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THE ANALYTICAL STUDY OF HIGHER ORDER ROGUE WAVE EQUATIONS

Ejinkonye I.O.

Department of Mathematics and Computer Science, Western Delta University, Oghara, Delta State, Nigeria.

Abstract

We consider the application of an analytic method based on the quasi-determinism theory for the space-time domain. In this paper we attempts to analyze the third order deterministic wave form. The analytical expressions of the third-order wave profile and velocity potential were obtained. Therefore the space-time evolution of a wave group, to the third-order in a Stokes expansion, when a very large crest occurs at a fixed time and location was investigated. This work will help to improve our understanding of the properties and dynamics for evolutions of the sea monster called rogue waves generally.

Keywords; Quasi-determinism, wave profile, velocity potential, rogue wave

1.0 Introduction

According to literature the most common definition of rogue waves are unusually large-amplitude waves that appear from nowhere in the open ocean. The evidence that such extremes can occur in nature have been provided, among others, by the Draupner and Andrea events, which have been extensively studied over the last decade [1–6]. Several physical mechanisms have been proposed to explain the occurrence of such waves [7]. From the observed evolution of rogue wave, we restrict our discussion to the case of unidirectional motion [8].

In this consideration, the initial state of wave development in deep water, short wave group are at the front of long wave group (wave packet). As time involves, the longer wave components with higher group velocity will overtake the shorter one. Consequently, the longer wave components will extract energy from the shorter components, thus, it will grow in size. The effectiveness of this mechanism had been justified not only by observation but by numerical simulation and by laboratory experimental modeling [8-10].

Boccoti [8-9] showed that in a Gaussian sea, if a high wave develops at some point in time and space; with a high probability, then a well defined quasi-deterministic wave group generates extreme wave profile (maximum or minimum structure in wave profile).

Boccoti [9] provided the quasi-determinism theory; this is a form of generalised Fourier series for wave group. Applying Stoke's expansion, the quasi-determinism wave group was extended to the second order in [11].

In this study, the theorem of quasi-determinism wave group will be extended to third order in Stoke's expansion. Numerical implications on wave crest height and trough depth will be better understood. This study will be concerned on the third order term of the same Stoke's expansion for wave profile and velocity potential. The wave steepness parameter will be introduced in the expansion. This is preferred because; steepness parameter enhances the convergence of the Stoke's expansion.

2 The basic idea of quasi-determinism theorem.

According to [12] quasi-determinism theory state that, a high wave crest with height H_c at the point x_o and at time t_o occurs with wave elevation $\eta(x_0, t_0)$. With the probability approaching unity, the subsequent elevation $\eta(x_0 + X, t_0 + T)$ as $H_c/\sigma \to \infty$ (σ = standard deviation calculated from discrete wave records of wave elevation time series) is obtained in [12] the deterministic form as

$$\overline{\eta}(x_0 + X, t_0 + T) = \frac{\psi(X, T, x_0)}{\psi(0,0,x_0)} H_c \qquad (1)$$

Where

$$\psi(X, T, x_0) = \frac{1}{T_c}\int_0^{T_c} \eta(x_0, t_0)\eta(x_0 + X, t_0 + T)dt_0 \qquad (2)$$

$\frac{H_c}{\sigma} \to \infty$ implies that in this study wave height H_c is large compared with mean wave height. Corresponding to equation (1), the deterministic velocity potential ϕ at depth z is given by

$$\psi(0,0,x_0) = \frac{1}{T_c}\int_0^{T_c} \eta^2(x_0, t_0)dt_0 \qquad (3)$$

Corresponding Author**:** Ejinkonye I.O., Email: ejinkonye.ifeoma@yahoo.com, Tel: +2348060297345

$$\bar{\phi}(x_0 + X, z, t_0 + T,) = \frac{\Phi(X, z, T; x_0)}{\psi(0,0; x)} H_c \qquad (4)$$

as $\frac{H_c}{\sigma} \to \infty$

$$\Phi(X, z, T, x_0) = \frac{1}{T_c} \int_0^{T_c} \eta(x_0, t)\phi(x_0 + X, z, t + T) dt_0 \qquad (5)$$

