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ODD GENERALIZED EXPONENTIAL-INVERSE LOMAX DISTRIBUTION: PROPERTIES AND APPLICATION

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Abstract

In this paper, we defined the probability density function (Pdf) and cumulative distribution function (Cdf) of the proposed distribution. Some mathematical properties of the proposed distribution such as quantile function, moments, order statistics and asymptotic behavior were derived. The estimation of the proposed distribution's parameters was conducted using the method of Maximum Likelihood Estimation (MLE) procedure, and finally we illustrate the usefulness of the proposed model by fitting the proposed distribution using a real life data and compared its performance to the performance of other distributions. It was found that the Odd Generalized Exponential-Inverse Lomax Distribution is a strong competitor in fitting real life data.

***Keywords*:** Inverse Lomax Distribution, Hazard Function, Survival Function, Moments, Maximum Likelihood Estimation, Odds Function, Generalized Exponential Family of Distributions.

1.0 Introduction

The Lomax or Pareto II (the shifted Pareto) distribution was proposed in [1]. This distribution has found wide applications especially in analysis of the business failure life time data, income and wealth inequality, size of cities, actuarial science, medical and biological sciences, engineering, lifetime and reliability modeling. In lifetime, the Lomax model belongs to the family of decreasing failure rate in [2].

Inverse Lomax distribution belongs to inverted family of distributions and found to be very flexible to analyze the situation where the non-monotonicity of the failure rate has been realized [3]. If a random variable Y has Lomax distribution, then $X = \frac{1}{Y}$ has an Inverse Lomax distribution (ILD) [4].The Inverse Lomax Distribution (ILD) has an application in stochastic modeling of decreasing failure rate life components. Like other distributions belonging to the family of generalized Beta distribution, the Inverse Lomax Distribution also has application in economics and actuarial sciences [5]. Inverse Lomax was implemented on geophysical databases [6] on the sizes of land fires in the California state [7], carried out research work regarding the statistical inference and Prediction on inverse Lomax distribution through Bayesian inferences. [8] Considered the Inverse Lomax distribution to possess the Lorenz ordering relationship between ordered statistics.

The Odd Generalized Exponential (OGE) family, is very flexible because it can have hazard rate shape such as increasing, decreasing, J, reversed-J, bathtub and upside-down bathtub. As such can be used for analyzing life time data of different types. The main reasons for using the OGE family are to make the kurtosis more flexible (compared to the baseline model) and also to construct heavy-tailed distributions that are not long-tailed for modeling real data. [9].

The rest of the paper is outlined as follows. In Section 2, we define the cumulative distribution function (Cdf), probability density function (Pdf), Survival function and hazard function of the Odd Generalized exponential-Inverse Lomax distribution (OGE-ILD). In Section 3, we introduce the statistical properties including, the quantile function, the median, moments, the order statistics and the asymptotic behavior. Section 4 estimates the parameters of the distribution using the Maximum Likelihood Estimation procedure. Finally, an application of OGE-IL using a real data set is presented in Section 5.

2.0 The OGE- Distribution

Given any continuous distribution with CDF $G(x;\Theta)$, pdf $g(x;\Theta)$ and survival function $(1-G(x;\Theta)$ [9] proposed the Odd Generalized Exponential (OGE) family of distributions that provides effective flexibility

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in modeling of real datasets and also used to analyses right-skewed datasets. The CDF of the OGE family of distributions according to [9] is defined as

$$F(x)=F(x;a,b,\Theta)=\left(1-\exp\left(-b\left(\frac{G(x;\Theta)}{1-G(x;\Theta)}\right)\right)\right)^{a} \quad (1)$$

where a> 0 and b > 0 are shape parameters respectively.
The pdf corresponding to (1) is given by

$$f(x;a,b,\Theta)=\frac{abg(x;\Theta)}{\left(1-G(x;\Theta)\right)^{2}}\exp\left(-b\left(\frac{G(x;\Theta)}{1-G(x;\Theta)}\right)\right)\left(1-\exp\left(-b\left(\frac{G(x;\Theta)}{1-G(x;\Theta)}\right)\right)\right)^{a-1} \quad (2)$$

where $g\left(x;\Theta\right)$ is the baseline Pdf.

