Journal of the Nigerian Association of Mathematical Physics
***Volume 47* (*July, 2018 Issue*), pp77 – 86**
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AN INVENTORY MODEL FOR NON-INSTANTANOUS DETERIORATING ITEM WITH TWO PHASE DEMAND RATE AND PARTIAL BACKLOGGING

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Abstract

This paper investigates an inventory model for non-instantaneous deteriorating items with two phase demand rate and partial backlogging. In real market situation, some items start to deteriorate as soon as they are placed on the shelf while others do not.It is often seen that the demand rate of newly lunched items such as electronic goods, mobile phone, computer and fashionable garments increases with time. The demand rate for such items is constant for some period of time and after that, when the items become popular in the market, the demand for the items increases due to the popularity of the items. Shortages are allowed and partially backlogged with constant rate for there could be situations, in which an economic advantage may be gained by allowing for shortages to occur. The objective of the model is to find the optimal cycle length and order quantity that minimizes the total average cost. Newton-Raphson method has been used to find the optimal cycle length and order quantity that minimizes the total average cost. The result is illustrated with numerical example. A sensitivity analysis of the optimal solution with respect to the parameters of the model is examined.

Keywords: Two–phase demand rate, non-instantaneous deterioration, partial backlogging.

1.0 Introduction

Most of the earlier inventory models consider demand rate to be constant. This is a feature of static environment, while in today's dynamic environment most of the things are not constant. In real market situations, the demand for items is constant for some time and after that, when the product becomes popular in the market, the demand for the items increases. Deterioration of items cannot be avoided in business scenarios, the items that are stored for future use always loose part of their value with passage of time. It is important to a supply manager in modern organization to control and maintain the inventories of deteriorating items. Generally, deterioration refers to damage, evaporation, and spoilage, loss of potential or utility and so on, of a product through time. In our daily life a lot of physical goods undergo decay or deterioration over time, such as vegetable, fruit, perfumes, foods, chemicals, pharmaceutical, electronic equipments and so on, or suffer from depletion by direct spoilage while stored. Highly volatile liquid such as gasoline, alcohol undergo physical depletion over time, through the process of evaporation. Electronic goods like computer, mobile phone and so on deteriorate through a gradual loss of potential or utility with the passage of time. So decay or deterioration of physical goods in stock is a very realistic feature. Therefore, there is need to consider deterioration when analyzing inventory models. Whitin [1] considered deterioration of the fashionable goods at the end of a prescribed shortage period. Ghare and Schrader [2] developed a model for exponentially decaying inventory. An order-level inventory model for items deteriorating at a constant rate was presented by Shah and Jaiswal [3]. Aggarwal [4] develop a note on an order level model for a system with constant deterioration.

All these models discussed above are based on the constant deterioration rate, constant demand, infinite replenishment and no shortages. At times behaviour of customer is unpredictable when the product becomes out of stock. The customer may quit the store and get product from other place, he may wait for the items to be in stock or he may take similar from the stock. According to Sharma [5] there could be situation, in which an economic advantage may be gained by allowing for shortages to occur. One advantage of allowing shortages is to increase the cycle time and hence spreading ordering cost over a longer period of time. Another advantage of shortage may be seen where the unit value of the inventory and hence the inventory carrying cost is high. Normally, the benefit due to reduced carrying cost or less number of orders in a planning period is less than the increase in the total inventory costs due to a shortage condition. Dave and Patel [6] develop (T, Si) policy inventory model for deteriorating items with

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time proportional demand. Roy [7] developed an EOQ model for deteriorating items where deterioration rate and holding cost are expressed as linearly increasing functions of time and demand rate is a function of selling price and shortages are allowed and completely backlogged. Baraya and Sani [8] develop an EOQ model for delayed deterioration items with inventory level dependent demand rate and partial backlogging. Antara [9] develops an EOQ model for time – dependent deterioration items with alternative demand rates allowing shortages by considering time – value of money. Datta and Kumar [10] develop a partial backlogging inventory model for deteriorating items with time dependent demand and constant holding cost. The review of the advances of deterioration inventory literature is presented by Janssen *et al*. [11].

There is a state of interest of studying time dependent demand rate. It is observed that the demand rate of newly lunched products such as electronic goods, mobile phones, computers and fashionable garments increases with time. Inventory problems involving time variable demand pattern has received attention of researchers. Many inventory models were developed assuming time dependent demand as either linear, quadratic, or exponential demand rate and so on. Silver and Meal [12] established approximate solution techniques of deterministic inventory model with time dependent demand rate. Donalson [13] introduced an inventory replenishment policy for a linear trend in demand-an analytical solution. Ritchie [14] presented practical inventory replenishment policies for a linear trend in demand followed by a period of steady demand. Datta and Pal [15] develop note on an inventory model with inventory-level dependent demand rate. Dave [16] develops on a heuristic inventory-replenishment rule for items with a linearly increasing demand incorporating shortages. Hariga and Goyal [17] studied an alternative procedure for determining the optimal policy for an inventory item having linear trend in demand. Bhunia and Maiti [18] developed two-warehouse inventory model for deteriorating items with linear trend in demand and shortages. Shinbsankar and Chaudhuri [19] develop an inventory model with time-dependent demand rate, inflation and time value money for a ware – house enterprises. Ghosh and Chaudhuri [20] presentedAn *EOQ* with a quadratic demand time – proportional deterioration and shortage in all cycle. Sunshil and Ravendra [21] develop deterministic inventory model for perishable items with time dependent demand and shortages. Amutha [22] developed an inventory model for deteriorating items with quadratic demand, time dependent holding cost and salvage values. Sandeep *et al.*[23] presented EOQ model for weibull distribution deteriorating items with exponential demand under linearly time dependent and shortage.

Inventory problem involving time variable demand pattern with deterioration have received attention from several researchers. Goswami and Chaudhuri [24] presented an EOQ model for deteriorating items with linear time-dependent demand rate and shortages under inflation and time discounting. Ghosh and Chaudhuri [25] developed an inventory model with two parameter weibull distribution deterioration rate, time quadratic demand and shortage. Khanra *et al.* [26] discussed an order – level inventory model for deterioration items with time dependent quadratic demand rateunder permissible delay in payment. Karabi *et al.* [27] develop an inventory model for two warehouses with deteriorating items with time dependent demand and shortages. Sarkar and Sakar [28] presented an inventory model with linear time dependent partial backlogging, time varying deterioration. Sunshil and Rajput [29] presented inventory model for deterioration items with time dependent partial backlogging. Preeti Malhotra [30] develop an inventory model for deteriorating items with an exponentially demand with time. Palani and Managatham [31] develop model for controllable deterioration rate time dependent demand and inventory holding cost. Rangarajan and Karthikeyan [32] a partially backlogged inventory model for non instantaneous deteriorating items.