T_c = period of the dominant wave group forming the system

2.1 The integral representation of wave profile

If $E(\omega)$ is the frequency amplitude spectrum calculated from the wave record, H_c is the elevation of the high wave crest at $x = x_0$ and time $t = t_0$, then alternative representation of equation (1) in the form of half range Fourier transformation is given by

$$\eta(x,t) = \frac{H_c}{\sigma^2} \int_0^{\infty} E(\omega)\cos(kx - \omega t) d\omega \qquad (6)$$

$$E(\omega) = \frac{2}{\pi} \int_0^{\infty} \eta(t)\cos \omega t dt \qquad (7)$$

$at \quad fixed \quad po\text{int}$

$$\sigma^2 = \int_0^{\infty} E(\omega) d\omega \qquad (8)$$

similary as function of the frequency spectrum, the velocity potential at a fixed point (x, z) and time t is given by

$$\phi(x,z,t) = \frac{gH_c}{\sigma^2} \int_0^{\infty} \frac{E(\omega)}{\omega} e^{-kz} \sin(kx - \omega t) d\omega \qquad (9)$$

where k is the wave number, in deep water

$$k = \frac{\omega^2}{g} \qquad (10)$$

equation (10) is derived from the deep water form of dispersion relation $\omega^2 = kg$

3 Fourier Stokes expansion for wave group

This section of the work is on the generalize Fourier series expansion for wave group. It is in line with usually Stoke`s expansion.

To the second order in Stokes expansion, in [12] it was stated that the free surface displacement and the velocity potential for the long crested deep-water waves are given respectively as follows;

$$\eta(x,t) = \eta_1 + \lambda\eta_2 = \sum_{n=1}^{N} a_n \cos\psi_n + \frac{\lambda}{4}\sum_{n=1}^{N}\sum_{m=1}^{M} a_n a_m \{(k_n + k_m)\cos(\psi_n + \psi_m) + |k_n - k_m|\cos(\psi_n - \psi_m)\} \qquad (11)$$

$$\phi(x,t) = \phi_1 + \lambda\phi_2 = g\sum_{n=1}^{N} a_n \omega_n^{-1} \exp(k_n z)\sin\psi_n +$$
$$\frac{\lambda}{4}\sum_{n=1}^{N}\sum_{m=1}^{M} a_n a_m \omega_n \exp[z|k_n - k_m|][(k_n + k_m)\sin(\psi_n + \psi_m) + |k_n - k_m|\sin(\psi_n - \psi_m)] \qquad (12)$$

$$where \ \ \psi_n = k_n x - \omega_n t + \varepsilon_n \qquad (13)$$

the coefficients a_m, a_n are amplitudes for the n_{th} and m_{th} mode components, k_n is the wave number for the n_{th} mode and in deep water $k = \frac{\omega^2}{g}, \lambda$ is the wave steepness parameter,

ε_n is phase angle for the n_{th} mode, ω_n is the frequency for the n_{th} mode

3.1 The higher order expansion

Our interest in this investigation is to extend this expansion to higher order. Firstly we start with third order term of this Stoke's expansion and our aim is to study its effects in relation to observed wave height for rogue wave event. The deduction of third order term is not the interesting part of it but its geophysical impart it will made on the development of rogue wave phenomenon. The third order free surface displacement and the velocity potential in the event of the long crested deep water waves group is given respectively as

$$\eta(x,t) = \eta_1 + \lambda\eta_2 + \lambda^2\eta_3 + O(\lambda^3) =$$
$$\sum_{n=1}^{N} a_n \cos\psi_n + \frac{\lambda}{4}\sum_{n=1}^{N}\sum_{m=1}^{M} a_n a_m \{(k_n + k_m)\cos(\psi_n + \psi_m) + |k_n - k_m|\cos(\psi_n - \psi_m)\} +$$
$$\frac{\lambda^2}{8}\sum_{n=1}^{N}\sum_{m=1}^{M} a_n^2 a_m \{(2k_n + k_m)\cos(2\psi_n + \psi_m) + |2k_n - k_m|\cos(2\psi_n - \psi_m)\} + O(\lambda^3) = h_0 + h_1 + h_2 + h_3 \qquad (14)$$

$$\phi(x,t) = \phi_1 + \lambda\phi_2 + \lambda^2\phi_3 + O(\lambda^3) = g\sum_{n=1}^{N} a_n \omega_n^{-1} \exp(k_n z)\sin\omega_n + \frac{\lambda}{4}\sum_{n=1}^{N}\sum_{m=1}^{M} a_n a_m \omega_n \exp[z(k_n - k_m)]\sin(\psi_n - \psi_m) +$$
$$\frac{\lambda^2}{8}\sum_{n=1}^{N}\sum_{m=1}^{M} a_n^2 a_m \omega_m^2 \exp[z|2k_n - k_m|][(2k_n + k_m)\sin(2\psi_n + \psi_m) + |2k_n - k_m|\sin(2\psi_n - \psi_m)] + O(\lambda^3) \qquad (15)$$

