FISHER INFORMATION MATRIX FOR THE LOG-LOGISTIC POISSON DISTRIBUTION

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Abstract

In this paper, the exact form of Fisher information matrix for Log-logistic Poisson (LLP) distribution with respect to censored data is determined. The LLP is a unimodal and decreasing distribution and is a family of widely applicable log-logistic distribution. Analytical prove of the existence and uniqueness of MLEs is also presented, and the approximate confidence interval for the parameters related to this distribution is also obtained.

***Keywords*:** Log-likelihood function; Fisher information matrix; Digamma function; Trigamma function.

1.0 INTRODUCTION

Information about the parameters usually obtained from a sample of data coming from the specific probability distribution, this normally done in the parameter estimation problems. However, to know how much information a sample of data can provide about the unknown parameter, we need a measure. Fisher information is one of the measures for such information, this measure of information can be used to find the bounds on the variance of estimators, and it can be used to obtain an approximate confidence interval when dealing with large sample, and it can also be used to approximate the sampling distribution of an estimator obtained from a large sample.

The basic principle of maximum likelihood estimation (MLE) says; let a random variable X with probability function $f(x,\theta)$, if θ were the true value of the parameter, the likelihood function should take a big value, or in another word, the derivative log-likelihood function should be close to zero. It is popular that Fisher information matrix used as a useful tool for derivation of variance-covariance matrix in the asymptotic distribution of MLEs. Moreover, under appropriate regularity conditions, the determinate (divided by the sample size) of asymptotic variance-covariance matrix of MLE reaches an optimal lower bound for the volume of the "spread ellipsoid" of joint estimators, [1 - 7]. In the univariate case, this optimal property of MLE is widely used in the "robustness versus efficiency" studies as a quantitative benchmark for efficiency considerations. We are motivated by the work of [2], where they introduced Fisher information matrix of feller-pareto distribution.

The paper organized as follows. In section 2, we give the definition of LLP and prove of the existence of MLEs with respect to censored data. In section 3, we provide the Fisher information matrix for LLP. The integrals related are presented in the appendix.

2.1 The Log-logistic Poisson distribution (LLP)

A random variable X is said to have log-logistics Poisson distribution with scale parameter $\lambda > 0$ and shape parameter $\alpha > 0$ if it probability density function is given by

$$f_X(x;\lambda,\alpha) = \alpha\lambda(e^{\lambda}-1)^{-1}\frac{x^{\alpha-1}}{(1+x^{\alpha})^2}e^{\frac{\lambda}{(1+x^{\alpha})}}, \qquad x > 0 \tag{1}$$

and zero otherwise.

The cumulative distribution function by

$$F(x) = \frac{e^{\lambda}-e^{\frac{\lambda}{(1+x^{\alpha})}}}{e^{\lambda}-1}, x > 0 \tag{2}$$

2.2 Maximum likelihood estimators

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample coming from LLP(λ,α). The log-likelihood function (of right censored data) is given by

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$$l(\lambda,\alpha)=\prod_{i=1}^{n} f^{\delta_i}(x_i,\theta)[1-F(x_i,\theta)]^{1-\delta_i}$$

$$=\sum_{i=1}^{n}\delta_i[\ln(\alpha)+\ln(\lambda)]+(\alpha-1)\sum_{i=1}^{n}\delta_i\ln(x_i)-2\sum_{i=1}^{n}\delta_i\ln(1+x_i{}^{\alpha})$$

$$+\lambda\sum_{i=1}^{n}\frac{\delta_i}{(1+x_i{}^{\alpha})}+\sum_{i=1}^{n}(1-\delta_i)\ln(e^{\frac{\lambda}{(1+x^{\alpha})}}-1)-n\ln(e^{\lambda}-1).$$

Where$\delta_i=0$, for complete observation and $\delta_i=1$, for censored observation.