1.1 The Inverse Lomax Distribution

[5] Defined the probability density function and cumulative distribution function of ILD as given by the following equations:

$$g(x;\alpha,\beta)=\frac{\alpha\beta}{x^{2}}\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)};\ x\geq 0,\alpha,\beta>0 \quad (3)$$

and

$$G\left(x;\alpha,\beta\right)=\left(1+\frac{\beta}{x}\right)^{-\alpha};\ x\geq 0,\alpha,\beta>0 \quad (4)$$

where, α is the shape parameter and β is the scale parameter of the distribution respectively.

3.0 The OGE-IL Distribution

In this section we define a four parameters distribution called Odd Generalized Exponential-Inverse Lomax Distribution with parameters a, b, α,β written as OGE-IL(Ω), where the vector Ω is defined by = (a, b, α,β) .
A random variable *X* is said to have OGE-IL with parameters a, b, α,β if its cumulative distribution function and probability density functions are given by:

$$F\left(x;\Omega\right)=\left(1-\exp\left(\frac{b}{1-\left(\frac{\beta}{x}+1\right)^{\alpha}}\right)\right)^{a} \quad (5)$$

$x\geq 0,a,b,\alpha,\beta>0$

It can also be re-written as:

$$F\left(x;\Omega\right)=\left(1-\exp\left(-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]\right)\right)^{a} \quad (6)$$

and

$$f(x;\Omega)=\frac{ab\alpha\beta\left(1+\frac{\beta}{x}\right)^{\alpha-1}\exp\left(\frac{b}{1-\left(\frac{\beta}{x}+1\right)^{\alpha}}\right)\left\{1-\exp\left(\left(\frac{b}{1-\left(\frac{\beta}{x}+1\right)^{\alpha}}\right)\right)\right\}^{a-1}}{x^{2}\left(\left(\frac{\beta}{x}+1\right)^{\alpha}-1\right)^{2}} \quad (7)$$

$x\geq 0,a,b,\alpha,\beta>0$

This can also be rewritten as:

$$f(x;\Omega)=\frac{ab\alpha\beta\left(1+\frac{\beta}{x}\right)^{\alpha-1}\exp\left(-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]\right)\left\{1-\exp\left(-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]\right)\right\}^{a-1}}{x^{2}\left(\left(\frac{\beta}{x}+1\right)^{\alpha}-1\right)^{2}} \quad (8)$$

where a and α are shape parameters, b is the rate (inverse scale) parameter, and β is the scale parameter.
A random variable X ~ OGE-IL (Ω) has survival function:

$$S(x)=1-\left(1-\exp\left(\frac{-b}{\left(\frac{\beta}{x}+1\right)^{\alpha}-1}\right)\right)^{a} \tag{9}$$

The hazard rate function of OGE-IL (Ω) is given by:

$$h(x)=\frac{ab\left(\alpha\beta\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)}\right)\exp\left(\frac{-b}{\left(\frac{\beta}{x}+1\right)^{\alpha}-1}\right)\left\{1-\exp\left(\frac{-b}{\left(\frac{\beta}{x}+1\right)^{\alpha}-1}\right)\right\}^{a-1}}{\left(x^{2}\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]^{2}\right)\left(1-\left(1-\exp\left(\frac{-b}{\left(\frac{\beta}{x}+1\right)^{\alpha}-1}\right)\right)^{a}\right)} \tag{10}$$

The plots of the pdf, cdf, survival function and hazard function are given in figures 1, 2, 3 and 4.

