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STABILITY OF THE NON-ZERO EQUILIBRIUM STATE IN A PREY-PREDATOR MODEL

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Abstract

The mathematical modeling of Prey-predator system by nonlinear autonomous equations is studied using of a system of first order nonlinear differential equations. The local and global stability of the system at the non-zero (endemic) equilibrium state is studied using the Jacobian method and Bell-Cooke's theorem respectively. One of the examples presented turned out to be globally asymptotically stable. This is mostly based on the environmental limitation factor to the growth of the prey and death rate of the predator. The ecosystem is stable for the Prey-predator interactions if the basic reproductive number $R_0 > 1$ otherwise it is unstable.

***Keywords*:** basic reproductive number; endemic (non-zero) state; prey; predator; differential equations, eigenvalues.

1.0 Introduction

The stability of endemic equilibrium state is the most important aspect in the study of a physical system such as ecosystem where there is Prey-predator relationship. However, an endemic equilibrium state has no physical significant unless it is stable. This shows that a steady state of a simple physical system corresponds to endemic equilibrium state in the phase plane [1].

An ecosystem (i.e. ecological situation) involving two species, one of which preys on the other (does not compete with it for food but actually preys on it) and the other lives on a different source of food is studied in this paper. The species who preys on the other is known as Predator, while the specie predator preys on is known as Prey. The relationship between the two species is known as Prey-predator system [1 – 9]. An example is foxes and rabbits in the bush. A Prey-predator problem is one of the fundamental problems of mathematical ecology [2].

Section 2 of this paper deals with formulation of the Prey-predator model, section 3 determines the non-trivial (i.e. non-zero or endemic) equilibrium state (or point), section 4 analyses the stability of the endemic equilibrium state (or point), section 5 considers numerical examples and section 6 deals with the conclusion and recommendation.

The aim of this study is to investigate the local and global stability of endemic equilibrium state of the system. The modeling of prey – predator system is more fully developed for two – dimensional autonomous equation using a system of differential equations. In this study the paper considered stability of unique endemic equilibrium state of prey – predator model.

Since a steady state of a simple physical system corresponds to equilibrium, the study of stability is very important in the prey – predator population, otherwise the population tends to extinction. Therefore, the mathematical descriptions of ecological systems are very important.

2.0 Model

Lokta-Volterra considered two species prey – predator modeled by the Lotka – Votlerra equation describing the two species that form a single vertical chain of relations. Lokta-Volterra or Prey-predator equations are important in ecological modeling. They have the form:

$$\frac{dx(t)}{dt} = f(x,y) = ax(t) - bx(t)y(t)$$
$$\frac{dy(t)}{dt} = g(x,y) = dx(t)y(t) - cy(t) \qquad (1)$$

Where $x(t)$ and $y(t)$ are the populations of prey and predator species respectively. The constants a, b, c and d are based on empirical observations and depends on the particular species that are being studies. In system (1), a denote the birth rate of prey, c denote the death rate of predators and the parameters b and d are the coefficients of interactions between predator and prey. $f(x,y)$ and $f(x,y)$ in system (1) are continuous and have first partial derivatives throughout the plane [1 – 4, 6 – 9].

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2.2 Formulation of A Prey-Predator Model

This paper derives a dynamic system of a Prey-predator system so as to study the dynamics of the encounters (i.e. interactions) between the prey and predator species in an ecosystem where both species cohabited. The prey population is denoted by $\boldsymbol{x(t)}$, while the predator population is denoted by $\boldsymbol{y(t)}$**.** Note that, α_1 is the environmental limitation factor to the growth of prey species, α_2 represents rate of interactions between the prey and the predator (i.e. the rate of aggression); β_1 and β_2 are the natural birth rates of x and y respectively; μ_1 and μ_2 are the natural mortality rates for prey and predator respectively and d_1 and d_2 are deaths due to hunting of prey and predator respectively by the hunters (i.e. human beings). And the f represents the combination of the efficiency of the predation and the mass ratio between prey and predator individuals.

2.3 Assumptions

The paper modeled the Prey-predator relation in an ecosystem using the following assumptions.

