Journal of the Nigerian Association of Mathematical Physics
Volume 47 (*July, 2018 Issue*), **pp323– 326**

ON CERTAIN PROPERTIES OF UNIVALENT FUNCTIONS ASSOCIATED WITH MULTIPLIER TRANSFORMATION

M. O. Oluwayemi*, E. O. Davids, K. O. Dopamu and J. O. Okoro

Department of Physical Sciences, Landmark University, Omu-Aran, Nigeria.

Abstract

In this work, the authors introduced and studied a subclass of univalent functions based on multiplier transformation. Certain characteristics of the function were also investigated.

***Keywords*:** Univalent functions, Hadamard product, partial sums, weighted and arithmetic means.
AMS Mathematics Subject Classification (2010): 30C45

1.0 Introduction and Preliminaries

1.1 Normalized Univalent Function

Let $\mathbb{U}$ be the unit disk $z \in \mathbb{C}: |z| < 1$, A be the class of functions analytic in U satisfying the conditions $f(0) = 0$ and $f^{'}(0) = 1$. Then each function $f \in A$ has the Taylor expansion

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1)$$

Further, by S we shall denote the class of all functions in A which are univalent in $\mathbb{U}$. We denote by T the subclass of A consisting of functions $f(z) \in A$ which are analytic and univalent in $\mathbb{U}$ and of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0. \qquad (2)$$

Multiplier transformation

Let $f \in T, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda, l \geq 0$, we define the multiplier transformation operator $D^n(\lambda, l)f(z)$ as follows:

$$D^n(\lambda, l)f(z) = z - \sum_{k=2}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^n a_k z^k. \qquad (3)$$

1.2 Definition

A function $f \in T$, defined by (1.2) with the multiplier transformation (1.3) is said to belong to the class $S_{\lambda,l}^{\mu}(\beta, \sigma, \omega, \xi)$ if

$$Re\left\{\frac{\frac{D^{n+m}(\lambda,l)f(z)}{D^n(\lambda,l)f(z)} - \mu}{\sigma\frac{D^{n+m}(\lambda,l)f(z)}{D^n(\lambda,l)f(z)} + \beta(\omega - \sigma)}\right\} > \xi \qquad (4)$$

where $m, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $0 \leq \sigma < 1$, $0 \leq \xi \leq 1$, $\omega \geq 1$, $\beta \geq 1$, and $\mu \geq 0$.

The object of this paper is to investigate a family of univalent functions. Authors such as [3], [4] and [5] also investigated certain classes of univalent functions. Motivated by [1], we define class $S_{\lambda,l}^{\mu}(\beta, \sigma, \omega, \xi)$ as a family of holomorphic univalent functions based on multiplier transformation.

2 Main Result

We now consider some theorems on the family of $S_{\lambda,l}^{\mu}(\beta, \sigma, \omega, \xi)$.

Theorem 2.1 *If a function $f(z)$ defined by (1.2) belongs to the the class of $S_{\lambda,l}^{\mu}(\beta, \sigma, \omega, \xi)$ then* $\sum_{k=2}^{\infty} [c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(l - \xi\sigma) - [\xi\beta(\omega-\sigma)+\mu]ak \leq [1-\mu-\xi\sigma+\beta(\omega-\sigma)]$ (5)

$$\sum_{k=2}^{\infty} a_k \leq \frac{[1 - \mu - \xi\sigma + \beta(\omega - \sigma)]}{[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1 - \xi\sigma) - [\xi\beta(\omega - \sigma) + \mu]\}}.$$

Proof. Let

$$c_k(l, \lambda) = \left(\frac{\lambda(k-1)+l+1}{l+1}\right) \qquad (6)$$

So that

$$[c_k(1, \lambda)]^n = \left(\frac{\lambda(k-1) + l + 1}{l + 1}\right)^n$$

Corresponding Author**:** Oluwayemi EM.O., Email: oluwayemi.matthew@imu.edu.ng, Tel: +2348032652678