Equations (14) and (15) above model asymptotically an evolution of wave group event. The third order term in the expansion is expected to involve three summation conventions. However, it is assumed that in the evolution of wave group, some of the modes are identical with same amplitude components, wave numbers and frequencies. In this case, only two summation conventions are required and this makes analytical manipulation quite clearer and less complicated.

3.2 The Coefficient In The Third Order Expansion

We take $\frac{\lambda^2}{8}a_n^2 a_m = R_{nm} + \overline{R}_{nm}$ being coefficient of third order expansion

Where $R_{nm} = \frac{10\cos L(3k_0 + 2k_n + k_m)}{(3k_0 + 2k_n + k_m)(3\omega_0 + 2\omega_n + \omega_m)(2k_n + k_m)}$

$\overline{R}_{nm} = \frac{10\cos L(3k_0 - 2k_n + k_m)}{(3k_0 - 2k_n + k_m)(3\omega_0 - 2\omega_n + \omega_m)(2k_n - k_m)}$ $\quad$ $where \ k_n = \frac{(n+1)\pi}{L}, k_m = \frac{(m+1)\pi}{L}, k_0 = \frac{\pi}{L}$

$R_{00} = \frac{5\cos 6k_0 L}{54k_0^2\omega_0}$ $\quad$ $\overline{R}_{00} = \frac{5\cos 2k_0 L}{4k_0^2\omega_0}$

$R_{10} = \frac{10\cos L(4k_0 + 2k_1)}{(4k_0 + k_1)(4\omega_0 + 2\omega_1)(2k_1 + k_0)}$ $\quad$ $\overline{R}_{10} = \frac{10\cos L(4k_0 - 2k_1)}{(2k_1 - k_0)(4\omega_0 - 2\omega_1)(4k_0 - 2k_1)}$

$R_{01} = \frac{10\cos L(5k_0 + k_1)}{(5k_0 + k_1)(5\omega_0 + \omega_1)(2k_0 + k_1)}$ $\quad$ $\overline{R}_{01} = \frac{10\cos L(k_0 + k_1)}{(2k_0 - k_1)(\omega_0 + \omega_1)(k_0 - k_1)}$

$R_{11} = \frac{10\cos L(k_0 + k_1)}{9k_1(k_0 + k_1)(\omega_0 + \omega_1)}$ $\quad$ $\overline{R}_{11} = \frac{10\cos L(3k_0 - k_1)}{(k_1)(3\omega_0 - \omega_1)(3k_0 - k_1)}$

$R_{12} = \frac{10\cos L(3k_0 + 2k_1 + k_2)}{(3k_0 + 2k_1 + k_2)(3\omega_0 + 2\omega_1 + \omega_2)(2k_1 + k_2)}$ $\quad$ $\overline{R}_{12} = \frac{10\cos L(k_0 - 2k_1 + k_2)}{(3k_0 - 2k_1 + k_2)(3\omega_0 - 2\omega_1 + \omega_2)(2k_1 - k_2)}$

$R_{21} = \frac{10\cos L(3k_0 + 2k_2 + k_1)}{(3k_0 + 2k_2 + k_1)(3\omega_0 + 2\omega_2 + \omega_1)(2k_2 + k_1)}$ $\quad$ $\overline{R}_{21} = \frac{10\cos L(k_0 - 2k_2 + k_1)}{(3k_0 - 2k_2 + k_1)(3\omega_0 - 2\omega_2 + \omega_1)(2k_2 - k_1)}$

$R_{22} = \frac{10\cos L(k_0 + k_2)}{9k_2(k_0 + k_2)(\omega_0 + \omega_2)}$ $\quad$ $\overline{R}_{22} = \frac{10\cos L(3k_0 - k_2)}{k_2(3\omega_0 - \omega_2)(3k_0 - k_2)}$

4.0 Extremization problem for third order term

The problem to second order was developed in [10] using equations (11) and (12). They assumed that the free surface displacement has an absolute maximum h at point $x = x_0$ and that this maximum occurs at time $t = t_0$ in [12]. In the further study in [11], attempt was made to establish the existence of absolute extreme crest for the profile of the rogue wave event. However, observed evolutional structure of rogue wave event suggests succession of crests and troughs. Thus, establishment of relative extreme appears to be more realistic exercise geophysical.

