Journal of the Nigerian Association of Mathematical Physics
Volume 47 (July, 2018 Issue), pp163 – 172
© J. of NAMP

THE EXPONENTIATED NEW WEIGHTED WEIBULL DISTRIBUTION: THEORY AND APPLICATION

UmarA.A [1,2*], David R.O[2], Falgore J.Y[2],Abubakar S.S[1,3],Abdullahi U.K[2], Mohammed A.S[2], and Damisa S.A[2]

[1]School of Mathematical Sciences, Universiti Sains Malaysia, Penang-Malaysia
[2]Department of Statistics, Ahmadu Bello University Zaria-Kaduna, Nigeria
[3]Department of Statistics, Kano University of Science and Technology Wudil, Nigeria

Abstract

This article generalized the New Weighted Weibull Distribution (NWWD) introduced by [1] by adding a single parameter using the exponentiated family of distribution to a four parameter(s) variant called Exponentiated New Weighted Weibull Distribution (E-NWWD). The probability density function (pdf), cumulative distribution function (cdf), hazard rate and survival function of the E-NWWD model were derived. Besides that, some of the mathematical and structural properties of the proposed model were studied. The parameter(s) of the E-NWWD model were estimated using the method of maximum likelihood estimate. The proposed E-NWWD model outperforms the competing Weibull and New Weighted Weibull Distributions in terms of the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). A real dataset on survival time was used to demonstrate the practical application of the proposed E-NWWD model.

***Keywords*:** New Weighted Weibull Distribution, Exponentiated Family of Distribution, Moment, Order Statistics, Asymptotic Behavior, Quantile Function, Maximum Likelihood Estimation

1.0 Introduction

The Weibull Distribution (WD) introduced in [2] is a well-known distribution in literature. An attempt to extend the Weibull distribution has received rapid attention from several researchers due to its impact over other existing distributions in term of flexibility in modelling lifetime data. In Recent literatures, studies to improve distribution(s) has received great attention by many researchers to address the challenges encountered on some real-life dataset in term of flexibility nature. This procedure includes induction of one or more additional shape parameter(s) to the baseline distribution to make it more flexible in examining the tail characteristic.

The idea of addition of parameter(s) to most existing distributions resulted to a better and more flexible new family of Distributions which cater for the setback when considering the baseline distribution alone in most cases. The modified approach in [3] of weighting distribution was adopted in [1] where he introduced the three parameter(s) New Weighted weibull distribution (NWWD) which is used as the baseline in this article. [4]Studied the class of Weighted Weibull Distribution from the approach in [3] techniques and emphasized that the weighting of existing distributions is of great impact to adjust the probabilities of the observed and documented records.

The generalization of the Weibull Distribution in [5] namely Exponentiated Weibull Distribution and further studied in[6] with some application to bus motor failure dataset and food. In addition [7, 8, 9] proposed a modification of the Weibull Distribution by multiplying the Weibull Cumulative Hazard Function using a different technique of the approach in [3]. [10, 11, 12] highlighted some special distribution(s) that can be achieved by the generalized family of distribution(s). The generalized family of distribution(s) are extensively suggested for construction of new family of distribution which receive great attentions by many researchers.

Corresponding Author**:** Umar A.A., Email: jamiluyf@gmail.com, Tel: +2348067778996, +2348030417330 (UAA)

Some of the family of distribution(s) in literature are the Kumaraswamy Generalized (KW-G) distribution proposed in[11], Beta Generalized (B-G) distribution introduced in[13], Marshall Olkin-G distributions due to [14], Exponentiated-G distributions proposed in[15], Gamma-G distributions in[16], weibull-G distributions introduced in[17], exponentiated generalized-G distributions in[10] and exponentiated exponential poisson-G distributions introduced in[18].
In this article, we intend to generalize the new weighted weibull distribution using the exponentiated generalized (E-G) family of distribution suggested in[15] to introduce a new family of distribution called exponentiated new weighted weibull distribution (E-NWWD).
The rest of this paper hereafter is organized as follows: In Section 2, the review of the related distribution models was briefly discussed. The explicit expression for density functions and some statistical properties of the proposed E-NWWD model are discussed in detail in Section 3. Section 4 present the estimation of parameter(s) and the performance evaluation is given in Section 5. In Section 6, a practical application to illustrate the implementation of the E-NWWD model is presented and Section 7 finally summarizes the article with some brief concluding remarks.

2.0 A review of the related distributions

This section presents the probability density function and cumulative distribution function techniques of the baseline new weighted Weibull distribution proposed in[1] which enormously enhance the performance of the Weibull distribution and that of the exponentiated generalized family of distribution introduced in[15].

2.1. The NWWD model

The probability density function (*pdf*) and cumulative distribution function (*cdf*) of the new weighted weibull distribution proposed by [1] are given as follows:

$$g(x;\alpha,\theta,\beta)=\left(1+\beta^{\theta}\right)\alpha\theta x^{\theta-1}\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)} \quad (1)$$

and

$$G(x;\alpha,\theta,\beta)=1-\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}, \quad (2)$$

where $x > 0$, $\alpha > 0$, $\theta > 0$, $\beta > 0$. Also, α represent the scale parameter while θ and β are the shape parameter(s).

