited in contrast to the restrained controls or limited controls which lie in a closed and bounded set.

For the delay system

$$\dot{x}(t) = L(t, x_t) + B(t)u(t), t > t_0 \quad (5)$$

$$x_{t_0} = \phi \in W_2^{(1)}([-h, 0], R^n) \equiv W_2^{(1)}$$

null controllability is not equivalent to controllability . For example, all nth order scalar differential difference equations of retarded type are null controllable**[20],** whereas they are never controllable**[20] .**It wasproved in **[21]** that if system(5) is controllable with unrestrained controls, and if

$$\dot{x}(t) = L(t, x_t) t > t_0 \quad (6)$$

is uniformly asymptotically stable, then system (5) is null controllable with constrained controls

(limited control powers).The problem was posed on whether the weaker the weaker condition of null controllability with unrestrained control powers and the uniform asymptotic stability assumption was sufficient for restrained null controllability. The issue is affirmatively settled in this paper and extends any other results in the literature.

The nonlinear infinite delay system of the form :

$$\dot{x}(t) = L(t, x_t) + B(t)u(t) + \int_{-\infty}^{0} A(\theta)\, x(t+\theta) d\theta + f\big(t, x_t, u(t)\big), t > t_0 \quad (7)$$

was studied in **[18]** and showed that system (7) is Euclidean null controllability if the linear base system

$$\dot{x}(t) = L(t, x_t) + B(t)u(t), t > t_0 \quad (8)$$

is proper and the free system

$$\dot{x}(t) = L(t, x_t) + \int_{-\infty}^{0} A(\theta)\, x(t+\theta) d\theta, t > t_0 \quad (9)$$

is uniformly asymptotically stable, provided that f satisfies some growth conditions. An analogous result was obtained in [**21]** for the delay system

$$\frac{d}{dt} D(t, x_t) = L(t, x_t) + B(t)u(t) + \int_{-\infty}^{0} A(\theta)\, x(t+\theta) d\theta + f\big(t, x_t, u(t)\big), t > t_0$$

$$x(t) = \phi(t), t \in (-\infty, 0)$$

This result was in **[17],** extended to perturbed delay systems with distributed delays in the control.

We shall now consider the system whose state is given by

$$\frac{d^n x(t)}{dt^n} = Ax(t) + \int_{-h}^{0} d_\theta H(t, \theta) u(t+\theta) + f\left(t, x(t), \int_{0}^{t} g\big(t, s, x(s)\big) ds\right) \quad (10)$$

where A is an nxn continuous matrix and $f\left(t, x(t), \int_0^t g\big(t, s, x(s)\big)ds\right)$ is continuous. The controls of interest, u, are square integrable with values in the unit cube $C^m$ $, given\ by$

$$C^m = \{ u \in C^m : |u_i| \le 1 ; i = 1, 2, \ldots, n \}$$

Here, we develop sufficient computable criteria for the null controllability of system(10)'

Our objective, therefore, is to study the controllability of the perturbed system described by system (10) through its linear base control system

$$\frac{d^n x(t)}{dt^n} = Ax(t) + \int_{-h}^{0} d_\theta H(t, \theta) u(t+\theta) \quad (11)$$

and it free system

$$\frac{d^n x(t)}{dt^n} = Ax(t) (12)$$

**Definition1.1 (Solution)**

*A continuous function* $x : [t_0 - \gamma, t_0 + a] \to R^n$ *is said to be a solution of system* (12)

*if there exists* $t_0 \in R$ , $a > 0$ *such that* $x \in B([t_0 - \gamma, t_0 + a], R^n)$ , $t \in (t_0, t_0 + a)$ *and* $x$ *satisfies system*(12) *on* $[t_0, t_0 + a]$ .
*Given* $t_0 \in R$ , $\phi \in B$ , *we say that* $x(t_0, \phi)$ *is a solution of system*(12) *with initial value* $(t_0, \phi)$ *if there exists an* $a > 0$ *such that* $x(t_0, \phi)$ *is a solution of system*(12) *on* $[t_0 - \gamma, t_0 + a]$ *and* $x_{t_0}(t_0, \phi) = \phi$.

## 2. PRELIMINARIES AND DEFINITIONS

Let n be a positive integer and $R = (-\infty, \infty)$ be the real line . Denote $R^n$ = the space of real n − tuples called the Euclidean space with norm denoted by |. |. If $J = [t_0, t_1]$ is any interval of R, $L_2$ is Lebesgue space of square integrable functions from J to $R^n$ written as $L_2([t_0, t_1], R^n)$. Let $h > 0$ be positive real number and let $C([t_0, t_1], R^n)$ be the Banach space of continuous functions with norm of uniform convergence defined by $\|\phi\| = \sup\phi(s); \phi \in C([t_0, t_1], R^n)$.
If x is a function from $[-h, \infty)$ to $R^n$ , then $x_t$ is a function defined on the delay interval $[-h, 0]$ given as :
$x_t(s) = x(t - s); s \in [-h, 0], t \in [0, \infty)$.

