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VORTEX FLOW IN THE INTERIOR REGION OF A CLOSED CYLINDER WITH DIRICHLET BOUNDARY CONDITIONS

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Abstract

The guiding equation here happens to be the Laplacian for Cylinder, the equation was obtained by using the known definition of differential operator for cylindrical coordinates (ρ, ϕ, z)*, it was then decomposed into three different. Ordinary Differential Equations (ODE) by variable separable methodand the first two solved independently and substituted in to the third to get the general solution. Each of the solutions* $Z_\lambda(z, z^{'})$ *and* $Q_\mu(\phi, \phi^{'})$ *was characterized as a component of the Green's function* $G_\Omega(\xi, \xi^{'})$ *which is the stream function for the flow. The parameters* z *and* $z^{'}$ *were considered by the angles they form with the* $xy-plane$*, while* λ*,* μ *and* l *are constants that assume positive values. Finally the contour plots of the solutions were presented, which describe vortex flow in the interior region of a closed Cylinder of arbitrary length* l*.*

Keywords: Vortex flow, stream function, Green's function, Bessel function

1. INTRODUCTION

The subject of vorticity and vortex flow in general is an area of interest, precisely in fluid mechanics, physics of fluid and engineering like mechanical, chemical and also in powder technology etc. [1]. The problem of description of point vortex motion has a long history dating as far back as the 19th century with Helmholtz initiating the two dimensional point vortex models and Leonardo da Vinci depicted very interesting drawings of various kinds of vortex and eddy flows.

Green's functions for a particular geometry such as sphere, Cylinder, square Cylinder, Ring Torus etc. describe the structure of vortex flow on the surface of the geometry considered. For example, Green's functions for the Laplace-Beltrami operator on a toroidal surface constructed in [2], and characterized it as the stream function in which he showed interesting point vortices by the contour plots. Green's functions in general are powerful tools and fundamental objects for solving partial differential equation [3], thus Green's functions are very crucial in Mathematics more especially in the area of fluid flow since most of the equations that describe fluid flow are partial differential equations. An experimental study of the turbulent near wake of a rotating circular Cylinder was made in [4] at a Reynolds number of 2000 for velocity ratio, λ between 0 and 0.27. Particle image velocimetry data were analyzed to study the effects of rotation on the flow structures behind the cylinder. The result indicates that the rotation of the cylinder causes significant changes in the vortex formation. Toroidal vortex structure (TVS) interaction in various media was investigated in [5]. They introduced the 3D hydrodynamics mathematical model and demonstrated the results of numerical experiments that allow them to make some useful conclusions on TVS. The derivation of Green's function for the heat conduction problem in a finite multi-layered hallow cylinder was presented in [6]. Consideringthe formulation and solution of the problem included an arbitrary number of the cylinder layer characterized by various thermal properties. Dynamics system of point vortices on Hyperboloid were considered in [7].They showed that this system has non-compact symmetry $SL(2,R)$ and a co-adjoint equivariant momentum map. Their aim was to study the stability of these relative Equilibria, motivated by the fact that $SL(2,R)$ is not compact. Vortex streets on a sphere were studied in [8]. Transported the well-known vortex streets from the plane to a curved two-dimensional manifold to the surface of a sphere. N Points vortices s_j of strengths K_j moving on a closed (compact, boudaryless, orientable) surface S with Riemannian metric g has been considered in [9]. They aimed at presenting an intrinsic geometric formulation for the general case, adding that since the pioneer works of Bogomolov [10] and Kimura/Okamoto [11] on the sphere S^2 it was known that stream function produced by a unit point vortex at $s_\circ \in S$ on a background uniform counter vorticity field is given by Green's function $G_g(s, s_\circ)$ of the Laplace-Beltrami operator $\Delta_g = div_g \circ grad_g$

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In this paper we consider the Laplacian in Cylindrical coordinate (ρ,ϕ,z), and present the solution by separating it in to three different partial differential equations (PDE), which help us to obtain Green's function for the interior region of a closed Cylinder, and hence characterize the Green's function obtained as the stream function for the region describing vortex flow, and using the fact that at every point $0<z<z^{'}<l$ on the Cylinder form an angle of vortex flow in the region. We obtained the contour plots for various values of the angle ϕ for $(0<\phi<2\pi)$ and shifting the origin along $z-axis$ as shown.