Singh *et al.* [33] considered an optimal inventor policy for deteriorating items with time proportional deterioration rate and constant and time dependent demand without shortage. In real market situation, some items start to deteriorate as soon as they are placed on the shelf while others do not. The items that do not start deterioration instantly include computers, televisions, branded android mobiles, automobiles, garments, electronic equipments and so on.In actual practice demand rate for such items is constant for some period of time and after that, when the items become popular in the market, the demand for the items increases due to the popularity of the items. The main objective of the paper is to determine the minimum cost inventory policy of an inventory model for non-instantaneous deteriorating items with two-phase demand rate in which shortages are allowed with constant partial backlogging rate.

2.0 Model Description and Formulation

The proposed model has been described by the following notation and assumptions.

2.1 Notation and Assumptions

Notation

$I(t)$ The inventory level at any time t

T The length of the replenishment cycle

μ The time point at which demand rate changes with time and when deterioration sets in

t_1 Time at which inventory depletes to zero

C_O The ordering cost per order

i Fraction of the inventory cost

C The inventory unit cost of item

K Shortage cost per unit per unit of time

W The maximum inventory level

M The backorder level during the shortage period

Q The order quantity during the cycle length

Assumptions

1. The inventory system involves only type of item.
2. Replenishment is instantaneous and planning horizon is infinite
3. The ordering cost, holding cost and unit cost remain constant.
4. The demand is deterministic and has two component forms of demand rate for the time horizon. During the time interval$[0,\mu]$, the demand rate is constant and during the time period $[\mu, t_1]$, the demand rate D (t) is linear function of time. This demand rate is defined as
$$D(t) = \begin{cases} a & 0 \le t \le \mu \\ a + b(t-\mu) & \mu \le t \le t_1 \end{cases}$$
5. Deterioration starts at t = μ with constant deterioration rate θ (0 < θ < 1)
6. Shortages are allowed to occur with constant partial backlogging rate δ $(0 < \delta < 1)$

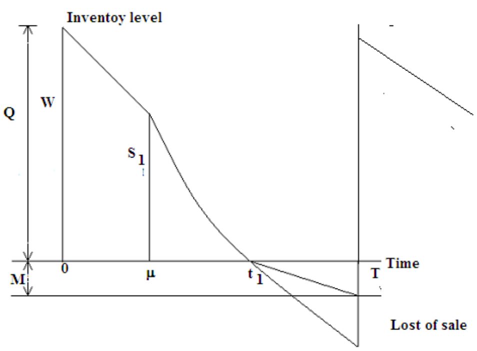

Figure 1: The graphical representation of the inventory level

2.2 Mathematical Formulation of the Model

The inventory level as depicted in figure 1 depletes during period [0,μ] due to market demand only. At time t = μ deterioration sets in and the depletion of inventory occurs due to the combined effects of demand and deterioration. Shortages start at time t = t_1 and the backorder is cleared at t = T. The systems of differential equations which describe the variation of the inventory level during [0, T] are given by the equations (1), (2) and (3) below.

$$\frac{dI(t)}{dt} = -a \quad , \quad 0 \le t \le \mu \quad (1)$$

with boundary conditions I(t) = W at t = 0 and I(t) = S_1 at t = μ .

$$\frac{dI(t)}{dt} + \theta I(t) = -[a + b(t-\mu)] \quad , \quad \mu \le t \le t_1 \quad (2)$$

with boundary conditions $I(\mu) = S_1,\ I(t_1) = 0$

$$\frac{dI(t)}{dt} = -a\delta \quad , \quad t_1 \le t \le T \quad (3)$$

with boundary conditions $I(t_1) = 0$ and $I(T) = \text{M}$.

The solution of equation (1) during the period [0,μ] is obtained as follows:

Equation (1) reduces to

$$\int dI(t) = -a \int dt$$

or

$$I(t) = -at + K_1 \ ,t_1 \le t \le T \quad (4)$$

where K_1 is a constant of integration.

Using the condition $I(t) = W$ at $t = 0$ in (4), we have

$W = K_1$

Now (4) becomes

$$I(t) = W - at \ , \ 0 \le t \le \mu \quad (5)$$

Applying the boundary condition $I(t) = S_1$ at $t = \mu$ in(5) , we get

$S_1 = W - a\mu$(6)

This implies at = μ , the inventory level has been reduced by $a\mu$

Equation (2) is first order linear differential equation whose integrating factor is $e^{\theta t}$

reduces to

$$I(t)e^{\theta t} = -\int e^{\theta t}(a + bt - b\mu)dt$$

$$= -a\int e^{\theta t}dt - b\int t\,e^{\theta t}dt + b\mu\int e^{\theta t}dt$$
$$= -\frac{a\,e^{\theta t}}{\theta} - b\left[\frac{te^{\theta t}}{\theta} - \frac{e^{\theta t}}{\theta^2}\right] + \frac{b\mu e^{\theta t}}{\theta} + K_2$$
$$= -\frac{a\,e^{\theta t}}{\theta} - \frac{bte^{\theta t}}{\theta} + \frac{be^{\theta t}}{\theta^2} + \frac{b\mu e^{\theta t}}{\theta} + K_2 \qquad (7)$$

where K_2 is a constant of integration.
Using condition $I(t_1) = 0$ into (7), we get
$$0 = -\frac{ae^{\theta t_1}}{\theta} - \frac{bt_1e^{\theta t_1}}{\theta} + \frac{be^{\theta t_1}}{\theta^2} + \frac{b\mu e^{\theta t_1}}{\theta} + K_2$$
or
$$K_2 = \frac{ae^{\theta t_1}}{\theta} + \frac{bt_1e^{\theta t_1}}{\theta} - \frac{be^{\theta t_1}}{\theta^2} - \frac{b\mu e^{\theta t_1}}{\theta}$$
Substituting K_2 into equation (7), we get
$$I(t)e^{\theta t} = \frac{ae^{\theta t_1}}{\theta} + \frac{bt_1e^{\theta t_1}}{\theta} - \frac{be^{\theta t_1}}{\theta^2} - \frac{b\mu e^{\theta t_1}}{\theta} - \frac{a\,e^{\theta t}}{\theta} - \frac{bte^{\theta t}}{\theta} + \frac{be^{\theta t}}{\theta^2} + \frac{b\mu e^{\theta t}}{\theta}$$
or
$$I(t) = \frac{b}{\theta^2}\left(1 - e^{\theta(t_1-t)}\right) + \frac{b\mu}{\theta}\left(1 - e^{\theta(t_1-t)}\right) - \frac{a}{\theta}\left(1 - e^{\theta(t_1-t)}\right) + \frac{b}{\theta}\left(t_1e^{\theta(t_1-t)} - t\right)$$
$$= \frac{\left(1 - e^{\theta(t_1-t)}\right)}{\theta}\left[\frac{b}{\theta} + b\mu - a\right] + \frac{b}{\theta}\left(t_1e^{\theta(t_1-t)} - t\right), \quad \mu \le t \le t_1 (8)$$
Putting the condition $I(\mu) = S_1$ into equation(8) , we get
$$S_1 = \frac{\left(1 - e^{\theta(t_1-\mu)}\right)}{\theta}\left[\frac{b}{\theta} + b\mu - a\right] + \frac{b}{\theta}\left(t_1e^{\theta(t_1-\mu)} - \mu\right) \qquad (9)$$

