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MATHEMATICAL MODEL OF AQUEOUS HUMOR FLOW THROUGH A CONSTRICTED CHANNEL

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Abstract

Rise in Intraocular Pressure (IOP) in human eye is pointed out as a major cause of Glaucoma. Despite this knowledge, the site where this increase occurs has not been definitely identified or stated. The junction between the endothelia-cell wall of the Schlemn's canal and the juxtacanalicular tissue is known to contain a funnel shaped pores. The flow of the AH through this point generates a funneling effect on the TM. This study is to investigate this effect on the flow resistance and the intraocular pressure (IOP) respectively considering different flow parameters.

Keywords: Constriction; Intraocular Pressure; Resistance; Slip Parameter; Stenosis.

1.0 Introduction

The aqueous humor (AH) is a fluid in the eye whose production is at the ciliary body (epithelium) [1]. This fluid among many of its function delivers nutrients to the avascular tissues, washes the lens, and affects the pressure in the eye. The aqueous humor has a route known as conventional aqueous outflow pathway which comprises of the trabecular meshwork (TM), the juxtacanalicular tissue (JCT), the Schlemm's canal (SC) and the endothelial lining of Schlemm's canal (SC) [2, 3]. The AH flow is also controlled by TM outflow facility [4]. See [5] for the structure of the TM and the SC where aqueous humor drift from the production site across the TM into the SC is described. This summarily describes the flow dynamics of aqueous humor in the conventional pathway. There are many more detailed studies of the aqueous humor and its flow dynamics [6, 7, 8] and AH flow in Schlemm's canal [9]. Changes in aqueous humor flow resistance affects intraocular pressure whose increase develops into glaucoma [10]. The present study is driven by the report of [2]. Some initial mathematical works on AH flow dynamics and its effect on the IOP in the eye has been carried out as well. These works include [11] who carried out a study of flow of aqueous humor in Schlemm's canal in the eye. They treated the canal segment along which the aqueous humor flows as an elliptic narrow tube. Further, they considered that the inner endothelial wall in contacts with TM is porous and collapsible, and that the inner wall of the SC is deformable. By considering that the same amount of aqueous humor drains through each collector channel is somehow unrealistic. In [12] a similar study on the SC was also carried out to determine its influence on Primary open angle glaucoma (POAG). In this study he considered aqueous humor flow from the anterior chamber across the TM into the SC. This he did using lubrication theory. After his study of the flow dynamics of the AH through the iris – lens channel [13] agrees that there is a pressure difference between the posterior and anterior chambers. See [11, 3] for other works on this subject.

Despite these works and their wonderful contribution to the study of AH flow and its pathway dynamics, no work has considered the TM or the SC as a channel with constriction. It is admitted in [2] that there is a "hydrodynamic interactions" existing between the walls of the canal in contact with the JCT of the TM which creates a funneling effect on the aqueous flow. This is due to the narrowing of the collector channels which is smaller in diameter than the canal segment of the flow. Though the major drainage resistance is produced near or in the wall of SC, it is believed that these funneling effect caused by the hydrodynamic interaction between the TM and SC regulate the resistance of AH flow [2]. To the best of our knowledge, no such work has been carried out on the AH flow Dynamics. Therefore we here try to investigate aqueous humor flow through the TM and SC as flow through a channel with constrictions. In the light of this, the equation governing the flow of aqueous humor in the Schlemms canal is given by

$$\rho \frac{D\bar{U}}{Dt} = -\bar{\nabla} P + \mu \bar{\nabla}^2 U + \bar{f}$$

where $\bar{U} = (u_1, u_2, u_3)$ is the velocity of the flow, $\frac{D\bar{U}}{Dt}$ is the instantaneous accelerations, $\bar{\nabla} P$ is the dynamic pressure gradient, i.e. the pressure drop across the channel length of flow and $\bar{f}$ is the external body force. Since there is no external body, $\bar{f} = 0$.

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(The aqueous flow is driven by the pressure drop in the channel, which is the difference between the flow at the entrance and exit of the channel). We shall assume that $(u_1, u_2, u_3) = (u_1, 0, 0)$, that is to say that there is only flow along the channel. Because of the funneling effect on aqueous flow caused by the JCT on the inner wall of the SC, the dynamics of the flow is considered as flow with constriction.