1. The prey (i.e. rabbits) population $x(t)$ tends to increase at a rate proportional to the number of prey $x(t)$.
2. The prey population $x(t)$ decreases at a rate proportional to $x(t)y(t)$, when foxes kill the rabbits.
3. The prey population $x(t)$ also decreases at a rate proportional to $x(t)$, due to natural deaths and when hunters kill the rabbits.
4. The predator population $y(t)$ increases at a rate proportional to encounters (i.e. interactions) between the two species $x(t)y(t)$, when foxes kill the rabbits.
5. The predator population $y(t)$ also decreases at a rate proportional to $y(t)$, due to natural deaths and when hunters kill the foxes.
6. Recruitment of prey and predator is through births.
7. There is additional deaths d_1 and d_2 induced by humans due to hunting activities.
8. The birth and death rates of prey and predators are not the same.

2.4 Schematic Diagram

In Figure 1, the circle and rectangle represent the populations of prey and predator. The thick arrows represent the progressions, while the arc represents the encounters (interactions) between the prey population and the predator population.

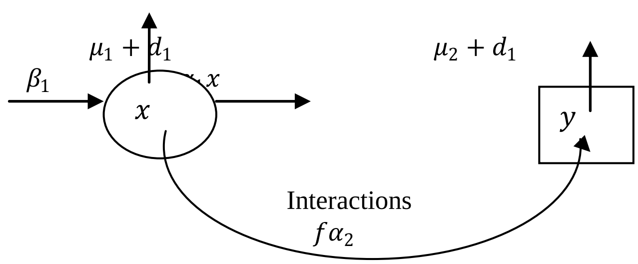

Figure 1 Schematic diagram showing probabilities and interactions between prey x and predator y populations.

2.5 Description of the Model

Let $x(t)$ denotes one of species (say rabbits) and $y(t)$ denotes another species (i.e. foxes). The rabbits have available to them, an unlimited supply of vegetation food, while foxes feed on the rabbits. There are other factors such as hunters decreasing the two populations, the encounters (i.e. interactions) between a rabbit and a fox, it results in the fox eating the rabbit at the rate $f\alpha_2$.

The prey population tends to increase at a rate proportional to the number of rabbits, but that it also decreases at the rate proportional to $x(t)y(t)$, when foxes kill the rabbits. The predator population increases at a rate proportional to encounters (i.e. interactions) between the two species, but decreases at a rate proportional to itself and this due to the absence of the prey. Thus

$$\frac{dx}{dt} = \beta_1 x - \alpha_1 x^2 - \alpha_2 xy - (\mu_1 + d_1)x$$
$$\frac{dy}{dt} = \beta_2 y + f\alpha_2 xy - (\mu_2 + d_2)y \qquad (2)$$

Where β_1 and β_2 are the birth rates of the prey x and predator y respectively, and α_1 the environmental limitation factor to the growth of the prey. The parameter $\mu_1 + d_1$ and $\mu_2 + d_2$ are the death rates of the prey and predator respectively from the natural causes and deaths due to hunting by human beings (i.e. hunters). The predator y will not survive without preying on x, with the rate of aggression α_2. The factor f represents the combination of efficiency of the predation and the average mass ratio between prey and predator individuals [1 – 9].

3.0 Determination of Equilibrium States

At equilibrium,

$x' = y' = 0$ (3)
$-\alpha_1 x^2 - \alpha_2 xy + k_1 x = 0$
$f\alpha_2 xy - k_2 y = 0$ (4)
where
$k_1 = \beta_1 - (\mu_1 + d_1)$ (5)
$k_2 = \mu_2 + d_2 - \beta_2$ (6)
From system (4),
$(-\alpha_1 x - \alpha_2 y + k_1)x = 0$ (7a)
$(f\alpha_2 x - k_2)y = 0$ (7b)
Solving (7a) and (7b) give:
$x = 0$ or $(-\alpha_1 x - \alpha_2 y + k_1) = 0$ (8a)
$y = 0$ or $(f\alpha_2 x - k_2) = 0$ (8b)
System (4) has zero (i.e. trivial) and non-zero (non-trivial or endemic equilibrium) states $(x^0, y^0) = (0,0)$ and $(x^*, y^*) = \left(\frac{k_2}{f\alpha_2}, \frac{k_1 f\alpha_2 - k_2\alpha_2}{f\alpha_2^2}\right)$ respectively.
Therefore, non-zero (endemic equilibrium) state is
$E^* = (x^*, y^*)$ (9)

3.1 Non-zero (Endemic Equilibrium) State

The non-zero (endemic equilibrium) state $E^* = (x^*, y^*) = \left(\frac{k_2}{f\alpha_2}, \frac{k_1 f\alpha_2 - k_2\alpha_2}{f\alpha_2^2}\right)$, is a steady state where the ecosystem is stable for the prey-predator system.