2.2. The E-G family of distribution

The Exponentiated Generalized (E-G) family of distribution technique was proposed by [15] to enhance the flexibility of the existing model for modelling life time dataset. Suppose a random variable X hasan arbitrary baseline distribution $G(x)$, the cdf and pdf of the E-G family of distribution are express in[15].

$$F(x)=\left[G(x)\right]^{\lambda} \quad (3)$$

and

$$f(x)=\lambda\left[G(x)\right]^{\lambda-1}g(x), \quad (4)$$

where $x > 0$ and $\lambda > 0$ is the shape parameter, $G(x)$ and $g(x)$ are the *cdf* and *pdf* of the baseline distribution given earlier in Eqs. (1) and (2) respectively.

3. A Proposed E-NWWD Model

This article incorporates the NWWD model suggested in[1] and the techniques of exponentiated generalized family of distribution suggested in[15] to derive some of the statistical properties of the proposed model namely exponentiated new weighted weibull distribution (E-NWWD). The *pdf* and *cdf* of the E-NWWD would be obtain by substituting Eqs. (1) and (2) into Eq. (4) as follows:

$$f(x;\alpha,\theta,\beta,\lambda)=\left(1+\beta^{\theta}\right)\lambda\alpha\theta x^{\theta-1}(1-\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)})^{\lambda-1}\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}, \quad (5)$$

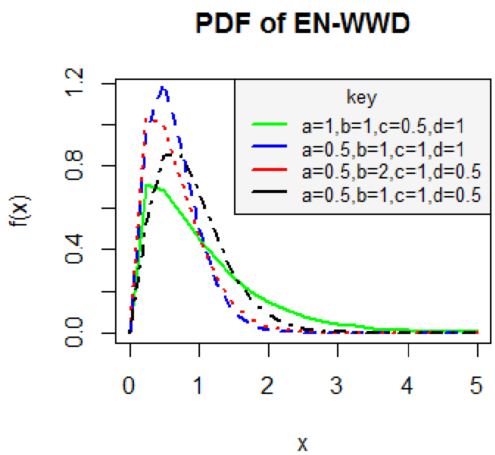
while substituting Eq. (2) into Eq. (3) gives the *cdf* by

$$F(x;\alpha,\theta,\beta,\lambda)=\left[1-\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}\right]^{\lambda} \quad (6)$$

we can further express Eq. (6) as

$$F(x;\alpha,\theta,\beta,\lambda)=1-\lambda\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}+\ell^{-\alpha x^{\theta}\lambda\left(1+\beta^{\theta}\right)}, \quad (7)$$

where $x > 0$, $\alpha > 0$, $\theta > 0$, $\beta > 0$ and $\lambda > 0$, also α is the scale parameter while θ, β and λ are the shape parameter(s). Besides that, the probability density function of any probability distribution is said to be valid if the function $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Moreover, given some values to the parameter(s), the possible shapes for the *pdf* and *cdf* of the E-NWWD model are presented as follows:



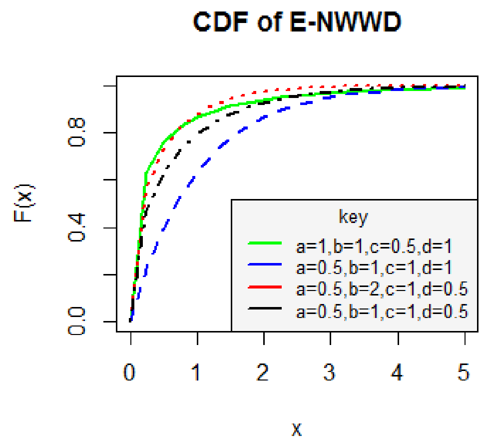


Fig. 1: The E-NWWD density function for some values of the parameter(s) α = a, β= b, θ = c,&λ =d.

Fig. 2: The E-NWWD cumulative distribution function for some values of the parameter(s) α = a, β= b, θ = c, λ = d.

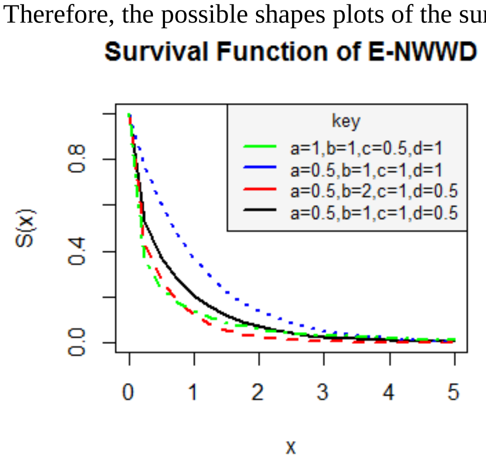
The reliability measure of the E-NWWD model are analyze by the survival and hazard functions. The survival function is described as the probability that an item under study will succeed within a period. Suppose that, a random variable X □ E-NWWD, the survival function of the proposed model can be written mathematically as follows:

$$S(x) = 1 - F(x) = 1 - \left[1 - \ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}\right]^{\lambda} \tag{8}$$

Also, the hazard rate function is an important quantity characterizing life phenomena and sometimes called the risk or failure function. It is the probability that a component may deteriorate over a period. The hazard function is obtained mathematically as the ratio of the *f(x)* in Eq. (5) to the *S(x)* in Eq. (8) is given by

$$H(x) = \frac{f(x)}{S(x)} = \frac{\left(1+\beta^{\theta}\right)\lambda\alpha\theta x^{\theta-1}\left(1-\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}\right)^{\lambda-1}\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}}{1-\left(1-\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}\right)^{\lambda}} \tag{9}$$