Suppose the inequality (2.1) holds for $|z| \leq 1$. Then by definition (1.2), we have

$$\left|\frac{\frac{D^{n+m}(\lambda,l)f(z)}{D^n(\lambda,l)f(z)}-\mu}{\sigma\frac{D^{n+m}(\lambda,l)f(z)}{D^n(\lambda,l)f(z)}+\beta(\omega-\sigma)}-1\right| \leq 1-\xi \qquad (7)$$

$$\left|\frac{\frac{D^{n+m}(\lambda,l)f(z)}{D^n(\lambda,l)f(z)}-\mu}{\sigma\frac{D^{n+m}(\lambda,l)f(z)}{D^n(\lambda,l)f(z)}+\beta(\omega-\sigma)}-1\right| =$$

$$\left|\frac{[1-\mu-\sigma-\beta(\omega-\sigma)]z-\sum_{k=2}^{\infty}[c_k(l,\lambda)]^n\{(1-\sigma)[c_k(l,\lambda)]^m-[\mu-\beta(\omega-\sigma)]\}a_k z^k}{[\sigma+\beta(\omega-\sigma)]z-\sum_{k=2}^{\infty}[c_k(l,\lambda)]^n\{\sigma[c_k(l,\lambda)]^m+\beta(\omega-\sigma)\}a_k z^k}\right| \leq 1-\xi.$$

Since $z \in U, then |z| \leq 1$ which thus implies definition (1.2). Hence, $f(z) \in S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$. Conversely, suppose $f(z) \in S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$, then

$$Re\left\{\frac{\frac{D^{n+m}(\lambda,l)f(z)}{D^n(\lambda,l)f(z)}-\mu}{\sigma\frac{D^{n+m}(\lambda,l)f(z)}{D^n(\lambda,l)f(z)}+\beta(\omega-\sigma)}\right\} =$$

$$Re\left\{\frac{1-\sum_{k=2}^{\infty}[c_k(1,\lambda)]^{n+m}a_k z^{k-1}-\mu+\sum_{k=2}^{\infty}\mu[c_k(l,\lambda)]^n a_k z^{k-1}}{[\sigma+\beta(\omega-\sigma)]-\sum_{k=2}^{\infty}[c_k(l,\lambda)]^{n+m}a_k z^{k-1}-\sum_{k=2}^{\infty}\beta(\omega-\sigma)[c_k(l,\lambda)]^n a_k z^{k-1}}\right\} \geq \xi; z \in U.$$

We thus remove the denominator and set $z \to 1^-$ so that

$$\sum_{k=2}^{\infty}[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]a_k \leq [1-\mu-\xi\sigma+\beta(\omega-\sigma)]\}.$$

It thus follows that

$$\sum_{k=2}^{\infty}a_k \leq \frac{[1-\mu-\xi\sigma+\beta(\omega-\sigma)]}{[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}}.$$

The result is sharp with the extremal function of

$$f(z) = z-\frac{[1-\mu-\xi\sigma+\beta(\omega-\sigma)]}{[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}}z^k.$$

2.1 Linear Combination for univalent functions in class $S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$

Definition 2.1*Let $f_j(z) = z-\sum_{k=2}^{\infty}a_{j,k}z^k, j = 1,2,\ldots m$ such that $f_j(z) \in S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$, then the linear combination of $f_j(j = 1,2\ldots m)$ is defined by*

$$G(z) = \sum_{j=1}^{m}k_j f_j(z); where \sum_{j=1}^{m}k_j = 1.$$

See [2] for detail.

Theorem 2.2*Let $f_1(z) = z-\sum_{k=2}^{\infty}a_{i,k}z^k, j = 1,2,\ldots m$ be the functions $S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$. Then the linear combination of $f_j(j = 1,2,\ldots m)$ defined as*

$$H(z) = \sum_{j=1}^{m}k_j f_j(z) \qquad (8)$$

where $\sum_{j=1}^{m}k_j = k_1+k_2\ldots k_m = 1$ also belongs to the class $S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$.

Proof: Let $f_1(z) = z-\sum_{k=2}^{\infty}a_{j,k}z^k, j = 1,2,\ldots m$ be the functions in the class $S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$.

By Theorem 2.1, for any $f \in S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$,

$$\sum_{k=2}^{\infty}[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}a_k \leq \xi|\varpi|+\sigma-1.$$

Thus,

$$\sum_{k=2}^{\infty}[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}a_{j,k} \leq \xi|\varpi|+\sigma-1.$$

$$H(z) = \sum_{j=1}^{m}k_j f_j(z)$$

$$\Rightarrow H(z) = \sum_{j=1}^{m}k_j\left(z-\sum_{k=2}^{\infty}a_{j,k}z^k\right)$$

$$H(z) = z-\sum_{k=2}^{\infty}\left(\sum_{j=1}^{m}k_j a_{j,k}\right)z^k.$$

Using the coefficient estimate for the class, we have that

$$\sum_{k=2}^{\infty}\left\{[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}\right\}\left(\sum_{j=1}^{m}k_j a_{j,k}\right)$$

$$= \sum_{j=1}^{m}k_j\left[\sum_{k=2}^{\infty}\left\{[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}\right\}a_{j,k}\right]$$

$$\leq \sum_{j=1}^{m} k_j\,[(k-1)+\xi|\varpi|] = [1-\mu-\xi\sigma+\beta(\omega-\sigma)].$$