The partial derivatives of the log-likelihood function with respect to parameters α and λ are

$$\frac{\partial l}{\partial\lambda}=\frac{\sum_{i=1}^{n}\delta_i}{\lambda}+\sum_{i=1}^{n}\frac{\delta_i}{1+x_i{}^{\alpha}}+\sum_{i=1}^{n}\frac{(1-\delta_i)e^{\frac{\lambda}{(1+x^{\alpha})}}}{(1+x_i{}^{\alpha})(e^{\frac{\lambda}{(1+x^{\alpha})}}-1)}-n\frac{e^{\lambda}}{e^{\lambda}-1} \quad (3)$$

$$\frac{\partial l}{\partial\alpha}=\frac{\sum_{i=1}^{n}\delta_i}{\alpha}+\sum_{i=1}^{n}\delta_i\ln(x_i)-\sum_{i=1}^{n}\frac{\delta_i x_i{}^{\alpha}\ln(x_i)(2+{}^{\lambda}/_{(1+x_i{}^{\alpha})})}{1+x_i{}^{\alpha}}$$

$$-\lambda\sum_{i=1}^{n}(1-\delta_i)\frac{x_i{}^{\alpha}\ln(x_i{}^{\alpha})e^{\frac{\lambda}{(1+x^{\alpha})}}}{(1+x_i{}^{\alpha})^2(e^{\frac{\lambda}{(1+x^{\alpha})}}-1)}$$

$$=\frac{\sum_{i=1}^{n}\delta_i}{\lambda}+\sum_{i=1}^{n}\delta_i\ln(x_i)-W_{\lambda,1}(\alpha,x^n)-W_{\lambda,2}(\alpha,x^n) \quad (4)$$

Where $W_{\lambda,1}(\alpha,x^n)=\frac{\delta_i x_i{}^{\alpha}\ln(x_i)(2+{}^{\lambda}/_{(1+x_i{}^{\alpha})})}{1+x_i{}^{\alpha}}$ and $W_{\lambda,2}(\alpha,x^n)=(1-\delta_i)\frac{x_i{}^{\alpha}\ln(x_i{}^{\alpha})e^{\frac{\lambda}{(1+x^{\alpha})}}}{(1+x_i{}^{\alpha})^2(e^{\frac{\lambda}{(1+x^{\alpha})}}-1)}$

To find the estimators, we set equation (3) and (4) to zero and solve simultaneously, but the equations are nonlinear, so it is very difficult to find the analytical solution by the way we try to prove the existence of MLEs given by the following theorems.

Theorem 2.1

*Let $h_{\alpha}(\lambda;x^n)$ be the right-hand-side of equation (3). Then for any fixed*e $\alpha>0$*, there exist a unique root of $\lambda;\hat{\lambda}$, i.e., $h_{\alpha}(\hat{\lambda};x^n)=0$ provided $n>\sum_{i=1}^{n}\frac{1}{(1+x_i{}^{\alpha})}$.*

Proof

$$h_{\alpha}(\lambda;x^n)=\frac{\sum_{i=1}^{n}\delta_i}{\lambda}+\sum_{i=1}^{n}\frac{\delta_i}{1+x_i{}^{\alpha}}+\sum_{i=1}^{n}\frac{(1-\delta_i)e^{\frac{\lambda}{(1+x^{\alpha})}}}{(1+x_i{}^{\alpha})(e^{\frac{\lambda}{(1+x^{\alpha})}}-1)}-n\frac{e^{\lambda}}{e^{\lambda}-1}$$

Let $\lambda\to\infty$

$$\lim_{\lambda\to\infty}h_{\alpha}(\lambda;x^n)=\sum_{i=1}^{n}\frac{\delta_i}{1+x_i{}^{\alpha}}+\sum_{i=1}^{n}\frac{(1-\delta_i)}{1+x_i{}^{\alpha}}-n$$