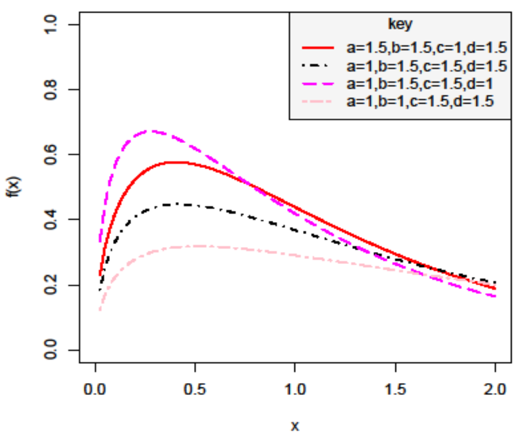

Figure 1: Pdf plot of OGE-ILD

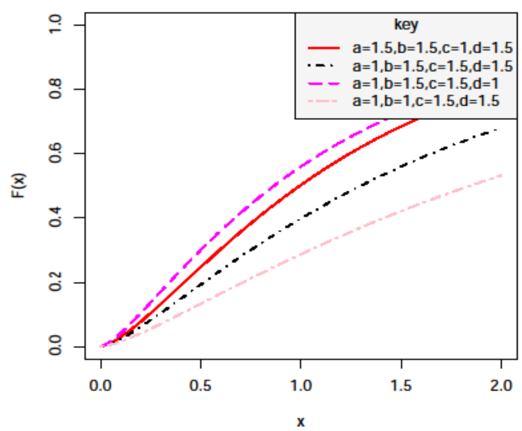

Figure 2: Cdf Plot of OGE-ILD

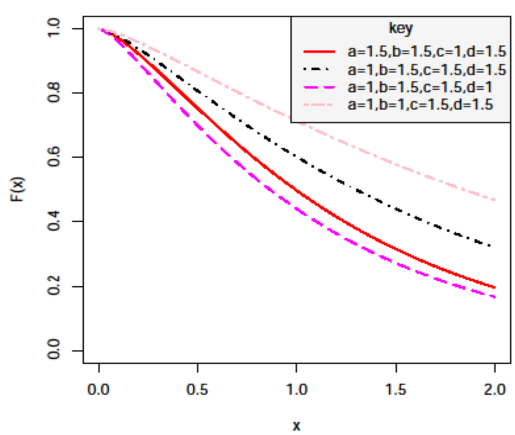

Figure 3: Survival plot of OGE-ILD

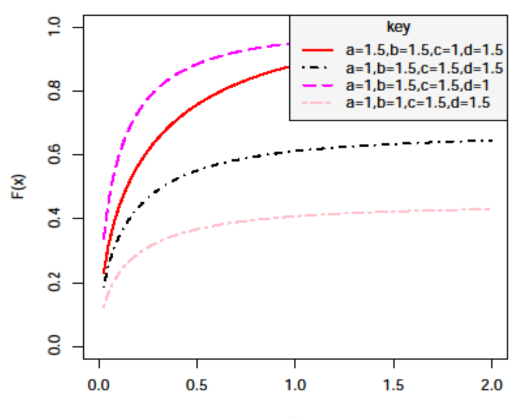

Figure 4: Hazard plot of OGE-ILD

3.0 Statistical Properties

In this section, we study some statistical properties of OGE-ILD, especially quantile, median, and moments.

3.1 Quantile Function and Median of OGE-IL

The quantile of OGE-ILD (Ω) is given by using

Let $F\left(x_q\right)=q$

$$\left(1-\exp\left(\frac{-b}{\left(\frac{\beta}{x_q}+1\right)^{\alpha}-1}\right)\right)^{a}=q \tag{11}$$

$$\exp\left(\frac{-b}{\left(\beta/x_q\right)^{\alpha}-1}\right)=1-q^{\frac{1}{a}} \tag{12}$$

after taking log of both sides and simplifying, we obtained:

$$x_q=\beta\left(\left(\left.l\,\mathrm{n}\left(1-q^{\frac{1}{a}}\right)-b\right/ l\,\mathrm{n}\left(1-q^{\frac{1}{a}}\right)\right)^{\frac{1}{\alpha}}-1\right)^{-1} \tag{14}$$

By setting q=0.5, we have the median as follows:

$$x_{0.5}=\text{Median}=\beta\left(\left(\frac{\ln\left(1-\left(\frac{1}{2}\right)^{\frac{1}{a}}\right)-b}{\ln\left(1-\left(\frac{1}{2}\right)^{\frac{1}{a}}\right)}\right)^{\frac{1}{\alpha}}-1\right)^{-1} \tag{15}$$

3.2. The Moments

Moments has a significance in statistical analysis. We can measure the central tendency of a set of observations with the help of moments, and also their variability, their asymmetry and the height of the peak of a curve of the distribution [10]. In this section, we will derive the r^{th} moments of the OGE-ILD (Ω) as infinite series expansion.