2.2 Partial Sums

Let $f \in T$ be a function of the form (1.2) and define a partial sums f_m defined by

$$f_m(z) = z - \sum_{k=2}^{\infty} a_k z^k (m \in \mathbb{N}\backslash\{1\}) \qquad (9)$$

Theorem 2.3*Let $f \in S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$ be given by (1.2) and define a partial sums $f_1(z)$ and $f_m(z)$ by*

$$f_1(z) = z and f_m(z) = z - \sum_{k=2}^{\infty} a_k z^k (m \in \mathbb{N}\backslash\{1\}) \quad (10)$$

Suppose also that

$$\sum_{k=2}^{\infty} d_k a_k \leq 1. \quad (11)$$

If

$$d_k \geq \frac{[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}}{1-\mu-\xi\sigma+\beta(\omega-\sigma)}; \quad (12)$$

$k = 2,3, \ldots m and k = m+1, m+2, m+3,$ .... Then $f \in S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$ and also,

$$Re\left(\frac{f(z)}{f_m(z)}\right) > 1 - \frac{1}{d_{m+1}}, \quad (13)$$

and

$$Re\left(\frac{f_m(z)}{f(z)}\right) > \frac{d_{m+1}}{1+d_{m+1}} \quad (14)$$

For the coefficients d_k given by (2.8), we have that

$$d_{k+1} > d_k > 1. \quad (15)$$

Hence,

$$\sum_{k=2}^{\infty} a_k + d_{m+1}\sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} d_k a_k \leq 1. \quad (16)$$

Using the hypothesis (2.8), setting $g_1(z) = d_{m+1}\left(\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right)\right)$,

$$= 1 + \frac{d_{m+1}\sum_{k=m+1}^{\infty} a_k z^{k-1}}{1-\sum_{k=2}^{\infty} a_k z^{k-1}}$$

it thus suffices to prove that $Re(g_1(z)) \geq 0 (z \in \mathbb{U})$ and using (2.12), we have that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq 1 (z \in \mathbb{U}).$$

Applying (2.12) we have that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_{m+1}\sum_{k=m+1}^{\infty} a_k}{2-2\sum_{k=2}^{\infty} a_k - d_{m+1}\sum_{k=m+1}^{\infty} a_k} \leq 1 (z \in \mathbb{U}).$$

Which yields the assertion (2.10) of Theorem 2.5 in order to see that

$$f(z) - \frac{z^{m+1}}{d_{m+1}}, \quad (17)$$

$$\frac{f(z)}{f_m(z)} = 1 - \frac{r^{m+1}}{d_{m+1}} \Rightarrow 1 - \frac{1}{d_{m+1}} as r \to 1^-.$$

Also taking and using (2.13), we have that $g_2(z) = (1+d_{m+1})\left(\frac{f_m(z)}{f(z)} - \frac{d_{m+1}}{1+d_{m+1}}\right)$,

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{d_{m+1}\sum_{k=m+1}^{\infty} a_k}{2-2\sum_{k=2}^{\infty} a_k - d_{m+1}\sum_{k=m+1}^{\infty} a_k} \leq 1. \quad (18)$$

Thus, the bound in (2.14) is sharp for each $m \in N$ with the extreme function $f(z)$ given by (2.13).

2.3 Hadamard Product

Theorem 2.4*If $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z - \sum_{k=2}^{\infty} b_k z^k$ belongs to $S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$. Then, the Hadamard product of $f(z)$ and $g(z)$ given by $(f*g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k$ also belongs to $S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$.*

Proof: Let $f(z)$ and $g(z)$ belongs to $S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$, then

$$\sum_{k=2}^{\infty} \frac{[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}}{1-\mu-\xi\sigma+\beta(\omega-\sigma)} a_k \leq 1$$

and

$$\sum_{k=2}^{\infty} \frac{[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}}{1-\mu-\xi\sigma+\beta(\omega-\sigma)} b_k \leq 1.$$

By Cauchy-Schwartz inequality, we have

$$\sum_{k=2}^{\infty} \frac{[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}}{1-\mu-\xi\sigma+\beta(\omega-\sigma)} a_k b_k$$

$$= \sum_{k=2}^{\infty} \left\{\frac{[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}}{1-\mu-\xi\sigma+\beta(\omega-\sigma)}\sqrt{a_k b_k}\right\}\sqrt{a_k b_k}$$

$$\leq \left(\sum_{k=2}^{\infty} \left\{\frac{[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}b_k}{1-\mu-\xi\sigma+\beta(\omega-\sigma)}\right\} a_k\right)^{\frac{1}{2}} .$$

$$\left(\sum_{k=2}^{\infty} \left\{\frac{[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}a_k}{\eta(2-\delta)+\xi-1}\right\} b_k\right)^{\frac{1}{2}} \leq 1$$

which completes the proof.