$=\sum_{i=1}^{n}\frac{1}{1+x_i{}^{\alpha}}-n<0$ Provided $n>\sum_{i=1}^{n}\frac{1}{(1+x_i{}^{\alpha})}$

Let $\lambda\to-\infty$

$$\lim_{\lambda\to-\infty}h_{\alpha}(\lambda;x^n)=\sum_{i=1}^{n}\frac{\delta_i}{1+x_i{}^{\alpha}}>0$$

This implies that there exists at least a solution for$h_{\alpha}(\lambda;x^n)$. To show the root is unique, we have

$$h'_{\alpha}(\lambda;x^n)=-\frac{\sum_{i=1}^{n}\delta_i}{\lambda^2}-\sum_{i=1}^{n}\frac{(1-\delta_i)e^{-\frac{\lambda}{(1+x^{\alpha})}}}{(1+x_i{}^{\alpha})^2(1-e^{-\frac{\lambda}{(1+x^{\alpha})}})^2}+n\frac{e^{-\lambda}}{(1-e^{-\lambda})^2}$$

$$=-\sum_{i=1}^{n}V_1^{\alpha}(\lambda;x_i)-\sum_{i=1}^{n}V_2^{\alpha}(\lambda;x_i)-n\frac{e^{-\lambda}}{1-e^{-\lambda}}$$

Where $i=1,2,\ldots,n$, $V_1^{\alpha}(\lambda;x_i)$ and $V_2^{\alpha}(\lambda;x_i)>0$ given in equations (5) and (6) respectively

$$V_1^{\alpha}(\lambda;x_i)=\delta_i\left(\frac{1}{\lambda^2}-\frac{e^{-\frac{\lambda}{(1+x^{\alpha})}}}{(1+x_i{}^{\alpha})^2\left(1-e^{-\frac{\lambda}{(1+x^{\alpha})}}\right)^2}\right) \quad (5)$$

$$V_2^{\alpha}(\lambda;x_i)=\frac{e^{-\frac{\lambda}{(1+x^{\alpha})}}}{(1+x_i{}^{\alpha})^2\left(1-e^{-\frac{\lambda}{(1+x^{\alpha})}}\right)^2} \quad (6)$$

Equation (6) is greater zero. Now consider the behavior of (5). Let$Z_i^{\alpha}(\lambda;x_i) = \lambda/2(1+x_i{}^{\alpha})$, than equation (5) can rewrite as

$$V_1^{\alpha}(\lambda;x_i) = \frac{(\sinh(Z_i^{\alpha}(\lambda;x_i)) - Z_i^{\alpha}(\lambda;x_i))(\sinh(Z_i^{\alpha}(\lambda;x_i)) + Z_i^{\alpha}(\lambda;x_i))}{\lambda^2(\sinh Z_i^{\alpha}(\lambda;x_i))^2}$$

The numerator can also reduce to $(\sinh Z_i^{\alpha}(\lambda;x_i))^2 - (Z_i^{\alpha}(\lambda;x_i))^2$, since $sinh^2(x) > x^2\ \forall x \in \mathbb{R}$, it follows that $V_1^{\alpha}(\lambda;x_i) > 0$.

Let $g(\lambda) = \frac{e^{\lambda}}{(e^{\lambda}-1)^2}$ then $(\ln g(\lambda))' g(\lambda) = g'(\lambda) = -\frac{e^{\lambda}}{(e^{\lambda}-1)^2}$ this implies that it is positive decreasing sequence and $g(\lambda) < \lim_{\lambda\to-\infty} g(\lambda) = 0$, then

$h'_{\alpha}(\lambda;x^n) < 0$. Thus, $h_{\alpha}(\lambda;x^n)$ is a decreasing function on λ, this implies that, there exists a unique root of $h_{\alpha}(\lambda;x^n)$.

Theorem 2.2

Let $h_{\lambda}(\alpha;x^n)$ be the right-hand-side of equation (4). Then there exist a unique root of α; $\hat{\alpha}$, i.e., $h_{\lambda}(\hat{\alpha};x^n) = 0$.