Combining (6) and (9), we have
$$W = a\mu + \frac{\left(1 - e^{\theta(t_1-\mu)}\right)}{\theta}\left[\frac{b}{\theta} + b\mu - a\right] + \frac{b}{\theta}\left(t_1e^{\theta(t_1-\mu)} - \mu\right) \qquad (10)$$
Q =Maximum inventory level +backorder
$$= a\mu + \frac{\left(1 - e^{\theta(t_1-\mu)}\right)}{\theta}\left[\frac{b}{\theta} + b\mu - a\right] + \frac{b}{\theta}\left(t_1e^{\theta(t_1-\mu)} - \mu\right) + \alpha\delta(T - t_1) \qquad (11)$$
Equation (3) is solved as follows:
$$I(t) = -\int a\,\delta\,dt$$
$$= -at\delta + K_3 \qquad (12)$$
where K_3 is constant of integration.
Substitute the boundary condition $I(t_1)$ = 0 into (12) to have
$$K_3 = at_1\delta$$
and substituting the value of K_3 into equation (12), we get

$$I(t) = -at\delta + at_1\delta$$
$$= a\delta(t_1 - t) \qquad ,t_1 \le t \le T \qquad (13)$$
The average total cost is composed of the following cost components:

(i) The inventory ordering cost= C_0

(ii) The inventory holding cost during the period $[0, T]$ is the sum of inventory holding cost during $[0, \mu]$ and inventory holding cost the during the period$[\mu, t_1]$which is given by
$$IHC = iC\int_0^{t_1} I(t)dt \quad = iC\int_0^{\mu} I(t)\,dt + iC\int_{\mu}^{t_1} I(t)dt$$
Substituting equation (5) and equation (8) into above expression, we get
$$IHC = iC\int_0^{\mu}(W - at)dt + iC\int_{\mu}^{t_1}\left(\frac{\left(1 - e^{\theta(t_1-t)}\right)}{\theta}\right)\left(\frac{b}{\theta} + b\mu - a\right)$$
$$+ \frac{biC}{\theta}\int_{\mu}^{t_1}\left(t_1e^{\theta(t_1-t)} - t\right)dt$$
$$= iC\int_0^{\mu}\left[a\mu + \frac{1}{\theta}\left(1 - e^{\theta(t_1-\mu)}\right)\left(\frac{b}{\theta} + b\mu - a\right) + \frac{b}{\theta}\left(t_1e^{\theta(t_1-\mu)} - \mu\right) - at\right]dt$$

$$+\frac{iC}{\theta}\int_{\mu}^{t_1}\left(1-e^{\theta(t_1-t)}\right)\left(\frac{b}{\theta}+b\mu-a\right)dt+\frac{biC}{\theta}\int_{\mu}^{t_1}\left(t_1e^{\theta(t_1-t)}-t\right)dt$$

$$=iC\left[a\mu t+\frac{1}{\theta}\left(\frac{b}{\theta}+b\mu-a\right)\left(t-te^{\theta(t_1-\mu)}\right)+\frac{b}{\theta}\left(t_1te^{\theta(t_1-\mu)}-\mu t\right)-\frac{at^2}{2}\right]_0^{\mu}$$

$$+\left[\frac{iC}{\theta}\left(\frac{b}{\theta}+b\mu-a\right)\left(t+\frac{e^{\theta(t_1-t)}}{\theta}\right)\right]_{\mu}^{t_1}+\frac{biC}{\theta}\left[\frac{-t_1e^{\theta(t_1-t)}}{\theta}-\frac{t^2}{2}\right]_{\mu}^{t_1}$$

$$=aiC\mu^2+\frac{iC}{\theta}\left(\frac{b}{\theta}+b\mu-a\right)\left(\mu-\mu e^{\theta(t_1-\mu)}\right)+\frac{biC}{\theta}\left(\mu t_1e^{\theta(t_1-\mu)}-\mu^2\right)-\frac{aiC\mu^2}{2}$$

$$+\frac{iC}{\theta}\left(\frac{b}{\theta}+b\mu-a\right)\left(t_1-\mu+\frac{1}{\theta}-\frac{e^{\theta(t_1-\mu)}}{\theta}\right)+\frac{biC}{\theta}\left(\frac{-t_1}{\theta}+\frac{t_1e^{\theta(t_1-\mu)}}{\theta}-\frac{t_1^2}{2}+\frac{\mu^2}{2}\right)$$

$$=a\mu^2iC-\frac{a\mu^2iC}{2}+\frac{iC}{\theta}\left(\frac{b}{\theta}+b\mu-a\right)\left(\mu-\mu e^{\theta(t_1-\mu)}\right)+\frac{biC}{\theta}\left(\mu t_1e^{\theta(t_1-\mu)}-\mu^2\right)$$

$$+\frac{iC}{\theta}\left(\frac{b}{\theta}+b\mu-a\right)\left(t_1-\mu+\frac{1}{\theta}-\frac{e^{\theta(t_1-\mu)}}{\theta}\right)+\frac{biC}{\theta}\left(\frac{-t_1}{\theta}+\frac{t_1e^{\theta(t_1-\mu)}}{\theta}-\frac{t_1^2}{2}+\frac{\mu^2}{2}\right)$$

$$=\frac{a\mu^2iC}{2}+\frac{b\mu iC}{\theta^2}-\frac{b\mu iC}{\theta^2}e^{\theta(t_1-\mu)}+\frac{b\mu^2iC}{\theta}-\frac{b\mu^2iC}{\theta}e^{\theta(t_1-\mu)}-\frac{a\mu iC}{\theta}$$

$$+\frac{a\mu iC}{\theta}e^{\theta(t_1-\mu)}+\frac{b\mu t_1iC}{\theta}e^{\theta(t_1-\mu)}-\frac{b\mu^2iC}{\theta}+\frac{bt_1iC}{\theta^2}-\frac{b\mu iC}{\theta^2}+\frac{biC}{\theta^3}-\frac{biC}{\theta^3}e^{\theta(t_1-\mu)}$$

$$+\frac{b\mu t_1iC}{\theta}-\frac{b\mu^2iC}{\theta}+\frac{b\mu iC}{\theta^2}-\frac{b\mu iC}{\theta^2}e^{\theta(t_1-\mu)}-\frac{at_1iC}{\theta}+\frac{a\mu iC}{\theta}-\frac{aiC}{\theta^2}+\frac{aiC}{\theta^2}e^{\theta(t_1-\mu)}$$

$$-\frac{bt_1iC}{\theta^2}+\frac{bt_1iC}{\theta^2}e^{\theta(t_1-\mu)}-\frac{bt_1^2iC}{2\theta}+\frac{b\mu^2iC}{2\theta}$$

$$=\frac{a\mu^2iC}{2}-\frac{2b\mu iC}{\theta^2}e^{\theta(t_1-\mu)}-\frac{biC\mu^2}{\theta}e^{\theta(t_1-\mu)}+\frac{aiC\mu}{\theta}e^{\theta(t_1-\mu)}+\frac{biC\mu t_1}{\theta}e^{\theta(t_1-\mu)}+\frac{b\mu iCt_1}{\theta}$$