2. DEFINITION OF PROBLEM

Consider the junction between the JCT and the inner wall of SC as a cylindrical tube or channel with stenosis, which is mild and symmetric and define the flow geometry of the channel as follows:

$$\frac{R_\delta(z)}{R(z)} = \begin{cases} 1 - \frac{\delta}{2R}(z-d); & d \le z \le d + l/2 \\ 1 - \frac{\delta}{2R}\left[1 + \cos\frac{2\pi}{l}\left(z - d - \frac{l}{2}\right)\right]; & d + \frac{l}{2} \le z \le d + l \\ 1; & \text{otherwise} \end{cases}$$

The flow channel is taken as shown below.

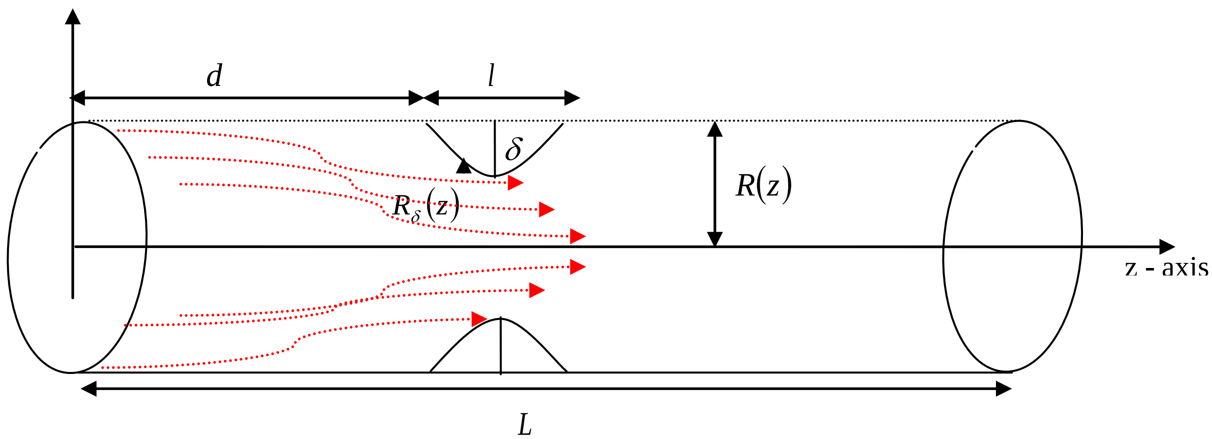

Figure 1: The flow Channel

2.1 MODEL EQUATIONS.

Consider that the aqueous humor is flowing through a composite stenosis in a channel of length L in such a way that z-axis is taken to be along (parallel to) the axis of the canal in the flow direction. The coordinate is cylindrical with velocity $\overline{U}(r, \theta, z)$, where r is the radial component and θ is the circumferential component of the directions respectively. We note here that the circumferential component of the velocity is zero in this case. The point $r = 0$ coincides with the axis of the symmetry of the channel. The stenosis is assumed mild, symmetric and in the arterial segment of the channel (canal). $R_\delta(z)$ and $R(z)$ are the radius of the channel with and without stenosis, L is the length of the channel, δ is the height of the stenosis at $z = \frac{l}{2}$ and l is the length of the stenosis where d locates the stenosis in the channel [14].

With these, the governing equations described above then reduces to

$$\frac{\partial p}{\partial z} = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{1}$$

The boundary conditions are as follows:

$$u = 0 \text{ at } r = R \tag{2}$$

$$\frac{\partial u}{\partial r} = u_\delta \text{ at } r = R_\delta \tag{3}$$

$$u = -\frac{R_\delta\sqrt{D_a}}{\alpha}u_\delta \text{ at } r = R_\delta. \tag{4}$$