2. THE LAPLACIAN

The Laplacian for the cylindrical surface is a scalar operator and can be obtained from the definition, using Cylindrical coordinates (ρ,ϕ,Z) as follow:

$$\nabla^2 G=\vec{\nabla}\cdot(\vec{\nabla}G)=\left(\rho\frac{\partial}{\partial\rho}+\frac{\phi}{\rho}\frac{\partial}{\partial\phi}+z\frac{\partial}{\partial z}\right)\cdot\left(\rho\frac{\partial G}{\partial\rho}+\frac{\phi}{\rho}\frac{\partial G}{\partial\phi}+z\frac{\partial G}{\partial z}\right) \tag{1}$$

$$=\rho\cdot\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}+\frac{\phi}{\rho}\frac{\partial G}{\partial\phi}+z\frac{\partial G}{\partial z}\right)+\frac{\phi}{\rho}\cdot\frac{\partial}{\partial\phi}\left(\rho\frac{\partial G}{\partial\rho}+\frac{\phi}{\rho}\frac{\partial G}{\partial\phi}+z\frac{\partial G}{\partial z}\right)+z\cdot\frac{\partial}{\partial z}\left(\rho\frac{\partial G}{\partial\rho}+\frac{\phi}{\rho}\frac{\partial G}{\partial\phi}+z\frac{\partial G}{\partial z}\right)$$

$$=\rho\cdot\left(\rho\frac{\partial^2 G}{\partial\rho^2}-\frac{\phi}{\rho^2}\frac{\partial G}{\partial\phi}+\frac{\phi}{\rho}\frac{\partial^2 G}{\partial\phi\partial\rho}+z\frac{\partial^2 G}{\partial z\partial\rho}\right)+\frac{\phi}{\rho}\cdot\left(\phi\frac{\partial G}{\partial\rho}+\rho\frac{\partial^2 G}{\partial\rho\partial\phi}-\frac{\rho}{\rho}\frac{\partial G}{\partial\phi}+\frac{\phi}{\rho}\frac{\partial^2 G}{\partial\phi^2}+z\frac{\partial^2 G}{\partial z\partial\phi}\right)$$

$$+z\cdot\left(\rho\frac{\partial^2 G}{\partial\rho\partial z}+\frac{\phi}{\rho}\frac{\partial^2 G}{\partial\phi\partial z}+z\frac{\partial^2 G}{\partial z^2}\right)$$

$$=\frac{\partial^2 G}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial G}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 G}{\partial\phi^2}+\frac{\partial^2 G}{\partial z^2}$$

On rearranging we have

$$=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial G}{\partial\phi^2}+\frac{\partial^2 G}{\partial z^2} \tag{2}$$

Therefore we can write the Laplacian Operator as;

$$\nabla^2=\frac{1}{\rho}\frac{\partial}{\partial\rho^2}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2} \tag{3}$$

3. DERIVATION OF THE GREEN'S FUNCTION

The Laplacian operator in (2) above is known to be variable separable [12] and considering ξ as the source and $\xi^{'}$ as the observation point respectively and knowing that a Green's function for the Dirichlet boundary condition (DBC) satisfies the following equation:

$$\nabla^2_{\Omega}G\left(\xi,\xi^{'}\right)=-4\pi\delta\left(\xi-\xi^{'}\right) \tag{4}$$

Where $G\left(\xi,\xi^{'}\right)=0$ whenever ξ is on $\partial\Omega$ the boundary of the cylinder.