Theorem 1: Let X be a continuous random variable from OGE-ILD, then the rth moment of X about the origin is given by:

$$E(X^r)=\sum_{i=0}^{a-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{a-1}{i}\binom{2+j}{k}(-1)^{i+j+k}\frac{(i+1)^j}{j!}\beta^r a\alpha b^{(1+j)}B\left(1-r,\alpha\left(i+j+k\right)+r\right) \tag{16}$$

Proof

The r^{th} moment of the random variable X with pdf f(x) is defined by:

$$E(X^r)=\int_{-\infty}^{\infty}x^r f(x)dx \tag{17}$$

Hence $E(X^r)$ for OGE-ILD is given by

$$E(X^r)=a\alpha\beta b\int_0^{\infty}x^{r-2}\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)}\exp\left(-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]\right)\left\{1-\exp\left(-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]\right)\right\}^{a-1}\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]^{-2}dx \tag{18}$$

since $1 < \exp\left(-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]\right) < 1$ then, using binomial expansion

$$\left\{1-\exp\left(-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]\right)\right\}^{a-1} = \sum_{i=0}^{a-1}\binom{a-1}{i}(-1)^i \exp\left(-bi\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]\right) \tag{19}$$

Substituting (17) in (12) gives:

$$E(X^r) = a\alpha\beta b\sum_{i=0}^{a-1}\binom{a-1}{i}(-1)^i \int_0^{\infty} x^{r-2}\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)} \exp\left(-b(i+1)\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]\right)\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]^{-2} dx \tag{20}$$

Using the series expansion $\exp\left(-b(i+1)\left[\frac{\left(1+\frac{\beta}{X}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{X}\right)^{-\alpha}}\right]\right) = \sum_{j=0}^{\infty}\frac{(-1)^j b^j (1+i)^j}{j!}\left[\frac{\left(1+\frac{\beta}{X}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{X}\right)^{-\alpha}}\right]^j$

Substituting (19) in (18) gives:

$$E(X^r) = a\alpha\beta b^{(1+j)}\sum_{j=0}^{\infty}\sum_{i=0}^{a-1}\binom{a-1}{i}(-1)^{(i+j)}(i+1)^j \int_0^{\infty} x^{r-2}\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)-\alpha j}\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]^{-2-j} dx \tag{22}$$

applying binomial expansion $\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]^{-(2+j)} = \sum_{k=0}^{\infty}(-1)^k\binom{2+j}{k}\left(1+\frac{\beta}{x}\right)^{-\alpha k}$ (23)

we have:

$$E(X^r) = a\alpha\beta b^{(1+j)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{a-1}\binom{a-1}{i}\binom{2+j}{k}\frac{(-1)^{(i+j)}(i+1)^j}{j!}\int_0^{\infty} X^{r-2}\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)-\alpha j-\alpha k} dx \tag{24}$$

Then,

$$E(X^r) = a\alpha\beta^r b^{(1+j)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{a-1}\binom{a-1}{i}\binom{2+j}{k}\frac{(-1)^{(i+j)}(i+1)^j}{j!}\int_0^{\infty} y^{-r}(1+y)^{-(1+\alpha)-\alpha j-\alpha k} dy \tag{25}$$

$$E(X^r) = a\alpha\beta^r b^{(1+j)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{a-1}\binom{a-1}{i}\binom{2+j}{k}\frac{(-1)^{(i+j)}(i+1)^j}{j!}\int_0^{\infty}\frac{y^{1-r-1}}{(1+y)^{1-r+\alpha(i+j+k)+r}} dy \tag{26}$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{a-1}\binom{a-1}{i}\binom{2+j}{k}\frac{(-1)^{(i+j)}(i+1)^j}{j!}\beta^r a\alpha b^{(1+j)} B\left(1-r, \alpha(i+j+k)+r\right) \tag{27}$$

From the moments, the mean, variance, skewness, kurtosis, and moment generating function can be derived.