***Definition 2.1*** ([7].)
*The Riemann − Liouville fractional integral operator of order* $\beta > 0$ *of function* $f \in C_n$ , $n \geq -1$ *is defined as*:

$$I^{\beta} f(t) = \frac{1}{\rho(\beta)} \int_0^t (t - s)^{\beta - 1} f(s) ds$$

***Definition 2.2*** (***fractional derivative***)
*If the function* $f \in C^m$ *and m is positive integer, then we can define the fractional derivative of* $f(t)$ *in the Caputo sense as*:

$$\frac{d^n f(t)}{dt^n} = \frac{1}{\rho(m-n)} \int_0^t (t - s)^{m-n-1} f^m(s) ds \; ; \; m - 1 < n \leq m.$$

*If* $m = 1$, *then* $m - 1 < n \leq m$ *becomes* $0 < n \leq 1$. *Then*

$$\frac{d^n f(t)}{dt^n} = \frac{1}{\rho(1 - n)} \int_0^t (t - s)^{1-n-1} f^1(s) ds = \frac{1}{\rho(1 - n)} \int_0^t (t - s)^{-n} f^1(s) ds$$

$$= \frac{1}{\rho(1 - n)} \int_0^t \frac{1}{(t - s)^n} f^1(s) ds = \frac{1}{\rho(1 - n)} \int_0^t \frac{f^1(s)}{(t - s)^n} ds,$$

*where* $f^1(s) = \dfrac{df(s)}{ds}$ *and* $f$ *is an abstract function with values in* $X = R$.

### 2.1. VARIATION OF CONSTANT FORMULA

*Consider the following system represented by the fractional Integro − differential equations in Banach spaces with distributed delays in the limited control powers of the form*:

$$\frac{d^n f(t)}{dt^n} = Ax(t) + \int_{-h}^{0} d_\theta H(t, \theta)\, u(t + \theta) + f\left(t, x(t), \int_0^t g\big(t, s, x(s)\big) ds\right) \quad (2.1)$$

$x(0) = x_0$ ; $t \in J = [t_0, t_1]$.
*where the state* $x(.)$ *takes values in the Banach space* $X$, $0 < n < 1$, *the control function* $u \in L_2([t_0, t_1], U)$ , *a Banach space of admissible control functions with* $U$ *as a Banach space*. $H(t, \theta)$ is an nxm matrix function continuous at t and of bounded variation in θ on $[-h, 0]$, $h > 0$ *for each* $t \in [t_0, t_1]$; $t_1 > t_0$. The integral is in the Lebesgue − Stieltjes sense and is denoted by the symbol $d_\theta$. And the nonlinear operators f: JxXxX → X , g: ΔxX → X are continuous; $\Delta = \{(t, s): 0 \leq s \leq t \leq t_1\}$.

If, $$Gx(t) = \int_{t_0}^{t} g\big(t, s, x(s)\big)\, ds, \quad (2.2)$$

then the equation (2.1)becomes equivalent to the following nonliear integral equation

$$x(t) = x_0 + \frac{1}{\rho(n)} \int_{t_0}^{t} (t - s)^{n-1} Ax(s) ds + \frac{1}{\rho(n)} \int_{t_0}^{t} (t - s)^{n-1} \left[\int_{-h}^{0} d_\theta H(t, \theta)\, u(t + \theta)\right] ds$$

$$+\frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1} f\big(t,x(t),Gx(s)\big)\mathrm{ds} \qquad (2.3)$$

And the mild solution of the system (2.1) is given by

$$\mathrm{x(t)} = T(t)x_0 + \frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}\, T(t-s)\left[\int_{-h}^{0} d_\theta H(t,\theta)\, u(t+\theta)\right]\mathrm{ds}$$

$$+\frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}\, T(t-s) f\big(t,x(t),Gx(s)\big)\mathrm{ds} \qquad (2.4)$$

which is similar to the concept defined in **[22].**

*For the limiting case,* $n \to 1$ *, the above system* (2.4) *reprsentation becomes*

$$\mathrm{x(t)} = T(t)x_0 + \int_{t_0}^{t} T(t-s)\int_{-h}^{0} d_\theta H(t,\theta)\, u(t+\theta)\mathrm{ds}$$

$$+\int_{t_0}^{t} T(t-s)\, f\big(t,x(t),Gx(s)\big)\mathrm{ds} \qquad (2.5)$$

.