Therefore, the possible shapes plots of the survival and hazard functions for the E-NWWD model are given as follows:



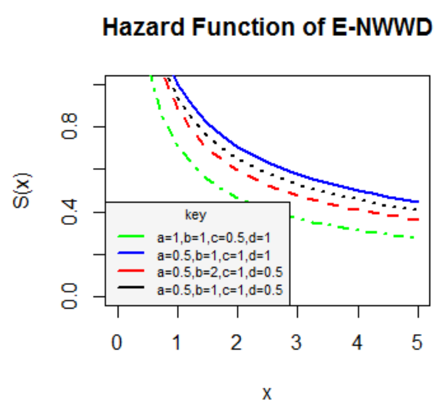


Fig. 3: The plot of survival function with parameter(s) α = a, β= b, θ = c, λ = d.

Fig. 4: The plot of hazard rate function for with parameter(s) α = a, β= b, θ = c, λ = d.

3.1 Order statistics

This is widely used in literature of many areas of statistical theory and practice, such as the outliers monitoring and detection. This section would derive the closed form expressions for the *pdf* of the i^{th} order statistics of the proposed E-NWWD model. Assume that X_1, X_2, …, X_n are random sample from the E-NWWD and also let $X_{1:n}$, $X_{2:n}$, …, $X_{n:n}$ denote the corresponding order statistics drawn from this sample. We now express the *pdf*, $f_{i:n}(x)$of the i^{th} order statistics as follows:

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-i}, \tag{10}$$

where the functions *f(x)* and *F(x)* are the *pdf* and *cdf* of E-NWWD model. Using the binomial expansion, we give the expression as follows:

$$\left[1-F(x)\right]^{n-i}=\sum_{r=0}^{n-i}\binom{n-i}{r}(-1)^r F(x)^r \tag{11}$$

By substituting Eq. (11) into Eq. (10), we obtain the expression as

$$f_{i:n}(x)=\frac{n!}{(i-1)!r!(n-i-r)!}f(x)F(x)^{i-1}\sum_{r=0}^{n-i}\binom{n-i}{r}(-1)^r F(x)^r, \tag{12}$$

and further simplify by

$$f_{i:n}(x)=\sum_{r=0}^{n-i}\frac{n!}{(i-1)!r!(n-i-r)!}(-1)^r f(x)F(x)^{i+r-1} \tag{13}$$

Therefore, substituting Eq. (5) as the *pdf* and Eq. (6) as *cdf* into Eq. (13) results to

$$f_{i:n}(x)=\sum_{r=0}^{n-1}\frac{n!}{r!(n-1-r)!}(-1)^r\left(1+\beta^\theta\right)\lambda\alpha\theta x^{\theta-1}\left(1-\ell^{-\alpha x^\theta\left(1+\beta^\theta\right)}\right)^{\lambda(1-r)-1}\ell^{-\alpha x^\theta\left(1+\beta^\theta\right)} \tag{14}$$

Hence, the pdf of the minimum and maximum order statistic denotes $X_{(1)}$ and $X_{(n)}$ of the proposed model written as

$$f_{1:n}(x)=\sum_{r=0}^{n-1}\frac{n!}{r!(n-1-r)!}(-1)^r\left(1+\beta^\theta\right)\lambda\alpha\theta x^{\theta-1}\left(1-\ell^{-\alpha x^\theta\left(1+\beta^\theta\right)}\right)^{\lambda(1-r)-1}\ell^{-\alpha x^\theta\left(1+\beta^\theta\right)} \tag{15}$$

Thus,

$$f_{n:n}(x)=n\lambda\alpha\theta\left(1+\beta^\theta\right)x^{\theta-1}\left(1-\ell^{-\alpha x^\theta\left(1+\beta^\theta\right)}\right)^{\lambda-1}\ell^{-\alpha x^\theta\left(1+\beta^\theta\right)}\left[\left(1-\ell^{-\alpha x^\theta\left(1+\beta^\theta\right)}\right)^{\lambda}\right]^{n-1} \tag{16}$$

3.2. Quantile function and median

The quantile function (x_q) of the E-NWWD model is the solution to the give mathematical equation

$$F(x_q)=q, \tag{17}$$

where q follows a uniform distribution with the interval $0 <q< 1$ and $F(x_q)$ remained the *cdf* of E-NWWD model. Making x_q the subject gives the quantile function as follows:

$$\left(1-\ell^{-\alpha x_q^\theta\left(1+\beta^\theta\right)}\right)^{\lambda}=q \tag{18}$$

By simplifying Eq. (18), we have

$$1-q^{\frac{1}{\lambda}}=\ell^{-\alpha x_q^\theta\left(1+\beta^\theta\right)}, \tag{19}$$

and taking the natural of Eq. (19) gives

$$\ln\left(1-q^{\frac{1}{\lambda}}\right)=\ln\ell^{-\alpha x_q^\theta\left(1+\beta^\theta\right)}=-\alpha x_q^\theta\left(1+\beta^\theta\right) \tag{20}$$

We now make x_q the subject of the formula to have

$$x_q=-\left[\ln\left(1-q^{\frac{1}{\lambda}}\right)\Big/\alpha\left(1+\beta^\theta\right)\right]^{\frac{1}{\theta}} \tag{21}$$