4.0 Stability Analysis

4.1 Theorem

The stability criteria for plane autonomous system
Let (x_0, y_0) be a critical point of the plane autonomous system
$x' = f(x, y)$
$y' = g(x, y)$ (10)
where $f(x, y)$ and $g(x, y)$ have continuous first-order partial derivatives in neighbourhood of (x_0, y_0). Let A = $\begin{bmatrix} f_x(x_0 y_0) & f_y(x_0 y_0) \\ g_x(x_0 y_0) & g_y(x_0 y_0) \end{bmatrix}$

i. If the eigenvalues of A have negative real parts, then (x_0, y_0) is a stable critical point of (1).
ii. If it has eigenvalues with positive real parts, then (x_0, y_0) is an unstable critical point of system of equations (10) [7].

To test whether the non-zero (i.e. endemic) equilibrium state is stable at long run or not, the stability analysis of non-zero (endemic) equilibrium state is carried out. The local stability and global stability of non-zero equilibrium state E^* using Jacobian method and the method in [10-14] will be established in sections 4.2 and 4.3 respectively.

4.2 Local Stability of Endemic Equilibrium State E^*

Proposition 1: The endemic (non-zero) equilibrium state E^* of model (2) is locally asymptotically stable if $R_0 > 1$.
Proof: By applying Jacobian method to the system of equations (4),

$$J(E^*) = \begin{vmatrix} -(2\alpha_1 x^* + \alpha_2 y^* - k_1 + \lambda) & -\alpha_2 x^* \\ f\alpha_2 y^* & f\alpha_2 x^* - k_2 - \lambda \end{vmatrix} = 0 \quad (11)$$

Therefore, the characteristic equation of (11) is

$$\lambda^2 + \frac{\alpha_1 k_2}{\alpha_2 f}\lambda + \frac{\alpha_1 k_2^2}{\alpha_2 f}(R_0 - 1) = 0 \quad (12)$$

where

$$R_0 = \frac{f\alpha_2 k_1}{\alpha_1 k_2} \quad (13)$$

The basic reproductive number R_0 is the survival rate for the prey species surviving the aggression by the predator species when one predator is introduced into the community.
Since $\frac{\alpha_1 k_2}{\alpha_2 f}$ and $\frac{\alpha_1 k_2^2}{\alpha_2 f}$ are positive, the eigenvalues of (12) have negative real parts, whenever $R_0 > 1$. By applying Routh-Hurwith criteria, the non-zero (endemic equilibrium) E^* state of model (2) is locally asymptotically stable (l.a.s) [10-18].

4.3 Global Stability of Non-zero (Endemic Equilibrium) State E^*

To analyse the global stability of non-zero (endemic) equilibrium state, apply the result of Bellman and Cooke (1963)[19] to system (4).

Proposition 2: The non-zero state E^* of model (2) is globally asymptotically stable if $R_0 > 1$.

Proof

$$H(\lambda) = \lambda^2 + \frac{\alpha_1 k_2}{\alpha_2 f}\lambda + \frac{\alpha_1 {k_2}^2}{\alpha_2 f}(R_0 - 1) \quad (14)$$

let $\lambda = iw$

$$H(iw) = -w^2 + \frac{\alpha_1 k_2}{\alpha_2 f} iw + \frac{\alpha_1 {k_2}^2}{\alpha_2 f}(R_0 - 1) \quad (15)$$