*Which is the mild solution of system* (2.1) *vis − avis system* (2.6) *given below*:

$$\frac{dx(t)}{dt} = Ax(t) + \int_{-h}^{0} d_\theta H(t,\theta)\, u(t+\theta) + f\big(t,x(t),Gx(s)\big) \qquad (2.6)$$

*with initial condition* $x(0) = x_0 \in X$.

*Analogus to the conventional controllability concept. A careful observation of the solution of the system* (2.1) *given as system* (2.4) *shows that the values of the control function* $u(t)$ *for* $t \in [-h, t_1]$ *enter the definition of complete state thereby creating the need for an explicit variation of constant formula. The control in the* 2nd *term of the formula* (2.4), *therefore, has to be separated in the intervals* $[-h, 0]$ *and* $[0, t_1]$.

*To achieve this that* 2nd *term of system* (2.4) *has to be transformed by applying the method of Klamka as contained in* [**23**].. *Finally, we interchange the order of integration using the Unsymmetric Fubuni theorem to have*

$$\mathrm{x(t)} = T(t)x_0 + \int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}\, T(t-s)H(s,\theta)u(s+\theta)ds\right)$$

$$+\quad \frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}\, T(t-s)f\big(t,x(t),Gx(s)\big)\mathrm{ds} \qquad (2.7)$$

$$\Rightarrow\ \mathrm{x(t)} = T(t)x_0 + \int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{t_0+\theta}^{t+\theta}(t-s)^{n-1}\, T(t-s)H(s-\theta,\theta)u(s-\theta+\theta)ds\right)$$

$$+\quad \frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}\, T(t-s)f\big(t,x(t),Gx(s)\big)\mathrm{ds} \qquad (2.8).$$

*Simplifying system* (2.8), *we have*

$$\mathrm{x(t)} = T(t)x_0 \quad + \quad \frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}\, T(t-s)f\big(t,x(t),Gx(s)\big)\mathrm{ds}$$

$$+\int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0+\theta}^{0}(t-s)^{n-1}\, T(t-s)H(s-\theta,\theta)u_0(s)ds\right)$$

$$+\int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0}^{t+\theta}(t-s)^{n-1}\, T(t-s)H(s-\theta,\theta)u(s)ds\right) \qquad (2.9)$$

*Using again the Unsymmetric Fubuni Theorm on the change of the order of integration and incorporating* $H^*$ *as defined below*:

$$H^*(s-\theta,\theta)=\begin{cases}H(s-\theta,\theta), for\ s\le t\\ 0 \quad\quad , fors\ge t\end{cases} \quad (2.10)$$

System (.2.9) becomes

$$\text{x(t)} = T(t)x_0 + \frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}\,T(t-s)f\big(t,x(t),Gx(s)\big)\text{ds}$$
$$+\int_{-h}^{0}d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0+\theta}^{0}(t-s)^{n-1}\,T(t-s)H(s-\theta,\theta)u_0(s)ds\right)$$
$$+\int_{t_0}^{t}\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}\,T(t-s)d_\theta H^*(s-\theta,\theta)u(s)\right]ds \quad (2.11)$$

*Integration is still in the Lebesgue Stieltjes sense in the variable $\theta$ in $H$.*
*For brevity, let*

$$\boldsymbol{\alpha(t,s)} = T(t)x_0 + \frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}\,T(t-s)f\big(t,x(t),Gx(s)\big)\text{ds} \quad (2.12)$$

$$\boldsymbol{\beta(t,s)} = \int_{-h}^{0}d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0+\theta}^{0}(t-s)^{n-1}\,T(t-s)H(s-\theta,\theta)u_0(s)ds\right) \quad (2.13)$$

$$\boldsymbol{Y(t,s)} = \frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}\,T(t-s)d_\theta H^*(s-\theta,\theta) \quad (2.14)$$

*Substituting equations (2.12), (2.13) and (2.14) in equation (2.11), we have a precise variation of constant formula for the system (2.1) as:*

$$\boldsymbol{x(t,x_0,u)} = \boldsymbol{\alpha(t,s)}+\boldsymbol{\beta(t,s)}+\int_{t_0}^{t}\boldsymbol{Y(t,s)u(s)ds} \ (2.15).$$

### 2.2. BASIC SET FUNCTIONS AND PROPERTIES

**Definition 2.2.1 (Reachable set)**

*The reachable set of the system (2.1) denoted by $R(t,t_0)$ is given as :*

$$R(t,t_0)=\left\{\int_{t_0}^{t}\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}\,T(t-s)d_\theta H^*(s-\theta,\theta)u(s)ds : u\in U; |u_j|\le 1\ ;\ j=1,2,\dots,m\right\}$$