Besides that, the median of the E-NWWD model is achieve by setting q = 0.5 in Eq. (21). This gives

$$\text{Median}\left(x_{0.5}\right) = -\left[\ln\left(1-(0.5)^{\frac{1}{\lambda}}\right)\Big/\alpha\left(1+\beta^{\theta}\right)\right]^{\frac{1}{\theta}} \tag{22}$$

3.3 The Moments

The measure of central tendency and dispersion of a population under study can be obtain with the aid of moments. The r^{th} moment of a continuous random variable X of the pdf of E-NWWD model is given by

$$\mu_r^{'} = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)dx, \tag{23}$$

where *f(x)* represent the *pdf* of E-NWWD in Eq. (5). Therefore,

$$\mu_r^{'} = \lambda\alpha\theta\left(1+\beta^{\theta}\right)\int_0^{\infty} x^{r+\theta-1}\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}\left(1-\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}\right)^{\lambda-1}dx \tag{24}$$

We now simplify Eq. (24) using binomial expansion as follows:

$$\left(1-\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}\right)^{\lambda-1} = \sum_{j=0}^{\infty}(-1)^j\binom{\lambda-1}{j}\ell^{-\alpha x^{\theta} j\left(1+\beta^{\theta}\right)}, \tag{25}$$

and substituting Eq. (25) into Eq. (24) to have the following expression

$$\mu_r^{'} = \lambda\alpha\theta\left(1+\beta^{\theta}\right)\int_0^{\infty} x^{r+\theta-1}\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}\sum_{j=0}^{\infty}(-1)^j\binom{\lambda-1}{j}\ell^{-\alpha x^{\theta} j\left(1+\beta^{\theta}\right)}dx \tag{26}$$

Also, Eq. (26) can further simplify as

$$\mu_r^{'} = \lambda\alpha\theta\left(1+\beta^{\theta}\right)\sum_{j=0}^{\infty}(-1)^j\binom{\lambda-1}{j}\int_0^{\infty} x^{r+\theta-1}\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)(1+j)}dx \tag{27}$$

The integral part of Eq. (27) can be writing as a gamma function by letting

$y = \alpha x^{\theta}\left(1+\beta^{\theta}\right)(1+j)$, $x = \left(\dfrac{y}{\alpha\left(1+\beta^{\theta}\right)(1+j)}\right)^{\frac{1}{\theta}}$, to have $dx = \dfrac{1}{\alpha\theta\left(1+\beta^{\theta}\right)(1+j)}\left(\dfrac{y}{\alpha\left(1+\beta^{\theta}\right)(1+j)}\right)^{\frac{1-\theta}{\theta}}dy$

Thus, Eq. (27) becomes

$$\mu_r^{'} = \lambda\alpha\theta\left(1+\beta^{\theta}\right)\sum_{j=0}^{\infty}(-1)^j\binom{\lambda-1}{j}\int_0^{\infty}\left(\frac{y}{\alpha\left(1+\beta^{\theta}\right)(1+j)}\right)^{\frac{r+\theta-1}{\theta}}\ell^{-y}\left(\frac{1}{\alpha\theta\left(1+\beta^{\theta}\right)(1+j)}\right)\left(\frac{y}{\alpha\left(1+\beta^{\theta}\right)(1+j)}\right)^{\frac{1-\theta}{\theta}}dy \tag{28}$$

and

$$\mu_r^{'} = \frac{\lambda}{(1+j)\left(\alpha\left(1+\beta^{\theta}\right)(1+j)\right)^{\frac{r+\theta-1}{\theta}}\left(\alpha\left(1+\beta^{\theta}\right)(1+j)\right)^{\frac{1-\theta}{\theta}}}\sum_{j=0}^{\infty}(-1)^j\binom{\lambda-1}{j}\int_0^{\infty} y^{\frac{r+\theta-1}{\theta}}\ell^{-y}y^{\frac{1-\theta}{\theta}}dy \tag{29}$$

The expression in Eq. (29) can be reduce to

$$\mu_r^{'} = \frac{\lambda}{(1+j)\left(\alpha\left(1+\beta^{\theta}\right)(1+j)\right)^{\frac{r}{\theta}}}\sum_{j=0}^{\infty}(-1)^j\binom{\lambda-1}{j}\int_0^{\infty} y^{\frac{r}{\theta}}\ell^{-y}dy \tag{30}$$

But,

$$\int_0^{\infty} y^{\frac{r}{\theta}}\ell^{-y}dy = \Gamma\left(\frac{r}{\theta}+1\right) = \Gamma\left(\frac{r+\theta}{\theta}\right) \tag{31}$$

By replacing the gamma function in Eq. (31) into Eq. (30), we achieved

$$\mu_r^{'} = \frac{\lambda(\lambda-1)!}{j!(\lambda-j-1)!(1+j)\left(\alpha\left(1+\beta^{\theta}\right)(1+j)\right)^{\frac{r}{\theta}}}(-1)^j\,\Gamma(\frac{r+\theta}{\theta}) \quad (32)$$

3.3.1 Moment generating function

The moment generating function (mgf) of the proposed model can be obtain as follows:

$$\mathrm{M_X}(t) = \mathrm{E}\left(\ell^{tx}\right) = \int_{-\infty}^{\infty} \ell^{tx} f(x)dx \quad (33)$$