We can then separate the corresponding Green's function as follows;

$$G\left(\xi,\xi^{'}\right)=G\left(\rho,\rho^{'},\phi,\phi^{'},z,z^{'}\right)=P\left(\rho,\rho^{'}\right)Q\left(\phi,\phi^{'}\right)Z\left(z,z^{'}\right) \tag{5}$$

By substituting (2) and (5) into (4) we obtain

$$\frac{1}{\rho}\frac{P^{'}}{P}+\frac{P^{"}}{P}+\frac{1}{\rho^2}\frac{Q^{"}}{Q}+\frac{Z^{"}}{Z}=0,\ \rho\neq 0 \tag{6}$$

Looking closely at (6) it is mathematically valid to assign constant number say λ^2 to the ordinary differential equation (ODE), $\frac{Z^{"}}{Z}$, where $\lambda\in N$ and $-\mu^2$ to the ordinary differential equation $\frac{Q^{"}}{Q}$ where $\mu\in N$, we obtain the following system of ordinary differential equations in (7), (8) and (9)

$$Z^{"}-\lambda^2 Z=0 \tag{7}$$

$$Q^{"}+\mu^2 Q=0 \tag{8}$$

$$\rho^2 P^{"}+\rho P^{'}+\left(\lambda^2\rho^2-\mu^2\right)P=0 \tag{9}$$

STEP I. We can rewrite equation (7) as;

$$Z_{zz}\left(z,z^{'}\right)-\lambda^2 Z\left(z,z^{'}\right)=0 \tag{10}$$

The general solution of equation (10) is given as;

$$Z_{\lambda}\left(z,z^{'}\right)=\begin{cases}\alpha_1(z^{'})Sinh(\lambda z)+\beta_1(z^{'})Cosh(\lambda z)0<z<z^{'}<l\\ \alpha_2(z^{'})Sinh(\lambda z)+\beta_2(z^{'})Cosh(\lambda z)0<z^{'}<z<l\end{cases} \tag{11}$$

Now applying the boundary condition $Z_{\lambda}(z,z^{'})\big|_{z=0}=0$ we see that $\beta_1(z^{'})=0$

On applying the second boundary condition $Z_{\lambda}(z,z^{'})\big|_{z=l}=0$

We have $\alpha_2(z') = -\coth(\lambda l)\beta_2(z')$ (12)

Also by the continuity theorem which states that Green's function is continuous around the source point, continuity is applicable to the components of the Green's function hence

$$Z_\lambda(z,z')\Big|_{z=z'^+} = Z_\lambda(z,z')\Big|_{z=z'^-} \tag{13}$$

With this fact we have

$$\alpha_1(z') = \beta_2(z')\left(\frac{Cosh(\lambda z')}{Sinh(\lambda z')} - \frac{Cosh(\lambda l)}{Sinh(\lambda l)}\right) = \beta_2(z')\left(\frac{Sinh(\lambda(l-z'))}{Sinh(\lambda z')Sinh(\lambda l)}\right) \tag{14}$$

In general Green's function between the source and observation points is known to be symmetry. Thus same holds for its components, hence we can write

$$G_\Omega(\xi,\xi') = G_\Omega(\xi',\xi) \tag{15}$$

As a result of this, we apply this rule to the system of solutions, by equating one solution in the system with the other one with its primed and unprimed parameters exchanged, as follows;

$$\beta_2(z')\frac{Sinh(\lambda(l-z'))}{Sinh(\lambda z')Sinh(\lambda l)} = \frac{Sinh(\lambda(l-z))}{Sinh(\lambda z)Sinh(\lambda l)}\beta_2(z) \tag{16}$$

(16) Implies

$$\beta_2(z)Sinh(\lambda z') = \beta_2(z')Sinh(\lambda z) \tag{17}$$

And thus $\beta_2(\xi) = Sinh(\lambda\xi)$ (18)