We now extend this development to third order term and in space-time domain. We now state the deterministic free surface displacement $\eta_3(x,t)$ at point $x_0 + X$ at time $t_0 + T$, when h is large in relation to the standard deviation σ of wave record.

$$\eta_3(x,t) = \frac{\lambda^2}{8}\sum_{n=1}^{N}\sum_{m=1}^{M} a_n^2 a_m \{(2k_n + k_m)\cos(2\psi_n + \psi_m) + |2k_n - k_m|\cos(2\psi_n - \psi_m)\} \qquad (14)$$

Extermization of $\eta_3(x,t)$ using equation (14) gives the following relations, for the local maximum at time $t = 0$ and the point $x = 0$.

Applying the function of two variables x,t to the method of extermization we have

$$when \ \psi_n = \psi_m = 0 \qquad (15)$$

$$\eta_3(0,0) = \frac{\lambda^2}{8}\sum_{nm}^{N} a_n^2 a_m [(2k_n + k_m) + |2k_n - k_m|] \qquad (16)$$

$$\frac{\partial \eta_3(x,t)}{\partial x} = 0 \quad when \quad \psi_n = \psi_m = 0 \qquad (17)$$

$$\frac{\partial \eta_3(x,t)}{\partial t} = 0 \quad when \ \psi_n = \psi_m = 0 \qquad (18)$$

equations (17) and (18) are the necessary condition for the development of relative extreme profile.

The sufficient conditions for the local high crest elevation are given as

$$\frac{\partial^2 \eta_3(x,t)}{\partial x^2} = -\frac{\lambda^2}{8}\sum_{n=1}^{N}\sum_{m=1}^{M} a_n^2 a_m \left[(2k_n + k_m)^3 + |2k_n - k_m|^3\right] < 0 \quad 2k_n > k_m \qquad (19)$$

$$\frac{\partial^2 \eta_3(x,t)}{\partial t^2} = -\frac{\lambda^2}{8}\sum_{n=1}^{N}\sum_{m=1}^{M} a_n^2 a_m \left[(2k_n + k_m)(2\omega_n + \omega_m)^2 + (2k_n - k_m)(2\omega_n - \omega_m)^2\right] < 0 \qquad (20)$$

$when \ x = t = 0$

and finally,

$$\left(\frac{\partial^2 \eta}{\partial x \partial t}\right)^2 - \frac{\partial^2 \eta}{\partial x^2}\frac{\partial^2 \eta}{\partial t^2} < 0 \qquad (21)$$

$when \ x = t = 0$

for the $(n,m)^{th}$ mode components and from equation (21) gives

$$-12k_n^2 w_n^2 - \left[4k_n^2(w_n^2 + 4w_n w_m) + 3k_m^2(2w_n^2 + w_m^2)\right] < 0 \qquad (22)$$

In the theory for the extreme values of the function of two variables $\eta(x,t)$ equation (17), (18), (19), (20) and (21) are the necessary and sufficient for the existence of maximum value of $\eta(x,t) \ at \ x = 0, t = 0$. For the minimum value corresponding to the trough, the phase angle is to be increased or decreased by $\pi \ for \cos\pi = -1$, with exception of equation (21), the signs of other inequalities will change correspondingly. Thus, it is deduced that the peak value of both first, second and third order crest heights correspond when $t = x = 0$. This conclusion appears to suggest strongly that the interactions among the quasi-monochromatic group are quite constructive. This development is obviously conducive to enhanced extremization in wave profile.