3. ANALYSIS

Using the boundary conditions (2) to (3) and in line with [15(13)], the solution (i.e. the velocity) to (1) is

$$u(r) = -\frac{p_z}{4\mu}\left\{R^2 - r^2 + \frac{\left(R_\delta{}^2 - R^2\right)}{\ln\left(\frac{R_\delta}{R}\right)}\ln\left(\frac{r}{R}\right) + \frac{R_\delta\sqrt{D_a}}{\ln\left(\frac{R_\delta}{R}\right)}\ln\left(\frac{r}{R}\right)\left[\frac{2R_\delta{}^2\ln\left(\frac{R_\delta}{R}\right) - \left(R_\delta{}^2 - R^2\right)}{\alpha R_\delta{}^2\ln\left(\frac{R_\delta}{R}\right) + R\sqrt{D_a}}\right]\right\}$$

$$= -\frac{p_z}{4\mu}\left\{R^2 - r^2 + \frac{\left(R_\delta{}^2 - R^2\right)}{\ln\left(\frac{R_\delta}{R}\right)}\ln\left(\frac{r}{R}\right) + \frac{\beta R_\delta\sqrt{D_a}}{\ln\left(\frac{R_\delta}{R}\right)}\ln\left(\frac{r}{R}\right)\right\} \tag{5}$$

where $\beta = \dfrac{2R_\delta{}^2\ln\left(\frac{R_\delta}{R}\right) - \left(R_\delta{}^2 - R^2\right)}{\alpha R_\delta{}^2\ln\left(\frac{R_\delta}{R}\right) + R\sqrt{D_a}}$.

Equation (5) gives the velocity of the flow in the canal.
The flux of the flow, Q, of the aqueous humor in the channel described above is given in [16] as

$$Q = 2\pi\int_0^{R_\delta} ur\,dr \tag{6}$$

The use of (5) in (6) obtains, after simplifying

$$Q = -\frac{\pi}{2\mu}\frac{\partial p}{\partial z}\int_0^{R_\delta}\left[rR^2 - r^3 + \frac{\left(R_\delta{}^2 - R^2\right)}{\ln\left(\frac{R_\delta}{R}\right)}r\ln\left(\frac{r}{R}\right) + \frac{\beta R_\delta\sqrt{D_a}}{\ln\left(\frac{R_\delta}{R}\right)}r\ln\left(\frac{r}{R}\right)\right]dz$$

$$= -\frac{\pi}{2\mu}\frac{\partial p}{\partial z}\left[\frac{r^2R^2}{2} - \frac{r^4}{4} + \frac{\left(R_\delta{}^2 - R^2\right)}{\ln\left(\frac{R_\delta}{R}\right)}\left(\frac{r^2\ln\left(\frac{r}{R}\right)}{2} - \frac{r^2}{4}\right) + \frac{\beta R_\delta\sqrt{D_a}}{\ln\left(\frac{R_\delta}{R}\right)}\left(\frac{r^2\ln\left(\frac{r}{R}\right)}{2} - \frac{r^2}{4}\right)\right]_0^{R_\delta}$$

$$= -\frac{\pi}{2\mu}\frac{\partial p}{\partial z}\left[\frac{R_\delta{}^2R^2}{2} - \frac{R_\delta{}^4}{4} + \left(\frac{\left(R_\delta{}^2 - R^2\right) + \beta R_\delta\sqrt{D_a}}{\ln\left(\frac{R_\delta}{R}\right)}\right)\left(\frac{R_\delta{}^2\ln\left(\frac{R_\delta}{R}\right)}{2} - \frac{R_\delta{}^2}{4}\right)\right]$$

$$= -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial z}\left\{2\left(\frac{R_\delta}{R}\right)^2 - \left(\frac{R_\delta}{R}\right)^4 + \left[\left(\frac{R_\delta}{R}\right)^4 - \left(\frac{R_\delta}{R}\right)^2 + \beta\left(\frac{R_\delta}{R}\right)^2\frac{\sqrt{D_a}}{R_\delta}\right]\left[2 - 1\Big/\ln\left(\frac{R_\delta}{R}\right)\right]\right\}$$

$$= -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial z}\left\{\left(\frac{R_\delta}{R}\right)^4\left[1 - \frac{1}{\ln\left(\frac{R_\delta}{R}\right)}\right] + \left(\frac{R_\delta}{R}\right)^2\left[\frac{1}{\ln\left(\frac{R_\delta}{R}\right)}\left(1 - \frac{\beta\sqrt{D_a}}{R_\delta}\right) + \frac{2\beta\sqrt{D_a}}{R_\delta}\right]\right\} \tag{7}$$