Substituting (12), (14) and (18) in to (11) we have the solution for the ordinary differential equation in (6) as;

$$Z_\lambda(z,z') = \begin{cases} \frac{Sinh(\lambda z)Sinh(\lambda(l-z))}{Sinh(\lambda l)} 0<z<z'<l \\ \frac{Sinh(\lambda(l-z))Sinh(\lambda z')}{Sinh(\lambda l)} 0<z'<z<l \end{cases} \tag{19}$$

This gives us a component of the Green's function which equally describes the component of the vortex flow on the interior region of the closed cylinder of length l. To get the contour plots of the flow at different points on l, we have to look at the parameters, l and z in a form of angles that the point of the flow makes with positive $z-axis$ on l with a maximum 2π since l is just a straight line

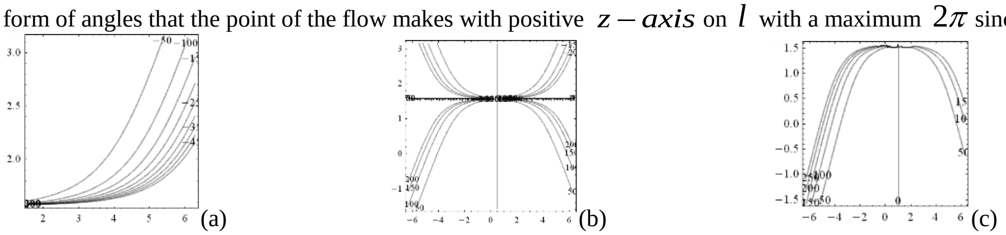


Figure1: Contour plots of the component of the stream function, showing component of the vortex flow in the interior region of a closed Cylinder at various points on l where $0<z<z'<l$

STEP II. Now rewriting (8) we have

$$Q_{\phi\phi}(\phi,\phi') + \mu^2 Q(\phi,\phi') = 0 \tag{20}$$

And the general solution of (20) can be written as

$$Q_\mu(\phi,\phi') = \begin{cases} \alpha_{1,\mu}(\phi')Sin(\mu\phi) + \beta_{1,\mu}(\phi')Cos(\mu\phi) 0<\phi<\phi'<2\pi \\ \alpha_{2,\mu}(\phi')Sin(\mu\phi) + \beta_{2,\mu}(\phi')Cos(\mu\phi) 0<\phi'<\phi<2\pi \end{cases} \tag{21}$$

Looking closely at (21) we can write

$$Q_\mu(\phi,\phi') = Cos(\phi-\phi') 0<\phi<\phi'<2\pi \tag{22}$$

And the value of $Q_\mu(\phi,\phi')$ must be periodic with period 2π thus

$$Q_\mu(\phi,\phi') = Q_\mu(\phi+2\pi,\phi') \tag{23}$$

And hence

$$Cos(\phi-\phi') = Cos(\mu(\phi-\phi')+2\pi\mu) \tag{24}$$

On substituting (24) into (22) we obtained (25)

$$Q_\mu(\phi,\phi') = Cos(\mu(\phi-\phi')+2\pi\mu) 0<\phi<\phi'<2\pi, \mu\in Z \tag{25}$$

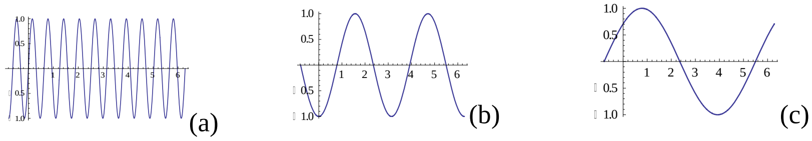


Figure 2: Showing the graph of the second component of the Green's function in question, as μ assumes positive values while $0<\phi<\phi'<2\pi$.

From the graph we can see that since $\mu \in Z$ and $0 < \phi < \phi^{'} < 2\pi$, the function $Q_{\mu}(\phi, \phi^{'})$ is just the cosine graph.