Proof

By examining the behavior of $h_{\lambda}(\alpha;x^n)$ on real positive line$(0,\infty)$, we have

$h_{\lambda}(\alpha;x^n) = \infty$ as $\alpha \to 0$ and

$h_{\lambda}(\alpha;x^n) = \sum_{(x_i<1)}^{n} \delta_i \ln(x_i) - \sum_{(x_i>1)}^{n} \delta_i \ln(x_i)$ as $\alpha\to\infty$

To show $h_{\lambda}(\alpha;x^n) < 0$, as $\alpha \to \infty$, we consider these cases below

Case I; when $\min(x_1,x_i) > 1$, then $h_{\lambda}(\alpha;x^n) = -\sum_{(x_i>1)}^{n} \delta_i \ln(x_i) < 0$

Case II; when $\max(x_1,x_i) < 1$, then$h_{\lambda}(\alpha;x^n) = \sum_{(x_i<1)}^{n} \delta_i \ln(x_i) < 0$

Case III; when$\min(x_1,x_i) < 1$ and $\max(x_1,x_i) > 1$, then $h_{\lambda}(\alpha;x^n) = \sum_{(x_i<1)}^{n} \delta_i \ln(x_i) - \sum_{(x_i>1)}^{n} \delta_i \ln(x_i) < 0$

It follows that $h_{\lambda}(\alpha;x^n) < 0$ if and only if $x_i \neq 1$ for some $1 \le i \le n$. Therefore, on $(0,\infty)$, $h_{\lambda}(\alpha;x^n)$ is continuous function which decreases monotonically to negative values from positive values. Hence, there exists at least one finite positive root of the $h_{\lambda}(\alpha;x^n) = 0$.

Also

$$h'_{\lambda}(\alpha;x^n) = -\frac{\sum_{i=1}^{n}\delta_i}{\alpha^2} - \sum_{i=1}^{n} \frac{\delta_i x_i{}^{\alpha} \ln^2(x_i)}{(1+x_i{}^{\alpha})^2}\left(2 + \lambda\frac{1-x_i{}^{\alpha}}{1+x_i{}^{\alpha}}\right)$$
$$-\lambda\sum_{i=1}^{n}(1-\delta_i)\frac{x_i{}^{\alpha}\ln^2(x_i)\, e^{\frac{\lambda}{(1+x^{\alpha})}}}{(1+x_i{}^{\alpha})^4(e^{\frac{\lambda}{(1+x^{\alpha})}}-1)}\left(\frac{\lambda x_i{}^{\alpha}}{(e^{\frac{\lambda}{(1+x^{\alpha})}}-1)} + 1 - x_i{}^{2\alpha}\right)$$

Consider the cases

Case I; for $\min(x_1,x_i) > 1$, and $\lambda < 0, h'_{\lambda}(\alpha;x^n) < 0$.

Case II; for $\max(x_1,x_i) < 1$, and $\lambda > 0, h'_{\lambda}(\alpha;x^n) < 0$.

This implies that in both cases $h'_{\lambda}(\alpha;x^n) < 0$ for all $\alpha > 0$. Hence, there exists a unique root of $h_{\lambda}(\alpha;x^n)$.

3.1 Fisher Information Matrix for LLP

Suppose X is a random variable with LLP probability function. Then the information matrix $I(\Theta)$ is an nxn matrix with elements given by

$$\boldsymbol{I_{ij}(\Theta)} = \boldsymbol{E}\left[-\frac{\partial^2 \ln f(x;\Theta)}{\partial\theta_i\partial\theta_j}\right],\ i,j = 1,2$$