Definition 2.4 *For $f \in A$, we define he integral transform*

$$V_\sigma(f)(z) = \int_0^1 v(t)\frac{f(tz)}{t}dt$$

for a real valued, non-negative weight function normalized σ so that $\int_0^1 \sigma(t)dt = 1.$ Since special case of $\sigma(t)$ are particularly interesting, such as $\sigma(t) = (1+c)t^c$, $c > -1$, for which V_σ is known as the Bernadi operator, and

$$V_\sigma(t) = \frac{(c+1)^\lambda}{v(\lambda)}t^c\left(log\frac{1}{t}\right)^{\lambda-1}, c > -1, \lambda \geq 0$$

which gives the Komatu operator. For detail, see Mur and $MurV$ for details.

We now show that the class $S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$ is closed under $V_\sigma(f)(z)$.

Theorem 2.5*Let $f \in S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$. Then, $V_\sigma(f)(z)$ also belongs to the class $S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$.*

Proof: From definition 2.4, it follows that

$$V_\sigma(f)(z) = \frac{(c+1)^\lambda}{\Gamma(\lambda)}\int_0^1 (-1)^{\lambda-1}t^c(logt)^{\lambda-1}\left(z+\sum_{k=2}^{\infty}|a_k|t^{k-1}\right)dt$$

$$= \frac{(-1)^{\lambda-1}(c+1)^\lambda}{\Gamma(\lambda)}\lim_{\to 0^+}\left[\int_0^1 (-1)^{\lambda-1}t^c(logt)^{\lambda-1}\left(z+\sum_{k=2}^{\infty}|a_k|t^{k-1}\right)dt\right].$$

So that

$$V_\sigma(f(z)) = z - \sum_{k=2}^{\infty}\left(\frac{c+1}{c+n}\right)^\lambda a_k z^k.$$

We now show that $V_\sigma(f(z)) \in S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$.

$$\sum_{k=2}^{\infty} \frac{[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}}{1-\mu-\xi\sigma+\beta(\omega-\sigma)}\left(\frac{c+1}{c+k}\right)^\lambda a_k \leq 1. \qquad (19)$$

Using Theorem 2.1, $f \in S_{\lambda,l}^{\mu}(\beta,\sigma,\omega,\xi)$ if and only if

$$\sum_{k=2}^{\infty} \frac{[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}}{1-\mu-\xi\sigma+\beta(\omega-\sigma)} a_k \leq 1.$$

Clearly, $\frac{c+1}{c+k} < 1$ for all $k \geq 2$. Thus,

$$\sum_{k=2}^{\infty} \frac{[c_k(l,\lambda)]^n\{[c_k(l,\lambda)]^m(1-\xi\sigma)-[\xi\beta(\omega-\sigma)+\mu]\}}{1-\mu-\xi\sigma+\beta(\omega-\sigma)}\left(\frac{c+1}{c+k}\right)^\lambda a_k \leq 1.$$

References

[1] Darwish H. E. (2007), Certain Subclasses of Analytic Functions with Negative Coefficients Defined by Generalized Salagean Operator, General Mathematics Vol. 15, No. 4 69-82.

[2] Jadhav P. G. (2017), Study of properties of certain family of univalent functions associated with subordination. International Journal of Recent Scientific Research 8(6), pp 17567 - 17573.

[3] Murugusundaramoorthy G. (2015), Certain subclasses of univalent functions associated with a unification of the Srivastava-Attiya and Cho-Saigo-Srivastava operators, Novi Sad J. Math. Vol. 45, No. 2, 59 - 76.

[4] Murugusundaramoorthy G., Vijaya K. and Deepa K. (2013), Holder inequalities for a subclass of univalent functions involving Dziok-Srivastava operator, Global Journal of Mathematical Analysis 1, 1 - 8.

[5] Oluwayemi M. O. and Fadipe-Joseph O. A. (2017), New subclasses of univalent functions defined by using a linear combination of a generalized Sălăgean and Ruscheweyh operators, International Journal of Mathematical Analysis and Optimisation, Vol. 2017, 187 - 200.