From (7) one obtains the pressure gradient for the flow as

$$\frac{\partial p}{\partial z} = \frac{8\mu Q}{\pi R_\delta{}^4}Y(z) \tag{8}$$

where $Y(z) = \dfrac{1}{H(z)}$ and

$$H(z) = \left(\frac{R_\delta}{R}\right)^4\left[1 - \frac{1}{\ln\left(\frac{R_\delta}{R}\right)}\right] + \left(\frac{R_\delta}{R}\right)^2\left[\frac{1}{\ln\left(\frac{R_\delta}{R}\right)}\left(1 - \frac{\beta\sqrt{D_a}}{R_\delta}\right) + \frac{2\beta\sqrt{D_a}}{R_\delta}\right]$$

To get the total pressure drop within the channel of flow, we integrate (8) as follows;

$$\Delta p = \int_0^L\left(\frac{\partial p}{\partial z}\right)dz = \frac{8\mu Q}{\pi R_\delta{}^4}\Theta \tag{9}$$

where $\Theta = \int_0^d Y(z)dz + \int_d^{d+l} Y(z)dz + \int_{d+l}^L Y(z)dz$.

While the second integral may not be easy to evaluate analytically (and shall be evaluated numerically), the first and the third can be evaluated to give

$$\frac{8\mu Q}{\pi R_\delta^{\ 4}}\left(\frac{L-l}{H}\right),$$

We obtained the dimensionless form of resistance using the argument of [16] as

$$\hat{\lambda} = \frac{8\mu}{\pi R_\delta^{\ 4}}\left(\frac{1-l/L}{\Pi} + \frac{1}{L}\left(\int_{d}^{d+l/2} Y(z)dz + \int_{d+l/2}^{d+l} Y(z)dz\right)\right), \tag{10}$$

where $\Pi = 2\left(1 + \frac{1}{\alpha + \sqrt{D_a}}\right)$, $\hat{\lambda} = \frac{\lambda}{\lambda_0}$ and $\lambda_0 = \frac{8\mu L}{\pi \Pi R_\delta^{\ 4}}$ is the resistance for a non-stenosed artery. We note here that the integrands in (10) give the pressure drop across the stenosis region. Another way to do this is to use the ratio of flow rate (generated by this pressure drop within the stenotic region) to the area of the stenotic region [2]. That is $p = \frac{Q}{A_\delta}$, where A_δ is the area of the stenotic region. However a measure of the flux within the stenotic region may not be easy to achieve. Despite that we shall not consider the stresses of the flow on the stenotic and non-stenotic region we have them in their dimensionless form as given below

$$\tau = -\mu \left.\frac{\partial u}{\partial r}\right|_{r = R_\delta}, \tag{11}$$

and that of the stress from non-stenotic region is

$$\tau = -\mu \left.\frac{\partial u}{\partial r}\right|_{r = R} \tag{12}$$

Furthermore, the pressure of the aqueous flow is calculated using

$$P = \frac{Q}{A} = \frac{Q}{\pi r^2}, \tag{13}$$

so that

$$P = \frac{8\mu}{R_\delta^{\ 2}}\left(\frac{L-l}{\Pi} + \int_{0}^{d+l/2} Y(z)dz + \int_{d+l/2}^{d+l} Y(z)dz\right) \tag{14}$$

4. NUMERICAL RESULTS AND DISCUSSIONS

The funneling effect of the region between the endothelia-cell wall of the Schlemm's canal and the JCT of the TM on the intraocular pressure is been pointed out as a major contributory factor to the IOP in the eye. Due to the resultant effect of increase in IOP we simulated results quantitatively to see clearer the effect of some of the flow parameters on the resistance and pressure against the stenosis height respectively. Thus, Fig 2 and 3 shows the effect of stenosis length on the resistance and pressure respectively versus the stenosis height. From the figure we see that both pressure and resistance increase with the stenosis height. Fig 3 in particular shows that for stenosis with length less than the channel length, the flow is almost steady which reflects in the resistance and pressure; but heights more than half of the channel length yields a more mobile flow as seen in their effect on the pressure and resistance.