Where $\Theta = (\lambda,\alpha)$, to find the entries, we first find all the second derivative of likelihood with respect the parameters, we have

$$\frac{\partial^2 l}{\partial\lambda^2} = -\frac{\sum_{i=1}^{n}\delta_i}{\lambda^2} - \sum_{i=1}^{n}\frac{(1-\delta_i)e^{-\frac{\lambda}{(1+x^{\alpha})}}}{(1+x_i{}^{\alpha})^2(1-e^{-\frac{\lambda}{(1+x^{\alpha})}})^2} + n\frac{e^{-\lambda}}{(1-e^{-\lambda})^2}$$

$$\frac{\partial^2 l}{\partial\alpha^2} = -\frac{\sum_{i=1}^{n}\delta_i}{\alpha^2} - \sum_{i=1}^{n}\frac{\delta_i x_i{}^{\alpha}\ln^2(x_i)}{(1+x_i{}^{\alpha})^2}\left(2 + \lambda\frac{1-x_i{}^{\alpha}}{1+x_i{}^{\alpha}}\right)$$
$$-\lambda\sum_{i=1}^{n}(1-\delta_i)\frac{x_i{}^{\alpha}\ln^2(x_i)\, e^{\frac{\lambda}{(1+x^{\alpha})}}}{(1+x_i{}^{\alpha})^4(e^{\frac{\lambda}{(1+x^{\alpha})}}-1)}\left(\frac{\lambda x_i{}^{\alpha}}{e^{\frac{\lambda}{(1+x^{\alpha})}}-1} + 1 - x_i{}^{2\alpha}\right)$$

$$\frac{\partial^2 l}{\partial\alpha\partial\lambda} = -\sum_{i=1}^{n}\frac{\delta_i x_i{}^{\alpha}\ln(x_i)}{(1+x_i{}^{\alpha})^2} + \lambda\sum_{i=1}^{n}\frac{(1-\delta_i)x_i{}^{\alpha}\ln(x_i)\, e^{-\frac{\lambda}{(1+x^{\alpha})}}}{(1+x_i{}^{\alpha})^3(1-e^{-\frac{\lambda}{(1+x^{\alpha})}})^2} - \sum_{i=1}^{n}\frac{(1-\delta_i)x_i{}^{\alpha}\ln(x_i)}{(1+x_i{}^{\alpha})^2(e^{\frac{\lambda}{(1+x^{\alpha})}}-1)}$$

$$I_{11} = \frac{\sum_{i=1}^{n}\delta_i}{\lambda^2} + n\frac{e^{-\lambda}}{(1-e^{-\lambda})^2} + \sum_{i=1}^{n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(1-\delta_i)\lambda^{k+1}(j+1)(-j)^k}{(e^{\lambda}-1)k!\,(k+3)}$$

$$I_{12} = I_{21} = \sum_{i=1}^{n}\sum_{k=0}^{\infty}\frac{\delta_i\lambda^{k+1}}{\alpha(e^{\lambda}-1)k!}A_1 - \sum_{i=1}^{n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(1-\delta_i)\lambda^{k+2}(j+1)(-j)^k}{\alpha(e^{\lambda}-1)k!}A_2 + \sum_{i=1}^{n}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\frac{(1-\delta_i)\lambda^{k+1}(1-j)^k}{\alpha(e^{\lambda}-1)k!}A_1$$

$$I_{22} = \frac{\sum_{i=1}^{n}\delta_i}{\alpha^2} + \sum_{i=1}^{n}\sum_{k=0}^{\infty}\frac{\delta_i\lambda^{k+1}}{\alpha^2(e^{\lambda}-1)k!}(\,2A_3+\lambda(A_4+A_5)) + \sum_{i=1}^{n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(1-\delta_i)\lambda^k}{\alpha^2(e^{\lambda}-1)k!}[(j+1)(-j)^k A_6 + \lambda(1-J)k(A_7-A_8)]$$

Finally, the exact expression of the elements of fisher information matrix in terms of digamma ($\psi(a)$) and trigamma and ($\psi'(a)$), [8] and $A_1 - A_8$, are given in appendix