$$-\frac{at_1iC}{\theta}-\frac{biC\mu^2}{\theta}+\frac{biC}{\theta^3}+\frac{biC\mu}{\theta^2}-\frac{aiC}{\theta^2}-\frac{biC}{\theta^3}e^{\theta(t_1-\mu)}+\frac{aiC}{\theta^2}e^{\theta(t_1-\mu)}+\frac{biCt_1}{\theta^2}-\frac{biCt_1^2}{2\theta}$$

$$+\frac{b\mu^2}{2\theta}$$

$$=\frac{biC}{\theta^3}\left[1-e^{\theta(t_1-\mu)}\right]+\frac{aiC}{\theta^2}\left[e^{\theta(t_1-\mu)}-1\right]+\frac{a\mu iC}{\theta}\left[e^{\theta(t_1-\mu)}+\frac{\mu\theta}{2}\right]$$

$$-\frac{b\mu^2iC}{\theta}\left[e^{\theta(t_1-\mu)}+\frac{1}{2}\right]+\frac{b\mu iC}{\theta^2}\left[1-2e^{\theta(t_1-\mu)}\right]+\frac{b\mu t_1iC}{\theta}\left[e^{\theta(t_1-\mu)}+1\right]-\frac{at_1iC}{\theta}$$

$$+\frac{bt_1iC}{\theta^2}e^{\theta(t_1-\mu)}-\frac{biCt_1^2}{2\theta}+\frac{b\mu^2iC}{2\theta} \tag{14}$$

(iii) The inventory deterioration cost during the period [0, T]

$$IDC = C\theta\int_{\mu}^{t_1}I(t)dt$$

Putting equation (5)in the above expression, we have

$$=\theta C\left[\int_{\mu}^{t_1}\left(\frac{\left(1-e^{\theta(t_1-t)}\right)}{\theta}\right)\left(\frac{b}{\theta}+b\mu-a\right)dt+\frac{b}{\theta}\int_{\mu}^{t_1}\left(t_1e^{\theta(t_1-t)}-t\right)\right]dt$$

$$=\theta C\left[\int_{\mu}^{t_1}\left(\frac{1}{\theta}\left(1-e^{\theta(t_1-t)}\right)\right)\left(\frac{b}{\theta}+b\mu-a\right)dt+\frac{b}{\theta}\int_{\mu}^{t_1}\left(t_1e^{\theta(t_1-t)}-t\right)\right]dt$$

$$=\theta C\left[\frac{1}{\theta}\left(\frac{b}{\theta}+b\mu-a\right)\left(t+\frac{e^{\theta(t_1-t)}}{\theta}\right)+\frac{b}{\theta}\left(\frac{-t_1e^{\theta(t_1-t)}}{\theta}-\frac{t^2}{2}\right)\right]_{\mu}^{t_1}$$

$$=\theta C\left[\frac{1}{\theta}\left(\frac{b}{\theta}+b\mu-a\right)\left(t_1-\mu+\frac{1}{\theta}-\frac{e^{\theta(t_1-\mu)}}{\theta}\right)\right]$$

$$+\frac{b\theta C}{\theta}\left[\left(\frac{-t_1}{\theta}+\frac{t_1}{\theta}e^{\theta(t_1-\mu)}-\frac{t_1^2}{2}+\frac{\mu^2}{2}\right)\right]$$

$$=\theta C\left[\frac{bt_1}{\theta^2}-\frac{b\mu}{\theta^2}+\frac{b}{\theta^3}-\frac{b}{\theta^3}e^{\theta(t_1-\mu)}+\frac{b\mu t_1}{\theta}-\frac{b\mu^2}{\theta}+\frac{b\mu}{\theta^2}-\frac{b\mu}{\theta^2}e^{\theta(t_1-\mu)}-\frac{at_1}{\theta}+\frac{a\mu}{\theta}-\frac{a}{\theta^2}+\frac{a}{\theta^2}e^{\theta(t_1-\mu)}-\frac{bt_1}{\theta^2}+\frac{bt_1}{\theta^2}e^{\theta(t_1-\mu)}-\frac{bt_1^2}{2\theta}+\frac{b\mu^2}{2\theta}\right]$$

$$=C\theta\left[\frac{b}{\theta^3}\left(1-e^{\theta(t_1-\mu)}\right)+\frac{a}{\theta^2}\left(e^{\theta(t_1-\mu)}-1\right)+\frac{bt_1}{\theta^2}\left(e^{\theta(t_1-\mu)}-\frac{\theta t_1}{2}\right)-\frac{b\mu e^{\theta(t_1-\mu)}}{\theta^2}-\frac{a(t_1-\mu)}{\theta}+\frac{b\mu t_1}{\theta}-\frac{b\mu^2}{2\theta}\right]$$

$$= C\left[\frac{b}{\theta^2}\left(1 - e^{\theta(t_1-\mu)}\right) + \frac{a}{\theta}\left(e^{\theta(t_1-\mu)} - 1 - \theta(t_1-\mu)\right) + \frac{bt_1}{\theta}(e^{\theta(t_1-\mu)} - \frac{\theta t_1}{2}) + b\mu t_1 - \frac{b\mu^2}{2}\right] \quad (15)$$

(iv)Shortage cost during this cycle is given by

$$ISC = K\delta\int_{t_1}^{T} -I(t)dt$$

$$= -K\delta\int_{t_1}^{T}[t_1 - t]dt$$

$$= -K\delta a\left[t_1 t + \frac{t^2}{2}\right]_{t_1}^{T}$$

$$= -K\delta a\left[t_1 T - t_1^2 - \frac{T^2}{2} + \frac{t_1^2}{2}\right]$$

$$= -K\delta a\frac{[2t_1 T - 2t_1^2 - T^2 + t_1^2]}{2}$$

$$= \frac{K\delta a}{2}[T^2 - 2t_1 T + t_1^2]$$

$$= \frac{K\delta a\,(T-t_1)^2}{2} \quad (16)$$

(iv) Lost sale during this cycle is given as

$$\text{LS} = \int_{t_1}^{T} a(1-\delta)dt$$

$$= a(1-\delta)(T-t_1) \quad (17)$$

(v) The total amount of backorder at the end of cycle is

$$\text{IBC} = \int_{t_1}^{T}\delta a dt$$

$$= [\delta\alpha t]_{t_1}^{T}$$

$$= \alpha\delta(T - t_1) \quad (18)$$

Now the total cost is a function of t_1 and is given by

ATC (t_1) = [holding cost + deterioration cost + shortage cost + backorder cost +lost sale]