STEP III

We start by rewriting (9) with P as a function of ρ and $\rho^{'}$ as follows;

$$\rho^2 P_{\rho\rho}(\rho, \rho^{'}) + \rho P_{\rho}(\rho, \rho^{'}) + (\lambda^2\rho^2 - \mu)P(\rho, \rho^{'}) = 0 \tag{26}$$

(26) Is a Bessel equation whose general solution is known to be in the form;

$$P_{\mu,\lambda}(\rho, \rho^{'}) = \begin{cases} \alpha_{1,\mu,\lambda}(\rho^{'})J_{\mu}(\lambda\rho) + \beta_{1,\mu,\lambda}(\rho^{'})Y_{\mu}(\lambda\rho) & 0<\rho<\rho^{'}<r \\ \alpha_{2,\mu,\lambda}(\rho^{'})J_{\mu}(\lambda\rho) + \beta_{2,\mu,\lambda}(\rho^{'})Y_{\mu}(\lambda\rho) & 0<\rho^{'}<\rho<r \end{cases} \tag{27}$$

In (27) $J_{\mu}(\lambda\rho)$ and $Y_{\mu}(\lambda\rho)$ are the Bessel functions of first and second kind respectively. The Bessel function of the second kind has a singularity at the origin thus its corresponding coefficient $\beta_{1,\mu,\lambda}(\lambda\rho^{'}) = 0$. Applying the continuity theorem as usual to the system of solutions in (27) it is clear that the coefficient $\beta_{2,\mu,\lambda}(\lambda\rho^{'}) = 0$ is as well due to linear independence of the Bessel function of the first and second kind.

Therefore we can rewrite (27) as follows;

$$P_{\mu,\lambda}(\rho, \rho^{'}) = \begin{cases} \alpha_{1,\mu,\lambda}(\rho^{'})J_{\mu}(\lambda\rho) & 0<\rho<\rho^{'}<r \\ \alpha_{2,\mu,\lambda}(\rho^{'})J_{\mu}(\lambda\rho) & 0<\rho^{'}<\rho<r \end{cases} \tag{28}$$

On applying the continuity theorem to the function $P_{\mu,\lambda}(\rho, \rho^{'})$ we have the following

$$P_{\mu,\lambda}(\rho, \rho^{'})\Big|_{\rho=\rho^{'+}} = P_{\mu,\lambda}(\rho, \rho^{'})\Big|_{\rho=\rho^{'-}} \tag{29}$$

This implies that

$$\alpha_{1,\mu,\lambda}(\rho^{'}) = \alpha_{2,\mu,\lambda}(\rho^{'}) \tag{30}$$

With the fact that Green's function is symmetry the coefficients $\alpha_{1,\mu,\lambda}(\rho^{'})$ and $\alpha_{2,\mu,\lambda}(\rho^{'})$ can be determined as follows

$$\alpha_{1,\mu,\lambda}(\rho^{'}) = \alpha_{2,\mu,\lambda}(\rho^{'}) = J_{\mu}(\lambda\rho^{'}) \tag{31}$$

We can proceed to rewrite (28) as follows

$$P_{\mu,\lambda}(\rho, \rho^{'}) = \begin{cases} J_{\mu}(\lambda\rho^{'})J_{\mu}(\lambda\rho) & 0<\rho<\rho^{'}<r \\ J_{\mu}(\lambda\rho^{'})J_{\mu}(\lambda\rho) & 0<\rho^{'}<\rho<r \end{cases} \tag{32}$$

Now applying the boundary condition we have

$$P_{\mu,\lambda}(\rho, \rho^{'})\Big|_{\rho=r} = J_{\mu}(\lambda\rho^{'})J_{\mu}(\lambda r) = 0 \tag{33}$$