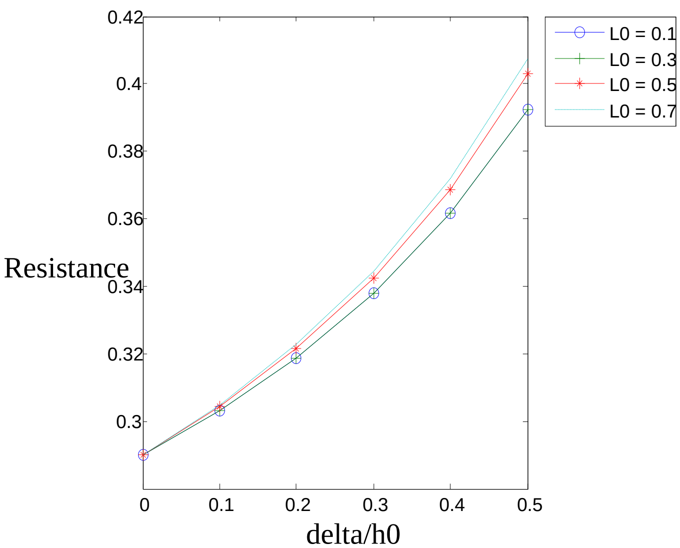

Fig 2: Effect of stenosis length on the flow resistance

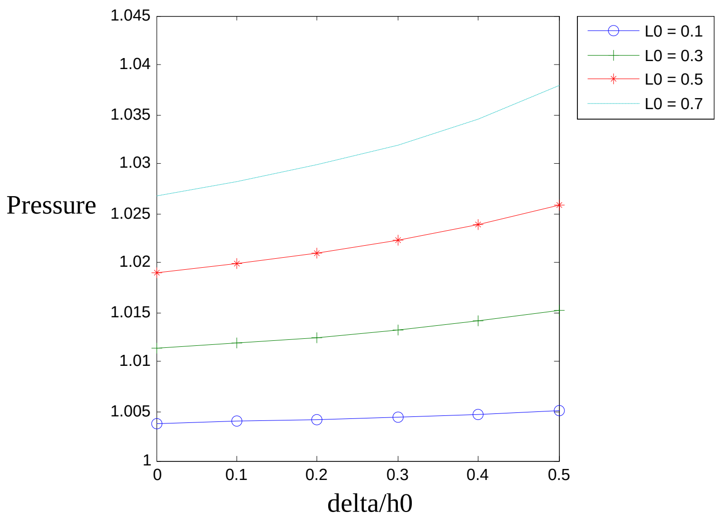

Fig 3: Effect of stenosis length on the flow pressure

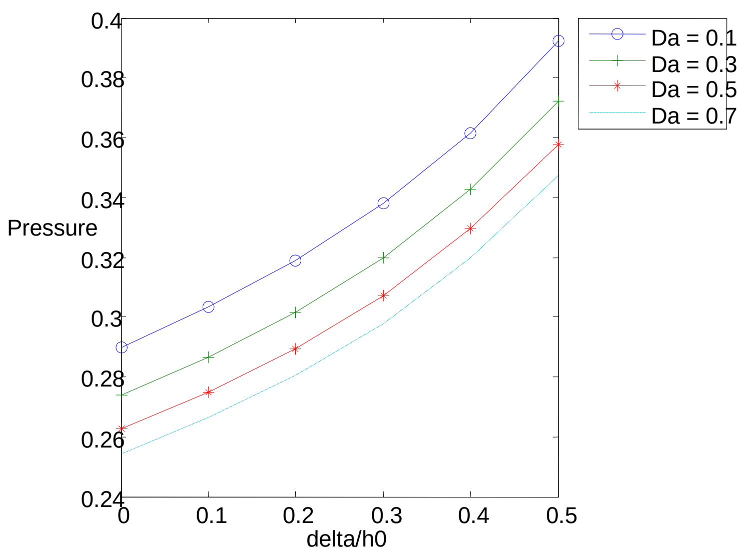

Fig 4: Effect of Darcy's number on the flow resistance

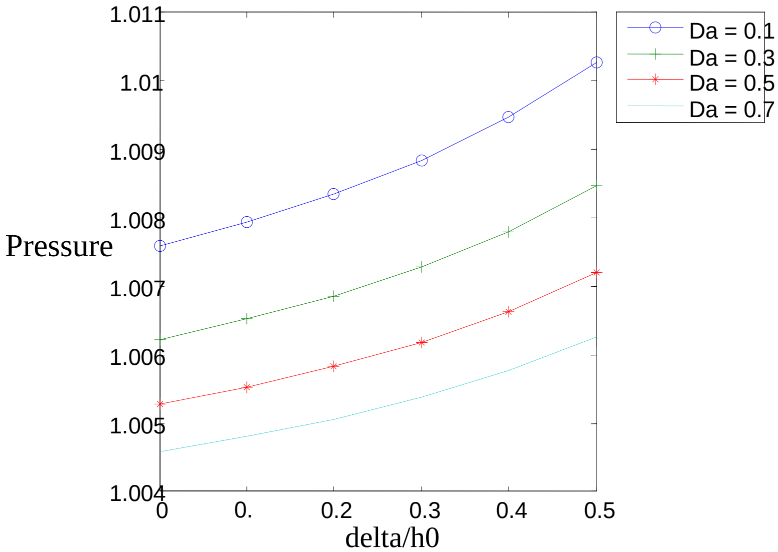

Fig 5: Effect of Darcy's number on the flow pressure

The effect of Darcy's number on the flow resistance and pressure is shown in figs 4 and 5 respectively. In these figs, we see that the lower the Darcy number the higher the resistance and pressure of flow. This is as expected since Darcy's number quantifies the relationship between the flow resistance and the specific hydraulic conductivity of the tissues involved in the channel. Since the hydraulic conductivity is a measure of ease with which a fluid passes through a tissue the pressure and resistance are inversely related to it.