Replacing x^r with e^{tx}in Eq. (27) gives

$$\mathrm{M_X}(t) = \lambda\alpha\theta\left(1+\beta^{\theta}\right)\sum_{j=0}^{\infty}(-1)^j\binom{\lambda-1}{j}\int_0^{\infty} x^{\theta-1}\ell^{tx}\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)(1+j)}dx \quad (34)$$

Using the Maclaurin's series expansion as

$$\ell^{tx} = \sum_{p=0}^{\infty}\frac{(tx)^p}{p!} = \sum_{p=0}^{\infty}\frac{t^p x^p}{p!}, \quad (35)$$

and substituting Eq. (35) into Eq. (34) resulted to

$$\mathrm{M_X}(t) = \lambda\alpha\theta\left(1+\beta^{\theta}\right)\sum_{j=0}^{\infty}(-1)^j\binom{\lambda-1}{j}\int_0^{\infty} x^{\theta-1}\sum_{p=0}^{\infty}\frac{t^p x^p}{p!}\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)(1+j)}dx \quad (36)$$

Thus, Eq. (36) can further simplified as

$$\mathrm{M_X}(t) = \sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\frac{\lambda\alpha\theta\left(1+\beta^{\theta}\right)t^p}{p!}(-1)^j\binom{\lambda-1}{j}\int_0^{\infty} x^{p+\theta-1}\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)(1+j)}dx \quad (37)$$

However, by letting

$$z = \alpha x^{\theta}\left(1+\beta^{\theta}\right)(1+j),\ x = \left(\frac{z}{\alpha\left(1+\beta^{\theta}\right)(1+j)}\right)^{\frac{1}{\theta}},\ \text{and}\ dx = \frac{1}{\alpha\theta\left(1+\beta^{\theta}\right)(1+j)}\left(\frac{z}{\alpha\left(1+\beta^{\theta}\right)(1+j)}\right)^{\frac{1-\theta}{\theta}}dz,$$

and substitute the quantities into Eq. (37), we obtain the following expression as

$$\begin{aligned}\mathrm{M_X}(t) = &\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\frac{\lambda\alpha\theta\left(1+\beta^{\theta}\right)t^p}{p!}(-1)^j\binom{\lambda-1}{j}\\ &\int_0^{\infty}\left[\left(\frac{z}{\alpha\left(1+\beta^{\theta}\right)(1+j)}\right)^{\frac{p+\theta-1}{\theta}}\ell^{-y}\left(\frac{1}{\alpha\theta\left(1+\beta^{\theta}\right)(1+j)}\right)\left(\frac{z}{\alpha\left(1+\beta^{\theta}\right)(1+j)}\right)^{\frac{1-\theta}{\theta}}dz\right]\end{aligned} \quad (38)$$

The expression can further simplify by

$$\mathrm{M_X}(t) = \sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\frac{\lambda t^p}{p!(1+j)\left(\alpha\left(1+\beta^{\theta}\right)(1+j)\right)^{\frac{p}{\theta}}}(-1)^j\binom{\lambda-1}{j}\int_0^{\infty} z^{\frac{p}{\theta}}\ell^{-z}dz \quad (39)$$

Expressing the integral part of Eq. (39) as a gamma function gives

$$\int_0^{\infty} z^{\frac{p}{\theta}}\ell^{-z}dz = \Gamma(\frac{p}{\theta}+1) = \Gamma(\frac{p+\theta}{\theta}) \quad (40)$$

Thus, Eq. (39) reduce to

$$\mathrm{M_X}(t) = \sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\frac{\lambda t^p}{p!(1+j)\left(\alpha\left(1+\beta^{\theta}\right)(1+j)\right)^{\frac{p}{\theta}}}(-1)^j\binom{\lambda-1}{j}\Gamma(\frac{p+\theta}{\theta}) \quad (41)$$

Beside that, Eq. (41) can further simplified as

$$\mathrm{M}_X(t)=\sum_{j=0}^{\infty}\sum_{p=0}^{\infty}\frac{\lambda t^p\left(\lambda-1\right)!}{j!(\lambda-j-1)!\,p!(1+j)\left(\alpha\left(1+\beta^{\theta}\right)(1+j)\right)^{\frac{p}{\theta}}}\left(-1\right)^j\Gamma\left(\frac{p+\theta}{\theta}\right) \tag{42}$$

3.3.2 Kurtosis and Skewness

The Bowley's skewness [19] is derived from the quantile function as follows:

$$S_k=\frac{Q(0.75)-2Q(0.5)+Q(0.25)}{Q(0.75)-Q(0.25)}, \tag{43}$$

while the Moor's kurtosis in [20] based on octiles is given as follows:

$$K_u=\frac{Q(0.875)-Q(0.625)+Q(0.375)+Q(0.125)}{Q(0.75)-Q(0.25)}, \tag{44}$$

where $Q(.)$ represent the function of quantile, Q(0.25) and Q(0.75) are the lower and upper quantile by setting q = 0.25 and 0.75 in Eq. (21) respectively.