*Where $U=\{u\in L_2([t_0,t_1],E^m)\}$*

**Definition 2.2.2(Attainable set)**

*The attainable set of the system (2.1) denoted by $A(t,t_0)$ is given as :*

$A(t,t_0)=\{x(t,x_0,u): u\in U; |u_j|\le 1\ ;\ j=1,2,\dots,m\}$, *where* $U=\{u\in L_2([t_0,t_1],R^m)\}$

**Definition 2.2.3 (Target set)**

*The Target set for the system(2.1) denoted by $G(t,t_0)$*is given by

$G(t,t_0)=\{x(t,x_0,u) : t\ge\tau>t_0 , for\ \ some\ fixed\ \ \tau\ and\ u\in U\}$

**Definition 2.2.4 (Controllability grammian or Map)**

*In system(2.11), we introduce the notation*

$\boldsymbol{Y(t,s)}=\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}\,T(t-s)d_\theta H^*(s-\theta,\theta)\right], t\ge s\ge t_0,$

*and define the controllability grammian orcontrollability map of the system (2.1) by*

$$\boldsymbol{W(t,t_0)}=\int_{t_0}^{t}\boldsymbol{Y(t,s)Y^T(t,s)ds},$$

*where $T$ denotes matrix transpose*

**Definition 2, 2.5 (Positive Definite)**

*The controllability grammian or map $W$ is said to be positive definite if $W$ varnishes only at the origin and $W(x)>0$ for all $x\ne 0, x\in D$, where $D=\{x\in E^n : \|x\|\le r ; r>0\}\subset R^n$*

### 2.3. RELATIONSHIP BETWEEN THE SET FUNCTIONS

*We shall first establish the relationship between the attainable set and the reachable set, to enable us see that once a property has been proved for one set function, then it is applicable to the other.* From equation (2.11),

$\boldsymbol{A(t,t_0) = [\eta(t) + R(t,t_0)]\,, for\ u \in U; t \in [t_0,t_1], where\ ,\eta(t) = \alpha(t\ ,s) + \beta(t\ ,s)}$.
This means that the attainable set is the translation of the reachable set through the $origin\ \eta \in E^n$. $Using\ the\ attainable\ set, therefore, it\ is\ easy\ to\ show\ that\ the\ set\ functions\ possess\ the\ properties\ of\ convexity, closedness, and\ compactness. Not\ alone,\ the\ set\ functions\ are\ continuous\ \ on\ [0,\infty) to\ the\ metric\ space\ of\ compact\ subsets\ of\ E^n$ $The\ impetus\ for\ adaptations\ of\ the\ proofs\ of\ these\ properties\ for\ system (2.1) wasgiven\ in$[**24**].

**Definition 2.3.1 (Relative controllability)**
$The\ system (2.1)\ is\ relatively\ controllable\ \ on\ the\ interval\ [t_0,t_1]\ if$
$A(t,t_0) \cap G(t,t_0) \neq \phi, t > t_0 \in [t_0,t_1]$'

**Definition 2.3.2 (Properness)**
$The\ system (2.1)\ is\ proper\ in\ E^n\ on\ the\ interval\ [t_0,t_1]\ if\ spanR(t\ ,t_0) = R^n$

$$i.e.if, \qquad C^T \int_{t_0}^{t} \left[ \frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} T(t-s) d_\theta H^*(s-\theta,\theta) \right] ds = 0\ \ a.e\ , \Rightarrow C = 0\ ; C \in R^n.$$

**Definition 2.3.3 (Complete state)**
$We\ denote\ the\ complete\ state\ of\ system (2.1)\ at\ time\ \ t\ \ by$
$\boldsymbol{z(t) = \{x(t), u_t\}}$.
$Then\ , the\ initial\ complete\ state\ of\ systm (2.1)\ at\ time\ t_0\ is\ given\ by$
$\boldsymbol{z(t_0) = \{x_0\ , u_{t_0}\}}$

**Definition 2.3.4 (Null controllability)**
$The\ system (2.1) is\ said\ to\ be\ null\ controllable\ on [t_0,t_1]\ if\ for\ each\ \phi \in B([-\gamma\ ,0]\ , R^n),$ $there\ exists\ \ a\ t_1 > t_0, u \in L_2([t_0,t_1]\ , P)\ , P\ a\ compact\ convex\ subset\ of\ R^m\ , such\ that$ $the\ solution\ x(t\ ,t_0\ ,\phi\ ,u\ ,f)\ of\ system (2.1) satisfies$
$x_{t_0}(t_0\ ,\phi\ ,u\ ,f) = \phi\ , and\ x(t_1\ ,t_0\ ,\phi\ ,u\ ,f) = 0.$

**Definition 2.3.5**
$The\ system (2.1) is\ said\ to\ be\ null\ controllable\ on [t_0,t_1]\ if\ for\ each\ function\ \phi\ and\ every$ $x_1 \in R^n$, there exists an admissible control function , u, such that a solution of $system (2.1)$ satisfies $\mathrm{x}(t_1) = x_1$ .