3.3 Order statistics

Order statistics are used in many areas of statistical theories and practices, for instance, detection of outlier in statistical quality control processes. In this section, we derive the closed form expressions for the pdf of the i^{th}order statistic of the OGE-ILD.

Suppose $x_1, x_2, ...x_n$ is a random sample from a distribution with pdf f(x) and Let $x_{1:n}, x_{2:n}, ...x_{n:n}$ denotes the corresponding order statistics obtained from this sample. Then

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x) F(x)^{i-1}\left[1-F(x)\right]^{n-1} \tag{28}$$

Where $f(x)$ and $F(x)$ are the pdf and cdf of OGE-ILD distribution.

Using the binomial expansion on $\left[1-F(x)\right]^{n-i}$, we have,

$$\left[1-F(x)\right]^{n-i}=\sum_{r=0}^{n-i}\binom{n-i}{r}(-1)^r F(x)^r \tag{29}$$

Substituting from above

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)\sum_{r=0}^{n-i}\binom{n-i}{r}(-1)^r F(x)^{r+i-1} \tag{30}$$

Substituting (5) and (6) in (28), we have the pdf of the i[th] order statistics of the OGE-IL distribution as:

$$f_{i:n}(x)=\sum_{r=0}^{n-i}\frac{n!(-1)^r}{(i-1)!(n-i-r)!r!}ab\alpha\beta\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)}x^{-2}\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]^{-2}e^{\left(\frac{b}{1-\left(\frac{\beta}{x}+1\right)^{\alpha}}\right)}\left\{1-e^{\left(\frac{b}{1-\left(\frac{\beta}{x}+1\right)^{\alpha}}\right)}\right\}^{a(r+i)-1} \tag{31}$$

Thus, $f_{i:n}(x)$ defined in (29) is the weighted average of the OGE-ILD with a and b shape parameters.

3.4 Asymptotic Behavior of Probability Density Function of OGE-ILD

$$\lim_{x\to\infty}f(x;a,b,\alpha,\beta)=\lim_{x\to\infty}a\alpha\beta b\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)}e^{\left(\frac{b}{1-\left(\frac{\beta}{x}+1\right)^{\alpha}}\right)}\left\{1-e^{\left(\frac{b}{1-\left(\frac{\beta}{x}+1\right)^{\alpha}}\right)}\right\}^{a-1}x^{-2}\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]^{-2} \tag{32}$$

Applying**L'Hôpital's rule,** $\lim_{x\to\infty}e^{\left(\frac{b}{1-\left(\frac{\beta}{x}+1\right)^{\alpha}}\right)}=e^{[0]}=1$ (33)

then,

$$\lim_{x\to\infty}f(x;a,b,\alpha,\beta)=0 \tag{34}$$

$$\lim_{x\to 0}f(x;a,b,\alpha,\beta)=a\alpha\beta b\left(1+\frac{\beta}{x}\right)^{-(1+\alpha)}e^{-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]}\left\{1-e^{-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]}\right\}^{a-1}x^{-2}\left[1-\left(1+\frac{\beta}{x}\right)^{-\alpha}\right]^{-2}$$

$$\lim_{x\to 0}f(x;a,b,\alpha,\beta)=0 \tag{35}$$

Hence, since

$$\lim_{x\to\infty}f(x;a,b,\alpha,\beta)=\lim_{x\to 0}f(x;a,b,\alpha,\beta)=0, \tag{36}$$

this indicate that the proposed distribution OGE-IL has at least one mode.

4. Maximum Likelihood Estimation

Let $x_1,x_2,......x_n$ be a random sample of size n from the OGE-ILD with unknown parameter vector $\Omega=(a,b,\alpha,\beta)^T$, then the Maximum Likelihood Function of OGE-ILD can be express as follows:

$$L(\Omega)=L(x_1,x_2,......x_n/\Omega)=\prod_{i=1}^{n}\left\{f(x,a,b,\alpha,\beta)\right\} \tag{37}$$

$$L(\Omega)=\prod_{i=1}^{n} ab\alpha\beta x_i^{-2}\left(\left(1+\frac{\beta}{x_i}\right)^{-(1+\alpha)}\right)e^{-b\left[\frac{\left(1+\frac{\beta}{x_i}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x_i}\right)^{-\alpha}}\right]}\left\{1-e^{-b\left[\frac{\left(1+\frac{\beta}{x_i}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x_i}\right)^{-\alpha}}\right]}\right\}^{a-1}\left[1-\left(1+\frac{\beta}{x_i}\right)^{-\alpha}\right]^{-2} \quad (38)$$