3.4 Asymptotic behavior of the model

This aimed to examine the behavior of the density function of the E-NWWD model above in Eq. (5) as $x\to 0$ &$x\to\infty$. Thus,

$$\lim_{x\to 0} f(x)=\lim_{x\to\infty} f(x)=0 \tag{45}$$

Now, we consider the left-hand side of Eq. (45) to have

$$\lim_{x\to 0} f(x)=\lim_{x\to\infty}\left[\lambda\alpha\theta\left(1+\beta^{\theta}\right)x^{\theta-1}\left(1-\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}\right)^{\lambda-1}\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}\right] \tag{46}$$

Taking $y = x$, as $x\to 0$ &$y\to\infty$, then we have

$$\lim_{x\to 0} f(x)=\lim_{y\to\infty}\left[\lambda\alpha\theta\left(1+\beta^{\theta}\right)y^{\theta-1}\left(1-\ell^{-\alpha y^{\theta}\left(1+\beta^{\theta}\right)}\right)^{\lambda-1}\ell^{-\alpha y^{\theta}\left(1+\beta^{\theta}\right)}\right] \tag{47}$$

Taking the limits of the function in Eq. (47) gives the expression as follows:

$$\lim_{x\to 0} f(x)=\lim_{y\to\infty}\left(\lambda\alpha\theta\left(1+\beta^{\theta}\right)y^{\theta-1}\ell^{-\alpha y^{\theta}\left(1+\beta^{\theta}\right)}\right)\lim_{y\to\infty}\left(1-\ell^{-\alpha y^{\theta}\left(1+\beta^{\theta}\right)}\right)^{\lambda-1} \tag{48}$$

From Eq. (48),

$$\lim_{y\to\infty}\left(\lambda\alpha\theta\left(1+\beta^{\theta}\right)y^{\theta-1}\ell^{-\alpha y^{\theta}\left(1+\beta^{\theta}\right)}\right)=\lim_{y\to\infty}\left(\frac{\lambda\alpha\theta\left(1+\beta^{\theta}\right)y^{\theta-1}}{\ell^{\alpha y^{\theta}\left(1+\beta^{\theta}\right)}}\right) \tag{49}$$

and solving Eq. (49) using the L'Hospital rule gives

$$\lim_{y\to\infty}\left(\frac{\lambda\alpha\theta\left(1+\beta^{\theta}\right)y^{\theta-1}}{\ell^{\alpha y^{\theta}\left(1+\beta^{\theta}\right)}}\right)=\lim_{y\to\infty}\left(\frac{\lambda\theta(\theta-1)y^{\theta-2}}{\ell^{\alpha y^{\theta}\left(1+\beta^{\theta}\right)}}\right)=0 \tag{50}$$

Substituting Eq. (50) into Eq. (48) results to

$$\lim_{x\to 0} f(x)=\left(0\right)\lim_{y\to\infty}\left(1-\ell^{-\alpha y^{\theta}\left(1+\beta^{\theta}\right)}\right)^{\lambda-1}=0 \tag{51}$$

Also, using the right-hand side of the same Eq. (45) and substitute the *pdf* of the E-NWWD in Eq. (5) as $x\to\infty$, we have

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\left[\lambda\alpha\theta\left(1+\beta^{\theta}\right)x^{\theta-1}\left(1-\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}\right)^{\lambda-1}\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}\right] \tag{52}$$

$$\lim_{x\to\infty} f(x)=0 \tag{53}$$

Therefore, the results obtained in Eqs. (51) and (53) satisfy the asymptotic behavior in Eq. (45) above. Hence, implies that the E-NWWD has at least one mode.

4. Estimation of parameter(s)

The parameter(s) estimation of the E-NWWD model is obtained using the method of maximum likelihood estimation (MLE). Suppose $X_1, X_2, \ldots, X_n$ be a random sample of size n independent and identically distribution random variables from the E-NWWD with unknown parameter(s) vector say $\boldsymbol{V} = (\alpha, \beta, \theta, \lambda)^T$, the log-likelihood function for $\boldsymbol{V}$ is obtained for the *pdf* in Eq. (5) is as follows:

$$L(V) = \prod_{i=1}^{n} f(x_i; \alpha, \beta, \theta, \lambda) \tag{54}$$

Substituting the *pdf* of the E-NWWD into Eq. (54) gives

$$L(V) = \prod_{i=1}^{n} \lambda\alpha\theta\left(1+\beta^{\theta}\right) x_i^{\theta-1} \ell^{-\alpha x_i^{\theta}\left(1+\beta^{\theta}\right)} \left(1-\ell^{-\alpha x^{\theta}\left(1+\beta^{\theta}\right)}\right)^{\lambda-1}, \tag{55}$$

taking the log-likelihood of Eq. (55) resulted to

$$\ln L(V) = n\ln\left[\lambda\alpha\theta\left(1+\beta^{\theta}\right)\right] + (\theta-1)\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\alpha x_i^{\theta}\left(1+\beta^{\theta}\right) + (\lambda-1)\sum_{i=1}^{n}\ln\left(1-\ell^{-\alpha x_i^{\theta}\left(1+\beta^{\theta}\right)}\right) \tag{56}$$

Eq. (56) is further simplify as

$$\ln L(V) = n\ln\lambda + n\ln\alpha + n\ln\theta + n\ln(1+\beta^{\theta}) + (\theta-1)\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\alpha x_i^{\theta}\left(1+\beta^{\theta}\right) + (\lambda-1)\sum_{i=1}^{n}\ln\left(1-\ell^{-\alpha x_i^{\theta}\left(1+\beta^{\theta}\right)}\right) \tag{57}$$