## 3. MAIN RESULT

$The\ issue\ of\ \ controllability\ of\ Neutral\ Volterra\ Integrodifferential$ $Equations\ have\ been\ settled\ in\ [12,25] as\ contained\ in$ [**14**].
$From\ the\ results\ of\ these\ studies\ the\ following\ equivalent\ statements\ emerge.$

**Theorem 3.1.(Necessary conditions )**
**Consider the system**

$$\frac{d^n x(t)}{dt^n} = Ax(t) + \int_{-h}^{0} d_\theta H(t,\theta) u(t+\theta) + f\left( t, x(t), \int_{0}^{t} g\big(t,s,x(s)\big) ds \right) \qquad (3.1)$$

$x(0) = x_0\ \ ; t \in J = [t_0\ ,t_1].$
$with\ the\ same\ conditions\ on\ the\ system' sparameters\ as\ in\ the\ system (2.1), then\ the\ following\ statements\ are\ equivalent:$
$(i). System (3.1) is\ relatively\ controllable\ on\ the\ interval\ J = [t_0\ ,t_1].$
$(ii). The\ controllability\ grammian\ \ \boldsymbol{W(t\ ,t_0)}\ of\ system (3.1)\ \ is\ non-singular.$
$(iii). System (3.1)\ is\ proper\ on\ the\ interval\ J = [t_0\ ,t_1].$

***PROOF***:
$\big((\boldsymbol{i}) = (\boldsymbol{ii}).\big)$
$Recall: The\ controllability\ grammian\ \boldsymbol{W(t\ ,t_0)}\ of\ the\ system (3.1)\ is\ non-singular, is$ $equivalent\ to\ saying\ that\ \boldsymbol{W(t\ ,t_0)}\ is\ positive\ definite, which\ in\ turn\ \ is\ equivalent\ to\ saying$ $that\ C^T times\ the\ controllability\ index\ of\ the\ system (3.1)\ is\ equal\ to\ zero\ almost\ everywhere$ $on\ the\ interval\ [t_0\ ,t_1]\ , implying\ that\ C = 0.; T\ denotes\ matrix\ transpose$

$$i.e.\, C^T \int_{t_0}^{t} \left[ \frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} T(t-s) d_\theta H^*(s-\theta,\theta) \right] = 0 \;\; a.e\,, \Rightarrow C = 0\,; C \in E^n,$$

*which is properness of the system*(3.1) *since the integral is non* − *negative.*
*This, therefore, showsed that* (***i***) *is equivalent to* (***ii***), ***or*** (***i***) = (***ii***).

***To show that*** (***ii***) ***and*** (***iii***) ***are equivalent.***
*By the definition of properness of the system*(3.1), *we have* (***ii***) *given as* :

$$C^T \int_{t_0}^{t} \left[ \frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} T(t-s) d_\theta H^*(s-\theta,\theta) \right] = 0 \;\; a.e\,, \Rightarrow C = 0\,; C \in E^n\,,$$

*for each* $s \in [t_0, t_1]$, *then*

$$\int_{t_0}^{t} C^T \left[ \frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} T(t-s) d_\theta H^*(s-\theta,\theta) \right] u(s) ds$$
$$= C^T \int_{t_0}^{t} \left[ \frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} T(t-s) d_\theta H^*(s-\theta,\theta) \right] u(s) ds = 0 \;\;, for\ u \in L_2 \quad (3.2)$$

It follows from this last equation (3.2) that C is orthogonal to the reachable set

$$R(t,t_0) = \left\{ \int_{t_0}^{t} \frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1}\, T(t-s) d_\theta H^*(s-\theta,\theta) u(s) ds : u \in C^m; \left|u_j\right| \le 1\,;\; j = 1,2,\dots,m \right\}$$

*If we assume the relative controllability of the system*(3.1) *now*, $R(t,t_0) = R^n$, *sothat*
$C = 0$, *showing that* (***iii***) *implies* (***ii***). *Or* (***i***) *is equivalent to* (***ii***) *and*
(***ii***) *is equivalent to* (***iii***) *and vis* − *a* − *vis* (***iii***) *is equivalent to* (***ii***) *is equivalent to* (***i***).
*Conversely, assume that system*(3.1) *is not controllable, sothat the*
*reachable* $R(t,t_0) \ne R^n$ *for* $t > t_0$. *Then, there exists* $C \ne 0, C \in R^n$, *such that*
$C^T R(t,t_0) = 0$.
*It follows that for all admissible controls* $u \in L_2$ *that*

$$0 = C^T \int_{t_0}^{t} \left[ \frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} T(t-s) d_\theta H^*(s-\theta,\theta) \right] u(s) ds \;\;, for\ u \in L_2$$
$$= \int_{t_0}^{t} C^T \left[ \frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} T(t-s) d_\theta H^*(s-\theta,\theta) \right] u(s) ds$$