While the likelihood function for Ω is given by:

$$\ell(\Omega)=n\ln a+n\ln b+n\ln\alpha+n\ln\beta-2\sum_{i=1}^{n}\ln x_i-(1+\alpha)\sum_{i=1}^{n}\ln\left(1+\frac{\beta}{x_i}\right)-b\sum_{i=1}^{n}\left[\frac{\left(1+\frac{\beta}{x_i}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x_i}\right)^{-\alpha}}\right]$$

$$+(a-1)\sum_{i=1}^{n}\ln\left(1-e^{-b\left[\frac{\left(1+\frac{\beta}{x_i}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x_i}\right)^{-\alpha}}\right]}\right)-2\sum_{i=1}^{n}\ln\left[1-\left(1+\frac{\beta}{x_i}\right)^{-\alpha}\right] \quad (39)$$

To obtain the MLEs of the parameters, we differentiate equation (37) partially with respect to $a, b, \alpha, and\, \beta$, and equate to zero. This gives:

$$\frac{\delta\ell(\Omega)}{\delta a}=\frac{n}{a}+\sum_{i=1}^{n}\ln\left(1-e^{-b\left[\frac{\left(1+\frac{\beta}{x_i}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x_i}\right)^{-\alpha}}\right]}\right) \quad (40)$$

$$\frac{\delta\ell(\Omega)}{\delta b}=\frac{n}{b}-\sum_{i=1}^{n}{}^{-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]}+(a-1)\sum_{i=1}^{n}\left\{\frac{-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]e^{-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]}}{1-e^{-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]}}\right\} \quad (41)$$

(42)

$$\frac{\delta\ell(\Omega)}{\delta\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\ln\left(1+\frac{\beta}{x}\right)+b\sum_{i=1}^{n}\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}\ln\left(1+\frac{\beta}{x}\right)}{\left(\left(1+\frac{\beta}{x}\right)^{-\alpha}-1\right)^{2}}+b(a-1)\sum_{i=1}^{n}\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}\ln\left(1+\frac{\beta}{x}\right)e^{-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]}}{\left(\left(1+\frac{\beta}{x}\right)^{\alpha}-1\right)^{2}\left(1-e^{-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]}\right)}-2\sum_{i=1}^{n}\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}\ln\left(1+\frac{\beta}{x}\right)}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}$$

$$\frac{\delta\ell(\Omega)}{\delta\beta}=\frac{n}{\beta}-(1-\alpha)\sum_{i=1}^{n}\left(\frac{1}{(\beta+x)}\right)+\alpha b\sum_{i=1}^{n}\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{\left(\left(1+\frac{\beta}{x}\right)^{-\alpha}-1\right)^{2}(\beta+x)}$$

$$+\alpha b(a-1)\sum_{i=1}^{n}\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}e^{-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]}}{(x+\beta)\left(\left(1+\frac{\beta}{x}\right)^{-\alpha}-1\right)^{2}\left(1-e^{-b\left[\frac{\left(1+\frac{\beta}{x}\right)^{-\alpha}}{1-\left(1+\frac{\beta}{x}\right)^{-\alpha}}\right]}\right)}+\frac{2\alpha\left(1+\frac{\beta}{x}\right)^{-\alpha}}{(x+\beta)\left(\left(1+\frac{\beta}{x}\right)^{-\alpha}-1\right)} \qquad (43)$$

These set of equations are non-linear in parameters and therefore cannot be solved analytically, hence we result to using numerical techniques such as Newton Raphson to solve the system of equations.