Moreover, differentiating Eq. (57) partially with respect to each of the parameters ($\alpha, \beta, \theta, \lambda$) and setting the results to zero yield the MLEs of the parameter(s) estimation as follows:

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} x_i^{\theta}(1+\beta^{\theta}) - (\lambda-1)\sum_{i=1}^{n}\frac{x_i^{\theta}(1+\beta^{\theta})}{\left(1-\ell^{-\alpha x_i^{\theta}(1+\beta^{\theta})}\right)} \tag{58}$$

$$\frac{\partial L}{\partial \beta} = \frac{n\theta\beta^{\theta-1}}{(1+\beta^{\theta})} - \sum_{i=1}^{n}\alpha x_i^{\theta}\theta\beta^{\theta-1} + (\lambda-1)\sum_{i=1}^{n}\frac{\alpha x_i^{\theta}\theta\beta^{\theta-1}\ell^{-\alpha x_i^{\theta}(1+\beta^{\theta})}}{\left(1-\ell^{-\alpha x_i^{\theta}(1+\beta^{\theta})}\right)} \tag{59}$$

$$\frac{\partial L}{\partial \theta} = \frac{n}{\theta} + \frac{n\beta^{\theta}\ln\beta}{(1+\beta^{\theta})} + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\alpha x_i^{\theta}(\ln x_i + \beta^{\theta}\ln(x_i\beta)) + (\lambda-1)\sum_{i=1}^{n}\frac{\alpha x_i^{\theta}(\ln x_i + \beta^{\theta}\ln(x_i\beta))}{\left(1-\ell^{-\alpha x_i^{\theta}(1+\beta^{\theta})}\right)} \tag{60}$$

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\ln\left(1-\ell^{-\alpha x_i^{\theta}(1+\beta^{\theta})}\right) \tag{61}$$

The solutions of the non-linear system of Eqs. (58), (59), (60) and (61) when setting to zero gives the MLEs of the parameter(s). However, the solution of those parameter(s) cannot be solved analytically rather numerically (such as the Newton Raphson techniques and other non-linear equation solver) with the aids of suitable statistical software such as R-package, SAS and MATLAB software's for any available datasets.

5. Performance evaluation

The conventional way of comparing the performance of distribution models is by measuring their ability in selecting the best fit model for future predictions. A distribution model at hand is preferable provided the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) values are smaller than the competing models. The kurtosis, skewness and descriptive statistics value of a lifetime dataset on survival time of some selected patient were given in Table 1. For example, the values of the mean of the survival time is 0.30, while the kurtosis and skewness values are 1.04318 and 3.402139. The AIC and BIC values are used to evaluate the performance of the models under comparison with their corresponding estimated parameter(s) values and are presented herein Table 2 and Table 3. For example, the estimated parameter(s) values of the E-NWWD model are ($\alpha, \beta, \theta, \lambda$) = (0.09292, 0.81656, 0.57784, and 1.60573) (see Table 2).

The R-studio software and MATLAB program were used for numerical computation of the E-NWWD, WD and NWWD models. The AIC and BIC values of the optimal E-NWWD model outperforms the WD and NWWD models for the all choice of parameter(s). For example, the AIC and BIC values of the E-NWWD is (1168.14, 1168.49), while that of the WD and NWWD models are (1173.72, 1170.24) and (1179.35, 1170.45) respectively. Therefore, the proposed model would have a better fit in future predictions than the competing models (see Table 3).

6. A Practical application

To demonstrate the implementation procedures of the proposed distribution model, a practical application using a real-life data set to evaluates the performance of the E-NWWD model to the competing WD and NWWD models in terms of the AIC and BIC criteria. The MLE of the estimated parameter(s) for the models under consideration were computed for comparisons purpose.

The survival time data of 121 patients affected with breast cancer are recorded from 1929 to 1938 was used in this study [21]. The real-life data are presented as follows: 0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3,11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5,17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0,31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0,41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0,54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0,78.0, 80.0,83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0,125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0. The positively skewed histogram of the survival time data set is provided in Fig. 5.

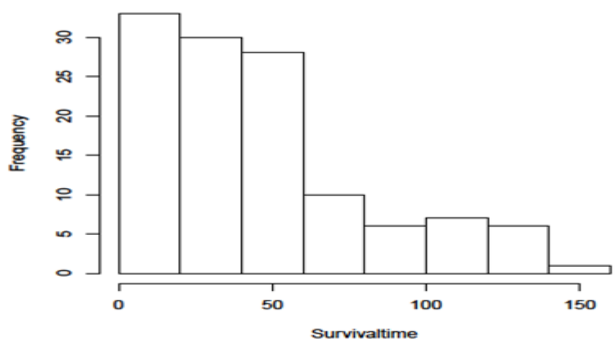


Fig. 5: The relative histogram of the Survival time data of 121 patients with breast cancer from 1929 to 1938 [21].

Table 1: Preliminary statistics.