3.2 Observe information matrix

The 2x2 unit information matrix is given by

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J_{\lambda\lambda}} & \boldsymbol{J_{\lambda\alpha}} \\ \boldsymbol{J_{\alpha\lambda}} & \boldsymbol{J_{\alpha\lambda}} \end{pmatrix}$$

The Asymptotic variance-covariance of the MLE for the parameters λ and α are the elements of the inverse of the fisher information matrix. However, the exact mathematical expressions are sometimes very difficult to obtain. Therefore, the approximate (observed) asymptotic variance-covariance matrix for MLE is given by

$$\begin{pmatrix} -\frac{\partial^2 l}{\partial\lambda^2} & -\frac{\partial^2 l}{\partial\lambda\partial\alpha} \\ -\frac{\partial^2 l}{\partial\alpha\partial\lambda} & -\frac{\partial^2 l}{\partial\alpha^2} \end{pmatrix}^{-1} = \begin{pmatrix} var(\hat{\lambda}) & cov(\hat{\lambda},\hat{\alpha}) \\ cov(\hat{\alpha},\hat{\lambda}) & var(\hat{\alpha}) \end{pmatrix}$$

The Asymptotic normality of the MLE can use to obtain the approximate confidence interval for parameters λ and α [9]. Therefore, (1-γ)100% confidence intervals for λ and α becomes

$\hat{\lambda} \pm Z_{\frac{\gamma}{2}}\sqrt{var(\hat{\lambda})}$ and $\hat{\alpha} \pm Z_{\frac{\gamma}{2}}\sqrt{var(\hat{\alpha})}$ where $Z_{\frac{\gamma}{2}}$ is a standard normal variate.

CONCLUSION

In this paper, the form of fisher information matrix in respect to right censored data is successfully derived. Proofs of the existence and uniqueness of MLEs are also established. The approximate confidence interval for the parameters also introduced.

Appendix

$$A_1 = \frac{1}{(k+1)}\int_0^{\infty}\frac{x^{\alpha}\ln x^{\alpha}}{(1+x^{\alpha})^2}(k+1)\alpha x^{\alpha-1}(1+x^{\alpha})^{-(k+2)}dx$$

$$= \frac{1}{(k+2)(k+3)}(\psi(k+2) - \psi(1) + \tfrac{k}{k+1})$$

$$A_2 = \frac{1}{(k+2)}\int_0^{\infty}\frac{x^{\alpha}\ln x^{\alpha}}{(1+x^{\alpha})^2}(k+2)\alpha x^{\alpha-1}(1+x^{\alpha})^{-(k+3)}dx$$

$$= \frac{1}{(k+3)(k+4)}(\psi(k+2) - \psi(1) + \tfrac{k+1}{k+2})$$

$$A_3 = \frac{1}{(k+1)}\int_0^{\infty}\frac{x^{\alpha}\ln^2(x^{\alpha})}{(1+x^{\alpha})^2}(k+1)\alpha x^{\alpha-1}(1+x^{\alpha})^{-(k+2)}dx$$

$$= \frac{1}{(k+2)(k+3)}\left(\psi'(k+1) - \psi'(1) + \left(\psi(k+1) - \psi(1) - \frac{k}{k+1}\right)^2 - \frac{(k+1)^2+1}{(k+1)^2}\right)$$

$$A_4 = \frac{1}{(k+2)}\int_0^\infty \frac{x^\alpha \ln^2(x^\alpha)}{(1+x^\alpha)^2}(k+2)\alpha x^{\alpha-1}(1+x^\alpha)^{-(k+3)}dx$$

$$= \frac{1}{(k+3)(k+4)}\left(\psi'(k+2) - \psi'(1) + \left(\psi(k+2) - \psi(1) - \frac{k+1}{k+2}\right)^2 - \frac{(k+2)^2+1}{(k+2)^2}\right)$$