Hence,

$$C^T \int_{t_0}^{t} \left[ \frac{1}{\rho(n)} \int_{-h}^{0} (t-s)^{n-1} T(t-s) d_\theta H^*(s-\theta,\theta) \right] u(s) ds = 0\,, a.e\,;\; s \in [t_0, t_1]\;, C \ne 0.$$

*By definition of properness, it implies that the system*(3.1) *is not proper, since*
$C \ne 0$. *Hence the system*(3.1) *is relatively controllable. And or controllable.*

**THEOREM 3.2(Sufficient condition)**
**Assume for system (10) that :**
(***i***). ***The constraint set U is an arbitrary compact subset of*** $\boldsymbol{R^n}$.
(***ii***). ***The system***(**12**) ***is uniformly asymptotically stable sothat the solution of system***(**12**) ***satisfies***
$\|\boldsymbol{x}(\boldsymbol{t},\boldsymbol{t_0},\boldsymbol{\phi},\boldsymbol{0},\boldsymbol{0})\| \le \boldsymbol{M}e^{-a(t-t_0)}\|\boldsymbol{\phi}\|$, ***for some*** $\boldsymbol{a} > 0, M > 0$.
(***iii***). ***The linear control system***(**11**) ***is proper in*** $\boldsymbol{R^n}$.
(***iv***). ***The continuous function f satisfies***
$\left|\boldsymbol{f}(\boldsymbol{t},\boldsymbol{x}(\boldsymbol{s}),\boldsymbol{u}(\boldsymbol{s}))\right| \le \boldsymbol{exp}(-\boldsymbol{bt})\boldsymbol{\pi}(\boldsymbol{x}(\boldsymbol{s}),\boldsymbol{u}(\boldsymbol{s}))$
***for all***$(\boldsymbol{t},\boldsymbol{x}(.),\boldsymbol{u}(.)) \in [\boldsymbol{t_0},\infty)\boldsymbol{xBxL_2}$, ***where*** $\int_{\boldsymbol{t_0}}^{\infty} \boldsymbol{\pi}(\boldsymbol{x}(\boldsymbol{s}),\boldsymbol{u}(\boldsymbol{s}))\boldsymbol{ds} \le \boldsymbol{k} < \infty$ ***and***
$\boldsymbol{b} - \boldsymbol{a} \ge \boldsymbol{0}$, ***then*** **system** (**10**) ***is null controllable.***

**PROOF**
*By theorem* 3.1 *above*, (*ii*) *and* (*iii*), $W^{-1}(t_1, t_0)$ *exists for each* $t_1 > t_0$.
*Suppose the pair of functions x, u form a solution pair to the set of integral equations*

$$\mathrm{x(t)} = T(t)x_0 + \frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}\,T(t-s)f\big(t,x(t),Gx(s)\big)\mathrm{ds}$$
$$+\int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0+\theta}^{0}(t-s)^{n-1}\,T(t-s)H(s-\theta,\theta)u_0(s)ds\right)$$
$$+\int_{t_0}^{t}\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}\,T(t-s)d_\theta H^*(s-\theta,\theta)u(s)\right]ds \qquad (3.3)$$

$$\mathrm{u(t)} = -\boldsymbol{Y^T(t_1,s)W^{-1}(t_1,t_0)}T(t)x_0 - \boldsymbol{Y^T(t_1,s)W^{-1}(t_1,t_0)}\frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}\,T(t-s)f\big(t,x(t),Gx(s)\big)\mathrm{ds}$$
$$-\boldsymbol{Y^T(t_1,s)W^{-1}(t_1,t_0)}\int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0+\theta}^{0}(t-s)^{n-1}\,T(t-s)H(s-\theta,\theta)u_0(s)ds\right) \quad (3.4)$$

*Then, u is square integrable on* $[t_0\,,t_1]$ *and x is a solution of system*(2.1)*corresponding to u with initial state* $\boldsymbol{x(t_0)} = \boldsymbol{\phi}$ *. Now,*