4.2 Third order integral representation.

Then, we deduce the following representations, $s_\omega[a_n^2]=E_1(2\omega_1)$, $s_\omega[a_m]=E_2(\omega_2)$; where $s_w[\ ]$ is frequency spectral, then the following third order continuous representation apply;

$$\eta_3=\frac{H_c^3}{8\sigma^6}\int_0^\infty\int_0^\infty E_1(2\omega_1)E_2(\omega_2)[(2k_1+k_2)+|2k_2-k_1|]d\omega_1 d\omega_2. \quad (23)$$

$$a_n=\frac{h}{\sigma^2}E(\omega)d\omega \quad (24)$$

η_3 is the third order term for observed wave elevation.

The free surface wave elevation when exceptionally high wave crest occurs at (x_0,t_0) is generally represented in the form.

$$\eta(x,t)=\eta_1+\lambda\eta_2+\lambda^2\frac{H_c^3}{8\sigma^6}\cdot$$

$$\int_0^\infty\int_0^\infty E_1(2\omega_1)E_2(\omega_2)[(2k_1+k_2)\cos(2\psi_1+\psi_2)+|2k_1-k_2|\cos(2\psi_1-\psi_2)]d\omega_1 d\omega_2 \quad (25)$$

equation (25) is Stoke's expansion for wave group in [9] but now extended to third order using the method in [13]in this study.

Where $\eta_1(x,t)$ is stated in equation (6) and

$$\eta_2(x,t)=\frac{H_c^2}{2\sigma^2}\int_0^\infty\int_0^\infty E_1(\omega_1)E_2(\omega_2)[(k_1+k_2)\cos(\psi_1+\psi_2)+|k_2-k_1|\cos(\psi_1-\psi_2)]d\omega_1 d\omega_2 \quad (26)$$

see (11)

$$\sigma^2=\int_0^\infty E(\omega)d\omega \quad (27)$$

σ^2 is still the variance calculated from $\eta_n(t)$ in time domain. Similarly the third order velocity potential in continuous from is derived as

$$\phi_3(X,T;z)=\frac{H_c^3}{\sigma^6}\int_0^\infty\int_0^\infty E_1(2\omega_1)E_2(\omega_2)[\exp|2k_1-k_2|z](2k_1+k_2)\sin(2\psi_1+\psi_2)+$$

$$+|2k_2-k_1|\sin(2\psi_1-\psi_2)d\omega_1 d\omega_2 \quad (28)$$

where k_1 and k_2 are parameters identified with wave numbers of the dominant wave group.

$\phi_2(X,T;z)$ was derived by (12) and stated as;

$$\phi_2(X,T;z)=\frac{H_c^2}{\sigma^4}\int_0^\infty\int_0^\infty E_1(\omega_1)E_2(\omega_2)[\exp|k_1-k_2|z](k_1+k_2)\sin(\psi_1+\psi_2)+$$

$$|k_1-k_2|\sin(\psi_1-\psi_2)d\omega_1 d\omega_2 \quad (29)$$

and thus,

$$\phi(X,T;z)=\phi_1(X,T;z)+\lambda\phi_2(X,T;z)+\lambda^2\phi_3(X,T;z)+O(\lambda^3) \quad (30)$$

4.3 **The third order particle velocity components**

The third order particle velocity components in the direction of x and z axis are U_3 and W_3 respectively and are calculated from equation (28) to give

$$U_3=\frac{H_c}{\sigma^6}\int_0^\infty\int_0^\infty E_1(2\omega_1)E_2(\omega_2)\omega[\exp-(2k_1-k_2)z](2k_1+k_2)\cos(2\psi_1+\psi_2)+$$

$$+|2k_1-k_2|^2\cos(2\psi_1-\psi_2)d\omega_1 d\omega_2 \quad (31)$$

$$W_3=\frac{H_c}{\sigma^6}\int_0^\infty\int_0^\infty E_1(2\omega_1)E_2(\omega_2)(2k_1-k_2)[\exp-(2k_1-k_2)z](2k_1+k_2)\sin(2\psi_1+\psi_2)+$$

$$+(2k_1-k_2)^2\sin(2\psi_1-\psi_2)d\omega_1 d\omega_2 \quad (32)$$

$k_1>k_2$; *thus* $2k_1-k_2>0,\ z>0$

Thus, third order velocity components decays with depth more rapidly than first and second order terms of particle velocity components. Again, like second order third order particle orbits are not closed.

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