$$A_5 = \frac{1}{(k+1)(k+2)}\int_0^\infty \frac{x^\alpha \ln^2(x^\alpha)}{(1+x^\alpha)^2}(k+2)(k+1)\alpha x^{2\alpha-1}(1+x^\alpha)^{-(k+3)}dx$$

$$= \frac{2}{(k+2)(k+3)(k+4)}\left(\psi'(k+1) - \psi'(2) + \left(\psi(k+1) - \psi(2) - \frac{k-1}{2(k+1)}\right)^2 - \frac{(k+1)^2+4}{4(k+1)^2}\right)$$

$$A_6 = \frac{1}{(k+3)(k+4)(k+5)}\int_0^\infty \frac{x^\alpha \ln^2(x^\alpha)}{(1+x^\alpha)^2}(k+2)(k+3)\alpha x^{2\alpha-1}(1+x^\alpha)^{-(k+4)}dx$$

$$= \frac{2}{(k+3)(k+4)(k+5)}\left(\psi'(k+2) - \psi'(2) + \left(\psi(k+2) - \psi(2) - \frac{k}{2(k+2)}\right)^2 - \frac{(k+2)^2+4}{4(k+2)^2}\right)$$

$$A_7 = \frac{1}{(k+3)}\int_0^\infty \frac{x^\alpha \ln^2(x^\alpha)}{(1+x^\alpha)^2}(k+3)\alpha x^{\alpha-1}(1+x^\alpha)^{-(k+4)}dx$$

$$= \frac{1}{(k+4)(k+5)}\left(\psi'(k+3) - \psi'(1) + \left(\psi(k+3) - \psi(1) - \frac{k+2}{k+3}\right)^2 - \frac{(k+3)^2+1}{(k+3)^2}\right)$$

$$A_8 = \frac{2}{(k+1)(k+2)(k+3)}\int_0^\infty \frac{x^\alpha \ln^2(x^\alpha)(k+1)(k+2)(k+3)\alpha x^{3\alpha-1}(1+x^\alpha)^{-(k+4)}}{2(1+x^\alpha)^2}dx$$

$$= \frac{6}{(k+2)(k+3)(k+4)(k+5)}\left(\psi'(k+1) - \psi'(3) + \left(\psi(k+1) - \psi(3) - \frac{2-k}{3(k+1)}\right)^2 - \frac{(k+1)^2+9}{9(k+1)^2}\right)$$

References

[1] Ahmed, A. S. (2005). Estimation of parameters of life from progressively censored data using burr-xii model. IEEE TRANSACTION ON RELIBILITY 54, 159-167.

[2] Brazauskas., V. (2002). Fisher information matrix for the feller-pareto distribution. Statistics and Probability Letters 59, 159-167.

[3] Aggarwala, R., Progressive censoring: a review, in Handbook of Statistics. vol. 20: Advances in Reliability, N. Balakrishnan and C. R. Rao, Eds., 2001, pp. 373-429.

[4] Balakrishnan, N. Ed. Newark, NJ: Gordon and Breach Publishers, 1999.

[5] Al-Hussaini, E. K. and Jaheen Z. F., Bayesian estimation of the parameters, reliability and failure rate functions of the Burr type XII failure model, • J. Statistical. Computation and Simulation, vol. 41, pp. 31-40, 1992.

[6] Al-Hussaini, E. K., Mousa, M. A. and Jaheen, Z. F. Estimation under the Burr type XII failure model based on censored data: a comparative study, Test, vol. 1, pp. 47-60, 1992.

[7] Kimber, A.C., 1983a. Trigamma in gamma samples. Applied Statistics 32, 7-14.

[8] Serfling, R.J., 1980. Approximation Theorems of Mathematical Statistics. Wiley, New York.

[9] Feller, W., 1971. An Introduction to Probability Theory and its Applications, Vol. 2, 2nd Edition. Wiley, New York.