$$x(t_1) = \mathrm{x(t)} = T(t_1)x_0 + \frac{1}{\rho(n)}\int_{t_0}^{t_1}(t_1-s)^{n-1}\,T(t_1-s)f\big(t_1,x(t_1),Gx(s)\big)\mathrm{ds}$$
$$+\int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0+\theta}^{0}(t_1-s)^{n-1}\,T(t_1-s)H(s-\theta,\theta)u_0(s)ds\right)$$
$$+\int_{t_0}^{t_1}[Y(\boldsymbol{t_1},\boldsymbol{s})].\,[-\boldsymbol{Y^T(t_1,s)W^{-1}(t_1,t_0)}T(t)x_0]$$
$$+\left[\int_{t_0}^{t_1}Y(\boldsymbol{t_1},\boldsymbol{s})\right]\left[-\boldsymbol{Y^T(t_1,s)W^{-1}(t_1,t_0)}\frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}\,T(t-s)f\big(t,x(t),Gx(s)\big)\mathrm{ds}\right]+$$
$$\int_{t_0}^{t_1}[Y(\boldsymbol{t_1},\boldsymbol{s})]\,[-\boldsymbol{Y^T(t_1,s)W^{-1}(t_1,t_0)}]\int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0+\theta}^{0}(t-s)^{n-1}\,T(t-s)H(s-\theta,\theta)u_0(s)ds\right)$$
$$(3.5)$$

But,

$$\int_{t_0}^{t_1}[Y(\boldsymbol{t_1},\boldsymbol{s})][-\boldsymbol{Y^T(t_1,s)W^{-1}(t_1,t_0)}] = \frac{-\int_{\boldsymbol{t_0}}^{\boldsymbol{t_1}}[Y(\boldsymbol{t_1},\boldsymbol{s})]\boldsymbol{Y^T(t_1,s)}}{\boldsymbol{W(t_1,t_0)}}$$

$$= \frac{-\int_{\boldsymbol{t_0}}^{\boldsymbol{t_1}}[Y(\boldsymbol{t_1},\boldsymbol{s})]\boldsymbol{Y^T(t_1,s)}}{-\int_{\boldsymbol{t_0}}^{\boldsymbol{t_1}}[Y(\boldsymbol{t_1},\boldsymbol{s})]\boldsymbol{Y^T(t_1,s)}} = \boldsymbol{-1}$$

*Then, equation*(3.5) *becomes*

$$x(t_1) = T(t_1)x_0 + \frac{1}{\rho(n)}\int_{t_0}^{t_1}(t_1-s)^{n-1}\,T(t_1-s)f\big(t_1,x(t_1),Gx(s)\big)\mathrm{ds}$$
$$+\int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0+\theta}^{0}(t_1-s)^{n-1}\,T(t_1-s)H(s-\theta,\theta)u_0(s)ds\right)$$
$$-T(t)x_0 - \left[\frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}\,T(t-s)f\big(t,x(t),Gx(s)\big)\mathrm{ds}\right]$$
$$-\int_{-h}^{0} d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0+\theta}^{0}(t-s)^{n-1}\,T(t-s)H(s-\theta,\theta)u_0(s)ds\right) = 0.$$

*It remains to prove that the function* $u\colon[t_0\,,t_1]\to C^m$ *is in the arbitrary compact constraint subset of* $R^m$*, that is* $|u|\le\rho_1$ *, for some constant* $\rho_1>0$.

*By part (ii) of theorem*3.2 *above,* $|\boldsymbol{Y^T(t_1,s)W^{-1}(t_1,t_0)}|\le\lambda_1$ , for some constant $\lambda_1>0$.

Thus, $|u(t)|\le\lambda_1[\lambda_2\exp\big(-\rho(\boldsymbol{t_1}-\boldsymbol{t_0})\big)]\int_{t_0}^{t_1}\lambda_3\exp\big[-\rho(t_1-s)exp(-ks)\pi\big(x(s),u(s)\big)ds\big]$

It follows that

$|u(t)|\le\lambda_1\big[\lambda_2\exp\big(-\rho(\boldsymbol{t_1}-\boldsymbol{t_0})\big)\big]+\lambda\lambda_3\exp(-\rho t_1)$(3.6)

*since* $k-\rho\ge0$ *and* $t\ge t_0\ge0$. *Thus, by taking t , sufficiently large, we have*

$|u(t)| \le \rho_1\,;t \in [t_0\,,t_1]\,,showing\ that\ u\ is\ an\ admissible\ control.$

Now, we need to prove the existence of a solution pair of the integral equations ( 3.3) and ( 3.4).