$= [C_O + IHC + IDC + ISC + IBC + LS]$

The total average cost per unit time is a function of t_1 and T and is given by

$\text{ATC}(t_{1,}T) = \frac{1}{T}[C_O + IHC + IDC + ISC + IBC + LS]$

Substituting the components of cost function, we have

$$ATC(t_{1,}T) = \frac{1}{T}\Bigg\{C_O + \frac{biC}{\theta^3}\left[1 - e^{\theta(t_1-\mu)}\right] + \frac{aiC}{\theta^2}\left[e^{\theta(t_1-\mu)} - 1\right] + \frac{b\mu iC}{\theta}\left[e^{\theta(t_1-\mu)} + \frac{\mu\theta}{2}\right] - \frac{b\mu^2 iC}{\theta}\left[e^{\theta(t_1-\mu)} + \frac{1}{2}\right]$$
$$+ \frac{biC\mu}{\theta^2}\left[1 - 2e^{\theta(t_1-\mu)}\right] + \frac{b\mu t_1 iC}{\theta}\left[e^{\theta(t_1-\mu)} + 1\right] - \frac{at_1 iC}{\theta} + \frac{bt_1 iC}{\theta^2}e^{\theta(t_1-\mu)} - \frac{bt_1 iC}{2\theta} + \frac{b\mu^2 iC}{2\theta} + \frac{bC}{\theta^2}\left[1 - e^{\theta(t_1-\mu)}\right]$$
$$+ \frac{aC}{\theta}\left[e^{\theta(t_1-\mu)} - 1 - \theta(t_1-\mu)\right] + \frac{bt_1 C}{\theta}\left[e^{\theta(t_1-\mu)} - \frac{\theta t_1^2}{2}\right] + b\mu t_1 C - \frac{b\mu^2}{\theta} + \frac{K\delta a(T-t_1)^2}{2} + \alpha\delta[T - t_1]$$
$$+ a[1-\delta][T-t_1]\Bigg\} \quad (19)$$

3.Optimal Decision

By letting $t_1 = \lambda T$ with$0 < \lambda < 1$, the necessary condition for the existence of optimum values of $t_1$and T is

$$\frac{dATC(T)}{dT} = 0 \quad (20a)$$

and sufficiency condition is

$$\frac{d^2ATC(T)}{dT} > 0 \quad (20b)$$

Substituting $t_1 = \lambda T$ in equation (19), we have

$$ATC(T)=\frac{C_O}{T}+\frac{biC}{T\theta^3}-\frac{biC}{T\theta^3}e^{\theta(\lambda T-\mu)}+\frac{aiC}{T\theta^2}e^{\theta(\lambda T-\mu)}-\frac{aiC}{T\theta^2}+\frac{b\mu iC}{T\theta}e^{\theta(\lambda T-\mu)}+\frac{b\mu^2 iC}{2T}$$
$$-\frac{b\mu^2 iC}{\theta T}e^{\theta(\lambda T-\mu)}-\frac{b\mu^2 iC}{2T\theta}+\frac{b\mu iC}{T\theta^2}-\frac{2b\mu iC}{T\theta^2}e^{\theta(\lambda T-\mu)}+\frac{b\mu iC\lambda}{\theta}e^{\theta(\lambda T-\mu)}-\frac{b\mu iC\lambda}{\theta}-\frac{aiC\lambda}{\theta}$$
$$+\frac{biC\lambda}{\theta^2}e^{\theta(\lambda T-\mu)}-\frac{biC\lambda}{2\theta}+\frac{biC\mu^2}{2\theta T}+\frac{bC}{T\theta^2}-\frac{bC}{T\theta^2}e^{\theta(\lambda T-\mu)}+\frac{aC}{T\theta}e^{\theta(\lambda T-\mu)}-\frac{aC}{T\theta}$$
$$-\frac{aC}{T}(\lambda T-\mu)+\frac{b\lambda C}{\theta}e^{\theta(\lambda T-\mu)}-\frac{bC\lambda^3T^2}{2}+bC\mu\lambda-\frac{bC\mu^2}{T\theta}+\frac{K\delta aT(1-\lambda)^2}{2}+a\delta(1-\lambda)$$
$$+a(1-\delta)(T-\lambda T) \quad (21)$$

Differentiating ATC (T) in(21) with respect to T, we have

$$\frac{dATC(T)}{dT}=-\frac{C_O}{T^2}-\frac{biC}{\theta^3T^2}+\frac{biC}{\theta^3T^2}e^{\theta(\lambda T-\mu)}-\frac{b\lambda iC}{\theta^2T}e^{\theta(\lambda T-\mu)}-\frac{aiC}{\theta^2T^2}e^{\theta(\lambda T-\mu)}$$
$$+\frac{aiC\lambda}{\theta T}e^{\theta(\lambda T-\mu)}+\frac{aiC}{\theta^2T^2}-\frac{b\mu iC}{\theta T^2}e^{\theta(\lambda T-\mu)}+\frac{b\mu\lambda iC}{T}e^{\theta(\lambda T-\mu)}-\frac{b\mu^2 iC}{2T^2}+\frac{biC\mu^2}{\theta T^2}e^{\theta(\lambda T-\mu)}$$
$$-\frac{biC\mu^2\lambda}{T}e^{\theta(\lambda T-\mu)}+\frac{b\mu^2 iC}{2\theta T^2}-\frac{b\mu iC}{\theta^2T^2}+\frac{2b\mu iC}{T^2\theta^2}e^{\theta(\lambda T-\mu)}-\frac{2b\mu iC}{\theta T}e^{\theta(\lambda T-\mu)}$$
$$+b\lambda^2\mu iCe^{\theta(\lambda T-\mu)}+\frac{biC\theta\lambda^2}{\theta}e^{\theta(\lambda T-\mu)}-\frac{biC\mu^2}{2\theta T^2}-\frac{bC}{T^2\theta^2}+\frac{bC}{T^2\theta^2}e^{\theta(\lambda T-\mu)}-\frac{bC\lambda}{\theta T}e^{\theta(\lambda T-\mu)}$$
$$-\frac{aC}{\theta T^2}e^{\theta(\lambda T-\mu)}+\frac{a\lambda C}{T}e^{\theta(\lambda T-\mu)}+\frac{aC}{\theta T^2}+\frac{aC}{T^2}(\lambda T-\mu)-\frac{a\lambda C}{T}+b\lambda^2Ce^{\theta(\lambda T-\mu)}$$
$$-b\lambda^3CT+\frac{bC\mu^2}{\theta T^2}+\frac{K\delta a(1-\lambda)^2}{2}$$
$$=-\frac{C_O}{T^2}-\frac{biC}{\theta^3T^2}+\frac{biC}{\theta^3T^2}e^{\theta(\lambda T-\mu)}-\frac{b\lambda iC}{\theta^2T}e^{\theta(\lambda T-\mu)}-\frac{aiC}{\theta^2T^2}e^{\theta(\lambda T-\mu)}+\frac{aiC\lambda}{\theta T}e^{\theta(\lambda T-\mu)}$$
$$+\frac{aiC}{\theta^2T^2}-\frac{b\mu iC}{\theta T^2}e^{\theta(\lambda T-\mu)}+\frac{b\mu\lambda iC}{T}e^{\theta(\lambda T-\mu)}-\frac{b\mu^2 iC}{2T^2}+\frac{biC\mu^2}{\theta T^2}e^{\theta(\lambda T-\mu)}-\frac{biC\mu^2\lambda}{T}e^{\theta(\lambda T-\mu)}$$
$$-\frac{b\mu iC}{\theta^2T^2}+\frac{2b\mu iC}{T^2\theta^2}e^{\theta(\lambda T-\mu)}-\frac{2b\mu iC}{\theta T}e^{\theta(\lambda T-\mu)}+b\lambda^2\mu iCe^{\theta(\lambda T-\mu)}+\frac{biC\theta\lambda^2}{\theta}e^{\theta(\lambda T-\mu)}-\frac{bC}{T^2\theta^2}$$
$$+\frac{bC}{T^2\theta^2}e^{\theta(\lambda T-\mu)}-\frac{bC\lambda}{\theta T}e^{\theta(\lambda T-\mu)}-\frac{aC}{\theta T^2}e^{\theta(\lambda T-\mu)}+\frac{a\lambda C}{T}e^{\theta(\lambda T-\mu)}+\frac{aC}{\theta T^2}+\frac{aC\lambda}{T}-\frac{aC\mu}{T^2}$$
$$-\frac{a\lambda C}{T}+b\lambda^2Ce^{\theta(\lambda T-\mu)}-b\lambda^3CT+\frac{bC\mu^2}{\theta T^2}+\frac{K\delta a(1-\lambda)^2}{2}$$
$$=-\frac{C_O}{T^2}-\frac{biC}{\theta^3T^2}+\frac{biC}{\theta^3T^2}e^{\theta(\lambda T-\mu)}-\frac{b\lambda iC}{\theta^2T}e^{\theta(\lambda T-\mu)}-\frac{aiC}{\theta^2T^2}e^{\theta(\lambda T-\mu)}+\frac{aiC\lambda}{\theta T}e^{\theta(\lambda T-\mu)}$$
$$+\frac{aiC}{\theta^2T^2}-\frac{b\mu iC}{\theta T^2}e^{\theta(\lambda T-\mu)}+\frac{b\mu\lambda iC}{T}e^{\theta(\lambda T-\mu)}-\frac{b\mu^2 iC}{2T^2}+\frac{biC\mu^2}{\theta T^2}e^{\theta(\lambda T-\mu)}-\frac{biC\mu^2\lambda}{T}e^{\theta(\lambda T-\mu)}$$
$$-\frac{b\mu iC}{\theta^2T^2}-\frac{bC}{T^2\theta^2}+\frac{bC}{T^2\theta^2}e^{\theta(\lambda T-\mu)}-\frac{bC\lambda}{\theta T}e^{\theta(\lambda T-\mu)}-\frac{aC}{\theta T^2}e^{\theta(\lambda T-\mu)}+\frac{a\lambda C}{T}e^{\theta(\lambda T-\mu)}+\frac{aC}{\theta T^2}$$
$$-\frac{aC\mu}{T^2}+b\lambda^2Ce^{\theta(\lambda T-\mu)}-b\lambda^3CT+\frac{bC\mu^2}{\theta T^2}+\frac{K\delta a(1-\lambda)^2}{2} \quad (22)$$