5. Application

In this section, we provide an application to real data to illustrate the performance of the Odd Genaralized Exponential Inverse Lomax Distribution (OGE-ILD). The performance of the distribution will be compared with the performances of Lomax, Inverse Lomax, and Odd Generalized Exponential Frechet (OGE-FR).We consider b = 1 for OGE-Fr due to the fact that one scale parameter is enough for fitting this univariate models. The MLEs of the parameters for these models are calculated and seven goodness-of-fit statistics are used to compare the new family with its sub-models. The data set to be used is for the survival times of 121 patients with breast cancer from a hospital from 1929 to 1938 [11]. The data was examined by [12] and also by [9]. The data are as follows:

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3,11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5,17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0,31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0,41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0,54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0,78.0, 80.0,83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0,125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0

Below is the histogram and summary of the data set.

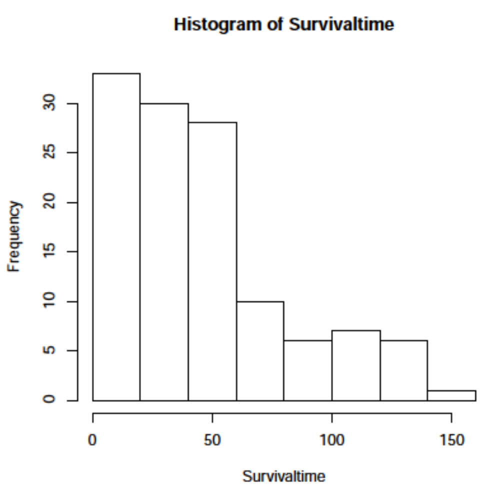

Fig.5: Histogram of Survival Time Data set

Table 1(Summary of Survival Time Data set)

Min.	Max.	1st Quart.	Median	Mean	3rd Quart.	Skewness	Kurtosis
0.30	154	17.50	40	46.33	60	1.04318	3.402139

The MLEs are computed using the Particle Swarm Optimization (PSO). The measures of goodness of fit includes the Akaike Information Criterion (AIC), Consistent Akaikes Information Criterion (CAIC), Bayesian Information Criterion (BIC) and Hannan-Quinn information criterion (HQIC), Anderson-Darling (A), Cramér–von Mises (W) and Kolmogrov-Smirnov (KS) statistics are computed to compare the fitted models. In general, the smaller the values of these statistics, the better the fit to the data. The required computations are carried out in the R-statistical software using the Adequecy Model package [13].

Table 2 and 3 below shows the MLEs and the Statistics (AIC, CAIC, BIC, KS, HQIC,Aand W).

Table 2 (MLE)

	A	b	α	β
OGE_FR	1.423487735	_	0.795687255	0.002629313
Exp.	0.02154913	_	_	_
Inv.lomax	1.904415	1.972035	_	_
Lomax	0.3724352	1.9571081	_	_
OGE_ILD	**1.36765455**	**0.03577866**	**1.26329683**	**1.01172807**

Table 3 Comparison

	AIC	CAIC	BIC	HQIC	VALUE	KS	W	P-Value	A
OGE_FR	1171.86	1172.065	1180.247	1175.267	582.9301	0.096815	0.07249608	0.2067	0.6119724
Inv.Lom	1337.191	1337.293	1342.783	1339.462	666.5956	0.62613	0.6661274	2.2e-16	4.134416
Lomax	1369.836	1369.937	1375.427	1372.107	682.9178	0.81768	0.5548926	2.2e-16	3.451177
OGE_IL	**1169.922**	**1170.267**	**1181.105**	**1174.464**	**580.9609**	**0.11773**	**0.06157787**	**0.06986**	**0.4359145**

We notice from the figures in Table 3 that OGE-ILD model have the lowest values of the AIC, CAIC, BIC, HQIC, KS, W and A statistics as compared to the sub models, suggesting that the OGE-ILD model provide the best fit.

5.1 Conclusion

We propose a new class of distributions called the Odd Generalized Exponential Inverse Lomax Distribution (OGE-ILD). We obtained some of the mathematical properties like the moments, quantile function, asymptotic behavior, and order statistics. The maximum likelihood method is employed to estimate the distribution parameters. We fit model of the proposed distribution to a real data sets to demonstrate the usefulness of the distribution. We use seven goodness-of-fit statistics in order to verify which distribution provides better fit to these data. We conclude that the propose model provide consistently better fits than other competing models.

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