Real life data	Min.	Max.	1st Quart.	Median	Mean	3rd Quart.	Skewness	Kurtosis
Survival data	0.30	154	17.50	40	46.33	60	1.04318	3.402139

Table 2: MLEs of the Models' parameter(s) of the E-NWWD, WD and NWWD

Parameter(s)	Models		
	WD (α,β)	NWWD (α,β,θ)	E-NWWD $(\alpha,\beta,\theta,\lambda)$
α	0.01774	0.00732	**0.09292**
β	1.02200	0.94068	**0.81656**
θ		1.08935	**0.57784**
λ			**1.60573**

Table 3: Performance Measures AIC, BIC of the E-NWWD, WD and NWWD

Performance Measures	Models		
	WD (α,β)	NWWD (α,β,θ)	E-NWWD $(\alpha,\beta,\theta,\lambda)$
AIC	1173.72	1170.24	**1168.14**
BIC	1179.35	1170.45	**1168.49**

7. Conclusion

This article introduced a four parameter(s) distribution called exponentiated new weighted weibull distribution (E-NWWD) using the generalized family of distribution suggested in[15]. Some of the mathematical and statistical properties of the proposed model such as quantile function, asymptotic behaviors, moment, kurtosis and skewness, survival and hazard functions, probability density and cumulative distribution functions were extensively discussed. The method of maximum likelihood estimates was used for parameters estimation of the proposed model. The plots of the model indicate that the E-NWWD has a positive skewed shape which implies that the proposed model maybe effective in modelling time events, where the survival rate reduces with time while hazard increases over period. The E-NWWD model outperforms the WD and NWWD models in terms of the AIC and BIC. Hence, the proposed E-NWWD model has some consistent better fits than the competing models especially for the positively skewed dataset and survival evaluation. An extensive use of the quadratic rank transmutation map (QRTM) scheme as suggested in [22] to further improve the baseline distribution can be an interesting future study.

Acknowledgements

The author Adamu Abubakar Umar is indebted to School of Mathematical Science, Universiti Sains Malaysia (USM), Penang, Malaysia and Department of Statistics, Faculty of Physical Sciences, Ahmadu Bello University (ABU), Zaria, Nigeria for providing excellent research facilities.

References

[1] Nasiru, S. (2015). Another weighted weibull distribution from azzalini’s family. *European Scientific Journal*, 11 (9):1857-7531.

[2] Weibull, W., (1961). Fatigue Testing and analysis of results. Pergamon Press LTD., Oxford.

[3] Azzalini, A. (1985). A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics*, 12(2): 171-178.

[4] Mahdy, M., 2013. A class of weighted weibull distributions and its properties. *Studies in Mathematical Sciences*, 6(1): 35-45.

[5] Mudholkar, G. S., Srivastava, D. K. and Freimer, M. (1995). The exponentiated Weibull family: A reanalysis of the bus-motor-failure data. *Technometrics*, 37(4): 436-445.

[6] Mudholkar, G. S. and Huston, A. D. (1996). The exponentiated Weibull family: Some properties and flood data application. *Communications in Statistic-Theory and Methods*, 25(12): 3059-3083.

[7] Lai, C. D., Xie, M. and Murthy, D. N. P. (2003). A modified Weibull distribution. *IEEE Transactions on Reliability*, 52(1): 33-37.

[8] Xie, M. and Lai, C. D. (1996). Reliability analysis using an additive Weibull model with bathtub shaped failure rate function. *Reliability Engineering and System Safety*, 52(1): 87-93.

[9] Zhang, T. and Xie, M. (2007). Failure data analysis with extended Weibull distribution. *Communications in Statistics-Simulation and Computation*, 36(3): 579-592.

[10] Cordeiro, G. M., Ortega, E. M. M. and da Cunha, D. C. C. (2013). The exponentiated generalized class of distributions. *Journal of Data Science*, 11:1-27.

[11] Cordeiro, G. M., and de Castro, M. (2011). A new family generalized distributions. *Journal of Computation and Simulation*, 81 (7):883-898.

[12] Nadarajah, S., Cordeiro, G.M. and Ortega, E.M.M. (2015). The Zografos-Balakrishnan–G family of distributions: Mathematical properties and applications, *Communications in Statistics- Theory and Methods:* 44(1): 186–215.

[13] Eugene, N., Lee, C., and Famoye, F. (2000). Beta-normal distribution and its applications. *Communication in Statistics-Theory and Methods*, 31 (4):497-512.

[14] Marshall, A. W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84(3):641-652.

[15] Gupta, R. C., Gupta, P. I. and Gupta, R. D. (1998). Modeling failure time data by Lehmann alternatives, *Communications in Statistics–Theory and Methods,* 27(4): 887–904.

[16] Zografos, K. and Balakrishnan, N. (2009). On families of beta and generalized gamma-generated distributions and associated inference. *Statistical Methodology,* 6 (4):344-362.

[17] Alzaatreh, A., Famoye, F. and Lee, C. (2013). Weibull-pareto distribution and its applications. *Communication in Statistics- Theory and Methods*, 42 (9):1673-1691.

[18] Ristic, M. M. and Nadarajah, S. (2012). A new lifetime distribution, *Journal of Statistical ComputationandSimulation*, 84(1): 135-150.

[19] Kenney, J. and Keeping, E. (1962). *Mathematics of Statistics*. (3rd ed.), Van Nostrand Company, Princeton.

[20] Moors, J.J.A. (1998). A quantile alternative for kurtosis. *Journal of the Royal Statistical Society, SeriesD (The Statistician)*, 37(1): 25-32.

[21] Lee, E. T. (1992). *Statistical Methods for Survival Data Analysis*. (2nd ed.). New York:John Wiley.

[22] Shaw, W. T. and Buckley, I. R. (2007). The alchemy of probability distributions: Beyond gram-charlier xpansions and a skew-kurtosis-normal distribution from a rank transmutation map.