Setting $\frac{dATC(T)}{dT}=0$, we have

$$-\frac{C_O}{T^2}-\frac{biC}{\theta^3T^2}+\frac{biC}{\theta^3T^2}e^{\theta(\lambda T-\mu)}-\frac{b\lambda iC}{\theta^2T}e^{\theta(\lambda T-\mu)}-\frac{aiC}{\theta^2T^2}e^{\theta(\lambda T-\mu)}+\frac{aiC\lambda}{\theta T}e^{\theta(\lambda T-\mu)}+\frac{aiC}{\theta^2T^2}$$
$$-\frac{b\mu iC}{\theta T^2}e^{\theta(\lambda T-\mu)}+\frac{b\mu\lambda iC}{T}e^{\theta(\lambda T-\mu)}-\frac{b\mu^2 iC}{2T^2}+\frac{biC\mu^2}{\theta T^2}e^{\theta(\lambda T-\mu)}-\frac{biC\mu^2\lambda}{T}e^{\theta(\lambda T-\mu)}-\frac{b\mu iC}{\theta^2T^2}$$
$$+\frac{2b\mu iC}{T^2\theta^2}e^{\theta(\lambda T-\mu)}-\frac{2b\mu iC}{\theta T}e^{\theta(\lambda T-\mu)}+b\lambda^2\mu iCe^{\theta(\lambda T-\mu)}+\frac{biC\theta\lambda^2}{\theta}e^{\theta(\lambda T-\mu)}-\frac{bC}{T^2\theta^2}$$
$$+\frac{bC}{T^2\theta^2}e^{\theta(\lambda T-\mu)}-\frac{bC\lambda}{\theta T}e^{\theta(\lambda T-\mu)}-\frac{aC}{\theta T^2}e^{\theta(\lambda T-\mu)}+\frac{a\lambda C}{T}e^{\theta(\lambda T-\mu)}+\frac{aC}{\theta T^2}-\frac{aC\mu}{T^2}$$
$$+b\lambda^2Ce^{\theta(\lambda T-\mu)}-b\lambda^3CT+\frac{bC\mu^2}{\theta T^2}+\frac{K\delta a(1-\lambda)^2}{2}=0 \quad (23)$$

$$\frac{d^2ATC(T)}{dT^2}=\frac{2C_O}{T^3}+\frac{2biC}{T^3\theta^3}-\frac{2biC}{\theta^3T^3}e^{\theta(\lambda T-\mu)}+\frac{b\lambda iC}{T^2\theta^2}e^{\theta(\lambda T-\mu)}+\frac{b\lambda iC}{T^2\theta^2}e^{\theta(\lambda T-\mu)}$$
$$-\frac{b\lambda^2 iC}{\theta T}e^{\theta(\lambda T-\mu)}+\frac{2aiC}{T^3\theta^2}e^{\theta(\lambda T-\mu)}-\frac{aiC\lambda}{\theta T^2}e^{\theta(\lambda T-\mu)}-\frac{aiC\lambda}{\theta T^2}e^{\theta(\lambda T-\mu)}+\frac{aiC\lambda^2}{T}e^{\theta(\lambda T-\mu)}$$