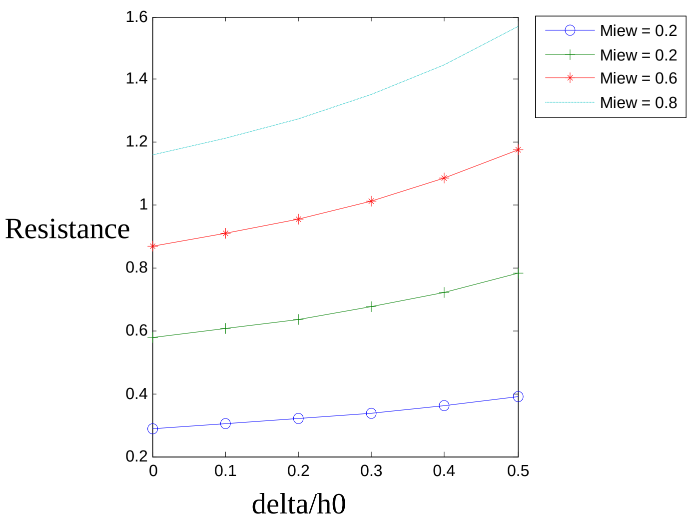

Fig 6: Effect of viscosity on the flow resistance

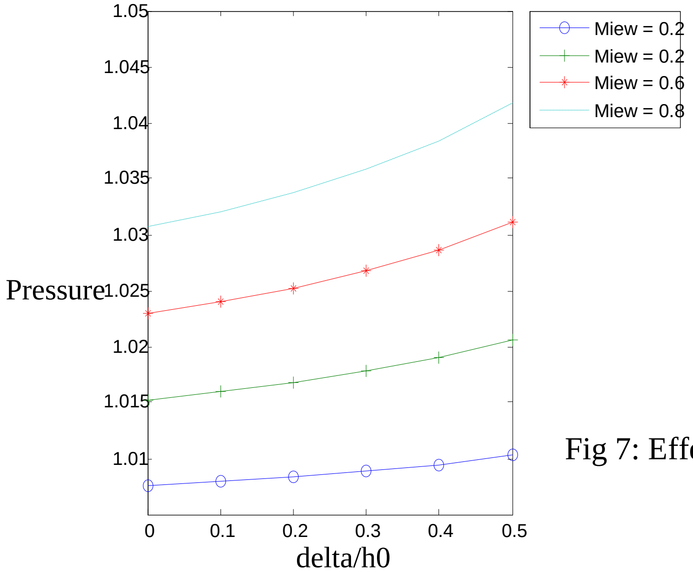

Fig 7: Effect of viscosity on the flow pressure

Fig 6 and 7 shows the effect of AH viscosity on the resistance and pressure. From these we see that both resistance and pressure vary directly with viscosity, a measure of resistance of fluid to flow due to frictional force existing between the particles of the fluid.

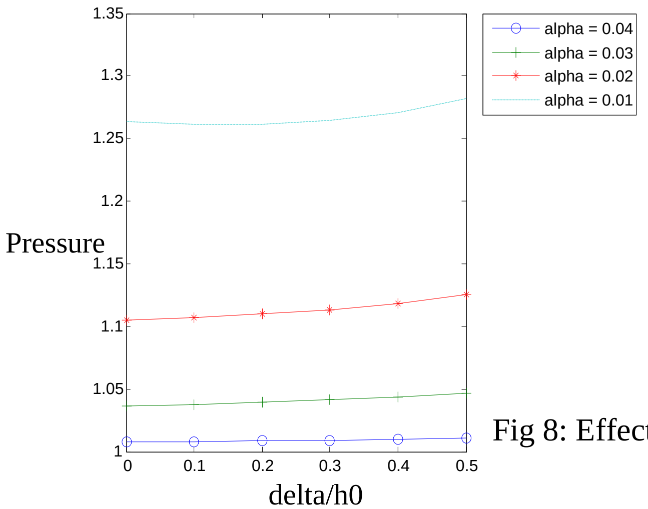

Fig 8: Effect of slip parameter on the flow pressure

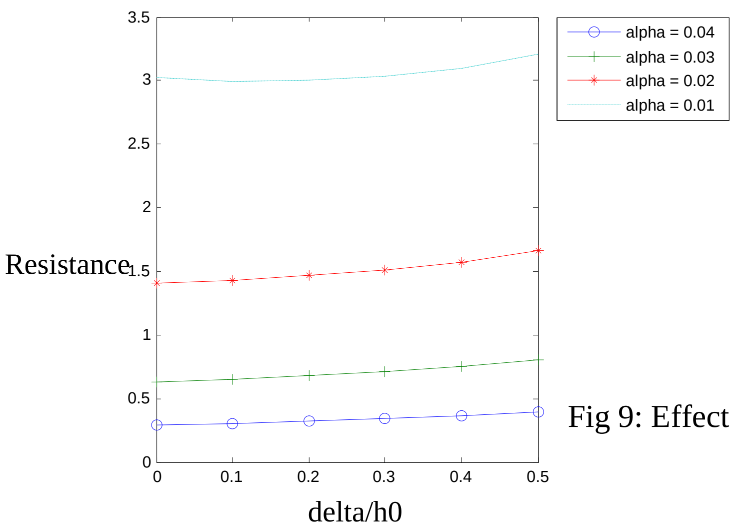

Fig 9: Effect of slip parameter on the flow resistance

From figure 9 it can be seen that the flow resistance increases with a decrease in the slip parameter showing that the funneling effect of the channel has a great influence on the IOP (as shown in fig 8) because the slip parameter measures the steepness or ease with which the AH flows in and out of the stenosis region. This agrees with [2] and the results of previous reports on constricted flow.

5 CONCLUSION

This study shows that the funneling effect existing between the funneling effect of the region between the endothelia-cell wall of the Schlemm's canal and the JCT of the TM plays a significant role in the IOP of the human eye. The result of the simulations shows that trying to reduce the stenosis height by increasing the length could reduce the risk of IOP. Despite this result, the authors worry about the combined effect of all such pores in the AH flow channel which are about thirty in number.

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