Resolving (15) into real and imaginary parts, we have

$$H(iw) = F(w) + iG(w) \quad (16)$$

Thus,

$$F(w) = -w^2 + \frac{\alpha_1 {k_2}^2}{\alpha_2 f}(R_0 - 1) \quad (17)$$

$$G(w) = \frac{\alpha_1 k_2}{\alpha_2 f} w \quad (18)$$

Differentiating (17) and (18) with respect to w, when $w = 0$, gives

$$F'(0) = 0 \quad (19)$$

$$G'(0) = \frac{\alpha_1 k_2}{\alpha_2 f} \quad (20)$$

Since $F'(0) = 0$, $G(0) = 0$, $F(0) \neq 0$, $G'(0) \neq 0$

Hence

$$F(0)G'(0) - F'(0)G(0) = \frac{{\alpha_1}^2 {k_2}^3}{{\alpha_2}^2 f^2}(R_0 - 1) > 0 \quad (21)$$

Therefore,

If $R_0 > 1$ then $F(0)G'(0) - F'(0)G(0) > 0$. Thus, non-zero state E^* is globally asymptotically stable [19]. This is the state where ecosystem is stable for the prey and predator populations.

5.0 Numerical Examples

The two numerical examples considered in this paper validate the model, while example 5.1 has no stable endemic (non-zero) equilibrium state and that only the prey species survive when $R_0 < 1$, example 5.2 has asymptotic stable endemic (i.e. non-zero) equilibrium state and that the two species survive the basic reproductive number $R_0 > 1$.

Example 5.1:

$$\frac{dX}{dt} = X - 3X^2 - 4XY, \quad \frac{dY}{dt} = -5Y + 4XY \quad X \geq 0, Y \geq 0 \quad (22)$$

The equilibrium point is $\left(\frac{5}{4}, -\frac{11}{16}\right)$ and $R_0 = \frac{4}{15}$ then, we have

$\lambda_{1,2} = -\frac{1}{8}\left[15 \pm \sqrt{1105}\right] = -\frac{15}{8} \pm \frac{\sqrt{1105}}{8}$,

$\lambda_1 = 2.28$ and $\lambda_2 = -6.03$.

Therefore, the system of equations (22) has no endemic (non-zero) equilibrium state and that only the prey species will survive since $\left(\frac{5}{4}, -\frac{11}{16}\right)$ is asymptotically unstable whenever $R_0 < 1$ and one of the eigenvalues of the characteristic equation of system (22) is positive [3,6].

Example 5.2:

$$\frac{dX}{dt} = 3X - X^2 - 2XY, \quad \frac{dY}{dt} = -2Y + 6XY \quad X \geq 0, Y \geq 0 \quad (23)$$

The endemic (non-zero) equilibrium state is $\left(\frac{1}{3}, \frac{4}{3}\right)$ and the basic reproductive number $R_0 = 9$ then, we have

$\lambda = -\frac{1}{6}\left[1 \pm \sqrt{1 - 192}\right] = -\frac{1}{6} \pm i\frac{\sqrt{191}}{6}$,

$\lambda_1 = -\frac{1}{6} - i\frac{\sqrt{191}}{6}$ and $\lambda_2 = -\frac{1}{6} + i\frac{\sqrt{191}}{6}$,

where $i = \sqrt{-1}$ is the imaginary number.

Therefore, the system of equations (23) has an endemic (non-zero) equilibrium state $\left(\frac{1}{3}, \frac{4}{3}\right)$, this state (position) is asymptotically stable and the two species can coexist since $R_0 = 9 > 1$ and the eigenvalues of characteristic equation of system (23) have negative real parts [3,6,9-17].

6.0 Conclusion and Recommendation

A system of two homogenous linear ordinary differential equations is used in formulating the prey-predator model. The basic reproduction number R_0 is determined using Jacobian Method. Existence of endemic equilibrium (ee) state was established. The local and global stabilities and endemic equilibrium (ee) state is analyzed and it was discovered that endemic equilibrium state are locally and globally asymptotically stable. The first example turned out to be asymptotically unstable when we have the basic reproductive number $R_0 < 1$ and also due to one of the eigenvalues of its characteristic equation having positive value, here environmental limitation factor to the growth of the prey and the death rate of the predator are very important factors to the stability of the system, while in the second example the system turned out to be asymptotically stable when we have, $R_0 > 1$ and also due to the eigenvalues of its characteristic equation having negative real parts. In conclusion, the local and global stabilities of endemic equilibrium show that the encounters (i.e. interactions) between the two species will not destabilized the ecosystem if $R_0 > 1$, otherwise the ecosystem will be destabilized. The paper recommends incorporation of population of hunters as a state rather than a parameter.