$$
\begin{aligned}
&+\frac{2b\mu iC}{\theta T^3}e^{\theta(\lambda T-\mu)}-\frac{b\mu iC\lambda}{T^2}-\frac{b\mu iC\lambda}{T^2}+\frac{b\mu iC\theta\lambda^2}{T}e^{\theta(\lambda T-\mu)}+\frac{b\mu iC}{T^3}-\frac{2b\mu^2 iC}{\theta T^3}e^{\theta(\lambda T-\mu)}\\
&-\frac{2b\mu^2 iC}{\theta T^3}e^{\theta(\lambda T-\mu)}+\frac{b\mu^2\lambda iC}{T^2}e^{\theta(\lambda T-\mu)}+\frac{b\mu^2 iC}{T^2}e^{\theta(\lambda T-\mu)}-\frac{b\mu^2\lambda\theta iC}{T}e^{\theta(\lambda T-\mu)}+\frac{2b\mu iC}{T^3\theta^2}\\
&-\frac{4b\mu iC}{T^3\theta^2}+\frac{2b\mu iC\lambda}{\theta T^2}+\frac{2b\mu iC\lambda}{\theta T^2}+\frac{2b\mu iC\lambda^2}{T}e^{\theta(\lambda T-\mu)}+biC\mu\lambda^3\theta e^{\theta(\lambda T-\mu)}+bC\lambda^3 e^{\theta(\lambda T-\mu)}\\
&+\frac{2bC}{T^3\theta^2}-\frac{2bC}{T^3\theta^2}e^{\theta(\lambda T-\mu)}+\frac{bC\lambda}{\theta T^2}e^{\theta(\lambda T-\mu)}+\frac{bC\lambda}{\theta T^2}e^{\theta(\lambda T-\mu)}-\frac{aC\lambda}{T}e^{\theta(\lambda T-\mu)}+\frac{2aC}{\theta T^3}e^{\theta(\lambda T-\mu)}\\
&-\frac{aC\lambda}{T^2}e^{\theta(\lambda T-\mu)}-\frac{aC\lambda}{T^2}e^{\theta(\lambda T-\mu)}+\frac{aC\theta\lambda^2}{T}+\frac{2aC}{\theta T^3}+\frac{2aC\mu}{T^3}+bC\theta\lambda^3 e^{\theta(\lambda T-\mu)}-bC\theta\lambda^3\\
&-\frac{2bC\mu^2}{\theta T^3}\\
&=\frac{2C_0}{T^3}+\frac{2biC}{T^3\theta^3}-\frac{2biC}{\theta^3T^3}e^{\theta(\lambda T-\mu)}+\frac{2b\lambda iC}{T^2\theta^2}e^{\theta(\lambda T-\mu)}-\frac{b\lambda^2 iC}{\theta T}e^{\theta(\lambda T-\mu)}+\frac{2aiC}{T^3\theta^2}e^{\theta(\lambda T-\mu)}\\
&-\frac{2aiC}{\theta^2T^3}-\frac{2aiC\lambda}{\theta T^2}+\frac{aiC\lambda^2}{T}e^{\theta(\lambda T-\mu)}+\frac{2b\mu iC}{\theta T^3}e^{\theta(\lambda T-\mu)}-\frac{2b\mu iC\lambda}{T^2}+\frac{b\mu iC\theta\lambda^2}{T}e^{\theta(\lambda T-\mu)}\\
&+\frac{b\mu^2 iC}{T^3}-\frac{2b\mu iC}{\theta T^3}e^{\theta(\lambda T-\mu)}+\frac{b\mu iC(\lambda+1)}{T^2}-\frac{b\mu^2\lambda\theta iC}{T}e^{\theta(\lambda T-\mu)}+\frac{2b\mu iC}{T^3\theta^2}-\frac{4b\mu iC}{T^3\theta^2}e^{\theta(\lambda T-\mu)}\\
&+\frac{4b\mu\lambda iC}{T^2}-\frac{2b\mu iC\lambda^2}{T}e^{\theta(\lambda T-\mu)}+b\mu iC\lambda^3(1+\theta)e^{\theta(\lambda T-\mu)}+\frac{2bC}{T^3\theta^2}-\frac{2bC}{T^3\theta^2}e^{\theta(\lambda T-\mu)}\\
&+\frac{2bC\lambda}{\theta T^2}e^{\theta(\lambda T-\mu)}-\frac{bC\lambda^2}{T}e^{\theta(\lambda T-\mu)}+\frac{2aC}{\theta T^3}e^{\theta(\lambda T-\mu)}-\frac{2aC\lambda}{T^2}e^{\theta(\lambda T-\mu)}+\frac{aC\theta\lambda^2}{T}e^{\theta(\lambda T-\mu)}\\
&+\frac{2aC}{\theta T^3}+\frac{2aC\mu}{T^3}+bC\theta\lambda^3 e^{\theta(\lambda T-\mu)}-bC\lambda^3-\frac{2bC\mu^2}{\theta T^3}\\
&=\frac{2C_0}{T^3}+\frac{2biC}{T^3\theta^3}-\frac{2biC}{\theta^3T^3}e^{\theta(\lambda T-\mu)}+\frac{2b\lambda iC}{T^2\theta^2}e^{\theta(\lambda T-\mu)}-\frac{2b\mu iC\lambda^2}{T}e^{\theta(\lambda T-\mu)}-\frac{biC\lambda^2}{\theta T}e^{\theta(\lambda T-\mu)}\\
&+\frac{2aiC}{\theta^2T^3}e^{\theta(\lambda T-\mu)}-\frac{2aiC}{\theta^2T^3}-\frac{2aiC\lambda}{\theta T^2}+\frac{aiC\lambda^2}{T}e^{\theta(\lambda T-\mu)}-\frac{b\mu^2\lambda\theta iC}{T}e^{\theta(\lambda T-\mu)}\\
&-\frac{2b\mu iC\lambda}{T^2}+\frac{b\mu iC\theta\lambda^2}{T}e^{\theta(\lambda T-\mu)}+\frac{b\mu^2 iC}{T^3}+\frac{2b\mu iC}{T^3\theta^2}-\frac{4b\mu iC}{T^3\theta^2}e^{\theta(\lambda T-\mu)}+\frac{4b\mu\lambda iC}{\theta T^2}\\
&+b\mu iC\lambda^3(1+\theta)e^{\theta(\lambda T-\mu)}+\frac{2bC}{T^3\theta^2}e^{\theta(\lambda T-\mu)}+\frac{2bC\lambda}{\theta T^2}e^{\theta(\lambda T-\mu)}-\frac{bC\lambda^2}{T}e^{\theta(\lambda T-\mu)}\\
&+\frac{2aC}{\theta T^3}e^{\theta(\lambda T-\mu)}-\frac{2aC\lambda}{T^2}e^{\theta(\lambda T-\mu)}-\frac{aC\lambda^2}{T}e^{\theta(\lambda T-\mu)}+\frac{2aC}{\theta T^3}-\frac{2bC\mu^2}{\theta T^3}+\frac{2aC\mu}{T^3}-bC\lambda^3\\
&+bC\theta\lambda^3 e^{\theta(\lambda T-\mu)}\\
&=\frac{2C_0}{T^3}+\frac{2biC}{T^3\theta^3}-\frac{2biC}{\theta^3T^3}e^{\theta(\lambda T-\mu)}+\frac{2b\lambda iC}{T^2\theta^2}e^{\theta(\lambda T-\mu)}\left(1-T\mu\lambda\theta-\frac{1}{2}\lambda\theta T\right)\\
&+\frac{2biC}{T^3\theta^3}\left(e^{\theta(\lambda T-\mu)}-1-\lambda\theta T\right)+\frac{iC\lambda}{T}(a\lambda-b\mu^2\theta)+\frac{b\mu iC}{T^3}e^{\theta(\lambda T-\mu)}(1+\lambda-2\lambda T)\\
&+\frac{b\mu iC\theta\lambda^2}{T}e^{\theta(\lambda T-\mu)}+\frac{b\mu^2 iC}{T^3}+\frac{2b\mu iC}{T^3\theta^2}-\frac{4b\mu iC}{T^3\theta^2}e^{\theta(\lambda T-\mu)}+\frac{4b\mu\lambda iC}{\theta T^2}\\
&+bC\theta\lambda^3(1+\theta)e^{\theta(\lambda T-\mu)}+\frac{2bC}{T^3\theta^2}e^{\theta(\lambda T-\mu)}+\frac{2bC\lambda}{\theta T^2}e^{\theta(\lambda T-\mu)}\left(1-\frac{1}{2}\lambda T\theta\right)\\
&+\frac{2aC}{\theta T^3}e^{\theta(\lambda T-\mu)}(1-\lambda T\theta)+\frac{aC\lambda^2}{T}e^{\theta(\lambda T-\mu)}+\frac{2aC}{\theta T^3}(a-b\mu^2)\\
&+\frac{2C}{T^3}\left(a\mu-\frac{1}{2}b\lambda^3T^3\right)+bC\lambda^3\theta e^{\theta(\lambda T-\mu)} \qquad (24)
\end{aligned}
$$

4. Solution procedure

Algorithm

In order to find the optimal solutions, we propose the following algorithm.