$Let\ N\ be\ the\ Banach\ space\ of\ all\ functions\ (x,u):[t_0-h\,,t_1]x[t_0-h\,,t_1] \to R^nR^m,$

$where\ x \in N([t_0-h\,,t_1]\,,R^n)\,;u \in L_2([t_0-h\,,t_1]\,,R^m)\ with\ the\ norm\ defined\ by$

$\|(x\,,u)\| = \|x\|_2 + \|u\|_2$

$$where\,,\quad \|x\|_2 = \left[\int_{t_0-h}^{t_1} |x(s)|^2\,ds\right]^{1/2}\ ;\ \|u\|_2 = \left[\int_{t_0-h}^{t_1} |u(s)|^2\,ds\right]^{1/2}.$$

$Define\ the\ operator\ \boldsymbol{T}:\boldsymbol{N}\to\boldsymbol{N}\ by\ T(x\,,u) = (y,v), where$

$$\text{v(t)} = -\boldsymbol{Y^T(t_1\,,s)W^{-1}(t_1\,,t_0)}T(t)x_0$$
$$-\boldsymbol{Y^T(t_1\,,s)W^{-1}(t_1\,,t_0)}\frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}\,T(t-s)f\big(t,x(t),Gx(s)\big)\text{ds}$$
$$-\boldsymbol{Y^T(t_1\,,s)W^{-1}(t_1\,,t_0)}\int_{-h}^{0}d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0+\theta}^{0}(t-s)^{n-1}\,T(t-s)H(s-\theta,\theta)u_0(s)ds\right)\quad(3.7)$$

$And\ v(t) = u(t), t \in [t_0-h\,,t_0].$

$$y(t) = T(t)x_0 + \frac{1}{\rho(n)}\int_{t_0}^{t}(t-s)^{n-1}\,T(t-s)f\big(t,x(t),Gx(s)\big)\text{ds}$$
$$+\int_{-h}^{0}d_{H_\theta}\left(\frac{1}{\rho(n)}\int_{0+\theta}^{0}(t-s)^{n-1}\,T(t-s)H(s-\theta,\theta)u_0(s)ds\right)$$
$$+\int_{t_0}^{t}\left[\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}\,T(t-s)d_\theta H^*(s-\theta,\theta)u(s)\right]ds\,,t \in\ [t_0\,,t_1]\qquad(3.8)$$

$And\ y(t) = \phi(\text{t})\,,for\ t \in [t_0-h\,,t_0].$

$It\ was\ already\ shown\ that$

$\boldsymbol{|v(t)| \le \rho_1\,, for\ t \in\ [t_0\,,t_1]\ and\ also\,, for\ the\ function\ v:[t_0-h\,,t_0] \to C^m}\,,$

$\boldsymbol{we\ have}$

$\boldsymbol{v(t)\ \le \rho_1}\,.$

$Thus\,,we\ have$

$$\boldsymbol{\|v(t)\|_2 \le \rho_1(t_1+h-t_0)^{1/2} = k_0}$$

$$And, |y(t)| \le \lambda_2 exp[-\rho(t-t_0)] + \lambda_4\int_{t_0}^{t}|v(s)|\,ds + \lambda\lambda_3 exp(-\rho t)$$

$$Put\quad \lambda_4 = sup\left|\frac{1}{\rho(n)}\int_{-h}^{0}(t-s)^{n-1}\,T(t-s)d_\theta H^*(s-\theta,\theta)\right|.$$

$Since\ \rho > 0; t \ge t_0 \ge 0, we\ have\ it\ that\ for\ \boldsymbol{t} \in\ [\boldsymbol{t_0}\,,\boldsymbol{t_1}]$

$\boldsymbol{|y(t)| \le \lambda_2 + \lambda_4\rho(t_1-t_0) + \lambda\lambda_3 = k_1}$

$\boldsymbol{And\ \ for\ t} \in [t_0-h\,,t_0]\,,we\ have$

$|y(t)| \le sup|\phi(t)| = \boldsymbol{\delta}$

$Thus, if\ \boldsymbol{\beta = max[k_1\,,\delta]}\,,then\ \boldsymbol{\|y(t)\|_2 \le \beta(t_1+h-t_0)^{1/2}} = k_2 < \infty.$

$Now, Let\ l = max\{k_0\,,k_2\}. Then\ if\ we\ put\ G(l) = \{(x\,,u) \in N: \|x\|_2 \le l\,,\|u\|_2 \le l\}\,,$

$we\ have\ \ T:G(l) \to G(l). Since\ G(l)\ is\ bounded, closed\ and\ convex, by\ Riesz\ theorem\ as$

$contained\ in[\boldsymbol{17}], it\ is\ relatively\ compact\ under\ the\ transformation\ T. Thus,$

$the\ Schauders'\ fix\ point\ theorem\ implies\ that\ the\ function\ T\ has\ a\ fixed\ point.$

$Hence, system(2.1)\ is\ null\ controllable(Euclidean\ null\ controllable).$

**4. CONCLUSION**

Necessary and sufficient conditions for the null controllability of perturbed Fractional Integro-differential System in Banach Spaces with Distributed Delays in the Limited Power Controls have been derived and established. These conditions are given with respect to the Stability of the free linear base system and the properness of the linear controllable base system, with the assumption that perturbation f satisfies some smoothness and growth conditions. Computable criteria for all these are reported. These results extended known results in the literature.

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