References

[1] George, F. S. (1983). Differential Equations with Applications and Historica Notes. Tata McGraw-Hill Company Ltd. New Delhi. Pp 302 – 303.

[2] Boyce, W. E. and DiPrima, R. (1986). Elementary Differential Equations and Boundary Value Problems. 4th ed., John Wiley & Sons, Inc. New York, USA. Pp 2, 325, 493 – 497.

[3] Kapur, J. N. (1988). Mathematical Modelling. 1st ed., Wiley Eastern Ltd., New Delhi, India. Pp 53 – 59.

[4] O'neil P. V. (1995). Advanced Engineering Mathematics. 4th ed., PWS-Kent Publishing Company. Boston, USA. Pp 529 – 538.

[5] Kreyszig, E. (1999). Advanced Engineering Mathematics. 8th ed., John Wiley & Sons, Inc., New Delhi, India. Pp 178 – 179.

[6] Achie, I. and Gajere, S. S. (2011). A System of Lotka – Volterra Equations Modelling Stability in A Prey-Predator System. *National Science and Technology Forum*. *192,* 1 – 9.

[7] Zill, D. G. and Cullen, M. R. (1989). Advanced Engineering Mathematics. Jones and Bartlett Publishers. Pp 692 – 693.

[8] Chon, T. (2003). Evolutionary Stability of Ecological Hierarchy. *Physical Review Letter.* https://arxiv.org/abs/cond-mat/0301619 Retrieved on 9/20/2018.

[9] Achie, I. and Gajere, S. S. (2011). A System of Lotka – Volterra Equations Modelling Stability in A Prey-Predator System. *National Science and Technology Forum. 192,* 1 – 9.

[10] Haruna, M. (2015). Development of mathematical model for the control of malaria and typhoid co-infection using susceptible-infectious-recovered-susceptible (sirs) approach. Ph.D. Thesis, Minna, Nigeria: Federal University of Technology.

[11] Adeboye, K. R. & Haruna, M. (2014). Mathematical modelling of co-existence of *plasmodium* and *salmonella typhi* in human. *Journal of Mathematical Sciences*. *3*(1) 641 - 656.

[12] Adeboye, K. R. & Haruna, M. (2015[a]). A mathematical model for the transmission and control of malaria and typhoid co-infection using *SIRS* approach. *Researchjournali's Journal of Mathematics*. 2(2), 1 – 24.

[13] Adeboye, K. R. & Haruna, M. (2015[b]). A mathematical model for the control of transmission of typhoid fever. *Journal of the Nigerian Association of Mathematical Physics*. *29*(1) 167 - 172.

[14] Diekmann O., Heesterbeek, J. A. P., & Metz, J. A. J. (1990). On the definition and computation of basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations. *Journal of Mathematical Biology*. *28*, 365-382.

[15] Feng, Z. & Jorge X. V. (1997). Competitive exclusion in a vector-host model for the dengue fever. *Journal of Mathematical Biology*. *35*, 523-544.

[16] Heffernan, J. M. Smith, R. J. & Wahl, L. M. (2005). Perspectives on the basic reproductive ratio. *Journal of Royal Society Interface*. *42,* 1-12. doi:10.1098/rsif.2005.0042

[17] Castillo-Chavez C, Feng, Z. & Huang, W. (2002). On the computation of R_0 and its role on global stability. In C. Castillo-Chavez, S. Blower, P. van der Driessche, D. Kirschner, A. Yakubu (Eds). *Mathematical Approaches for Emerging and Reemerging Infectious Diseases*: *Vol 125 An Introduction. The IMA Volumes in Mathematics and its Applications* (*pp* 229 – 250), New York: Springer Verlag.

[18] Abdulrahman, S. (2014). A mathematical model for the transmission dynamics of hepatitis B virus. Minna, Nigeria: PhD Thesis, Federal University of Technology.

[19] Bellman, R. and Cooke, R (1963). Differential Equations. Academic Press. Pp 11 – 45.