Step1:Input the value of the parameters

Step2:Using Newton – Raphson Method determine the value of T from equation (23)

Step3:Compare T withμ , If $T > \mu$, go to step 4, otherwise T is infeasible

Step4: Compute the corresponding average total cost ATC (T), maximum inventory level S_1 , ordering quantity Q $t_1$from (10), (11) and (21) respectively.

5.Numerical Example

It is difficult to find closed form solution to (23), we then numerically determine optimal solution by Newton-Raphson method. Assume the inventory system represented has the following values of parameters: $C_O = \$80.0$, $i = \$0.50$, $C = 18.0$ / units $a = 20$ units, $b = 0.2$ $\mu = 0.4$, $\theta = 0.02$, $\lambda = 0.6$, $\delta = 0.4/unit$, $K = 0.04$

Applying the solution procedure described above the optimal values obtained is as follows. The optimal cycle length $T^* = 3.5865$ days using expression (23) which satisfies the sufficient condition $\frac{d^2\,ATC(T)}{dT^2} = 437.2564$. Using the value of T* in $t_1 = \lambda T$ we obtain $t_1^* = 2.1519$ day. The optimum ordering quantity, Q*=30.3785 units using expression (11). The maximum inventory level W^* =22.3785units are determined using the equation (10) and the corresponding average total cost is \$103.0963.

6. Sensitivity Analysis

We now study the effect of changes in the values of various parameters. The sensitivity analysis is performed by changing each parameter by -3% to 3%, taking one parameter at a time and keeping remaining parameters unchanged.

Table 1: Percentage change in the parameter values with respect to the decision variable.

Variation Parameters	Percentage change in Parameters	Change in decision Variables from -3% to 3%				
a		t_1^*	T^*	W^*	Q^*	ATC(T^*)
	-3	1.31	1.31	-0.95	-1.49	-2.03
	-2	0.87	0.87	-0.63	-0.99	-1.35
	-1	0.43	0.43	-0.32	-0.50	-0.68
	1	-0.43	-0.43	0.32	0.50	0.68
	2	-0.85	-0.85	0.65	1.00	1.35
	3	-1.27	-1.27	0.98	1.51	2.02
b	-3	-0.05	-0.05	-1.12	-0.82	0.01
	-2	-0.03	-0.03	-0.74	-0.55	0.00
	-1	-0.02	-0.01	-0.37	-0.27	0.00
	1	0.02	0.02	0.37	0.27	-0.00
	2	0.04	0.03	0.74	0.55	-0.01
	3	0.07	0.07	1.49	1.10	-0.01
θ	-3	-0.00	-0.00	1.04	0.77	0.00
	-2	-0.00	-0.00	0.69	0.51	0.00
	-1	-0.00	-0.00	0.34	0.25	0.00
	1	0.00	0.00	-0.33	-0.24	-0.00
	2	0.00	0.00	-0.66	-0.49	-0.00
	3	0.00	0.00	-0.98	-0.72	-0.00
C	-1	1.63	1.63	1.09	0.80	-2.55
	-2	1.08	1.08	0.72	0.53	-1.70
	-3	0.54	0.54	0.36	0.26	-0.85
	1	-0.53	-0.53	-0.35	-0.26	0.85
	2	-1.05	-1.05	-0.70	-0.52	1.70
	3	-1.56	-1.56	-1.04	-0.77	2.55
i	-3	1.85	1.85	1.24	0.91	-0.35
	-2	1.23	1.23	0.82	0.60	-0.23
	-1	0.61	0.61	0.41	0.30	-0.12
	1	-0.60	-0.60	-0.40	-0.30	0.12
	2	-1.19	-1.19	-0.80	-0.59	0.23
	3	-1.77	-1.77	-1.18	-0.87	0.35
δ	-3	1.31	1.31	0.88	0.65	-1.83
	-2	0.87	0.87	0.58	0.43	-1.22
	-1	0.22	0.22	0.14	0.11	-0.30
	1	-0.43	-0.43	-0.29	-0.21	0.61
	2	-0.85	-0.85	-0.57	-0.41	1.22
	3	-1.27	-1.27	-0.85	-0.62	1.83

7.0 Discussion of Sensitivity Analysis

a) With increase in parameter a (constant demand) there is decrease in t_1^*time at which the inventory level falls zero, replenishment cycle timeT^* and there is also corresponding increase in maximum inventory level W^*, maximum order quantity Q^*, and average total cost ATC(T^*) .This is clearly expected since increases order quantity Q^*will affect the depletion time and replenishment cycle time.

b) When the value of parameter b increases, there is corresponding increase in time at which the inventory level falls zero t_1^*, replenishment cycle timeT^*, the maximum inventory level W^*, maximum order quantity Q^*, and decrease in average total cost ATC (T^*). This is expected since higher values of b result in higher value of t_1^*, W^*,Q*, T*.

c) As deterioration rate increases, the maximum inventory level W^*,order quantity Q^* and average total cost ATC (T^*) decrease and t_1^*time at which the inventory level falls zero replenishment cycle timeT^*increases.

d) When the unit cost increases, there is corresponding increase total cost and the inventory depletion period t_1^*, cycle length (T*), the maximum inventory level W^* , the order quantity Q^* decreases.

e) When value of parameter i increases, there is corresponding increase total cost and the inventory depletion period t_1^*, cycle length (T*), the maximum inventory level W^* , the order quantity decreases.

f) Increase in the backlogging parameter result in decreases of order quantity Q*, maximum inventory level W^*, depletion period t_1^*, replenishment cycle time$T^*$with increases in average total cost ATC (T^*).

7. Conclusion

In this paper an inventory model is developed for deteriorating item with two phase demand rate. Demand rate is constant in the first part of the cycle and linear time dependent in the second part of the cycle. When new products such computers, televisions, android mobiles, and so on, are launched in the market, demand for them become constant for some time after that the demand increases due popularity of the product. Shortages are allowed and partially backlogged with constant rate. There could also be situations, in which an economic advantage may be gained by allowing shortages to occur. The optimal cycle time and optimal order quantity have been derived by minimizing the total average cost. A solution procedure is provided to illustrate the proposed model. The result is illustrated with numerical example, followed by sensitivity analysis.

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