(33) Implies that

$$J_{\mu}(\lambda r) = 0 \Rightarrow \lambda r = \Gamma_{\mu n} \Rightarrow \lambda = \frac{\Gamma_{\mu n}}{r} (n \in N, \mu \in Z) \tag{34}$$

Where $\Gamma_{\mu n}$ is the n^{th} root of the Bessel function of the first kind and μ^{th} order

Thus the Green's function in question can be written as

$$G_{\Omega}(\xi, \xi^{'}) = \sum_{\substack{\mu=-\infty \\ \mu \in Z}}^{+\infty} \sum_{\substack{n=1 \\ n \in N}}^{+\infty} \Theta_{\mu n} P_{\mu n}(\rho, \rho^{'}) Z_{\mu n}(z, z^{'}) Q_{\mu n}(\phi, \phi^{'}) \tag{35}$$

Looking at equation (4), the Dirac delta function in the right-hand- side of the equation can be separated in cylindrical coordinate system as follows:

$$\delta(\xi - \xi^{'}) = \frac{\delta(\rho - \rho^{'})\delta(\phi - \phi^{'})Z(z - z^{'})}{\rho} \tag{36}$$

Substituting (36) in (4) and integrating both sides of the outcome over the infinitesimal interval of $(z^{'-}, z^{'+})$, we have

$$\int_{z=z^{'-}}^{z=z^{'+}} \nabla^2 G(\xi, \xi^{'})dz = -\int \frac{\delta(\rho - \rho^{'})\delta(\phi - \varphi^{'})\delta(z - z^{'})}{\rho} dz \tag{37}$$

This implies

$$\int_{z=z^{'-}}^{z=z^{'+}} \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial G}{\partial\phi^2} + \frac{\partial^2 G}{\partial z^2} dz = \int_{z=z^{'-}}^{z=z^{'+}} \frac{\partial^2 G}{\partial z^2} dz \tag{38}$$

$$\Rightarrow \frac{\partial G}{\partial z}\bigg|_{z=z^{'-}} - \frac{\partial G}{\partial z}\bigg|_{z=z^{'+}} = -\frac{\delta(\rho - \rho^{'})\delta(\phi - \phi^{'})}{\rho} \tag{39}$$

$$\frac{\partial G}{\partial z} = \frac{\partial}{\partial z}\left(\sum_{\substack{\mu=-\infty \\ \mu \in Z}}^{+\infty} \sum_{\substack{n=1 \\ n \in N}}^{+\infty} \Theta_{\mu n} P_{\mu n}(\rho, \rho^{'}) Z_{\mu n}(z, z^{'}) Q_{\mu n}(\phi, \phi^{'})\right) \tag{40}$$

$$\frac{\partial G}{\partial z}=\sum_{\substack{\mu=-\infty\\ \mu\in Z}}^{+\infty}\sum_{\substack{n=1\\ n\in N}}^{+\infty}\Theta_{\mu n}P_{\mu n}(\rho,\rho^{'})Cos(\mu(\phi-\phi^{'}))\frac{\partial}{\partial z}Z_{\mu n}(z,z^{'}) \tag{41}$$

And thus

$$\frac{\partial}{\partial z}Z_{\mu n}(z,z^{'})=\left.\frac{\partial(Z_{\mu,n}(z,z^{'})}{\partial Z}\right|_{z=z^{'+}}-\left.\frac{\partial(Z_{\mu,n}(z,z^{'})}{\partial Z}\right|_{z=z^{'-}} \tag{42}$$

From (19) on replacing the value of λ as in (34) we can write (39) as follows

$$\sum_{\substack{\mu=-\infty\\ \mu\in Z}}^{+\infty}\sum_{\substack{n=1\\ n\in N}}^{+\infty}\Theta_{\mu n}P_{\mu n}(\rho,\rho^{'})Cos(\mu(\phi-\phi^{'}))\left(\frac{\Gamma_{\mu n}}{r}\right)=\frac{\partial(\rho-\rho^{'})\partial(\phi-\phi^{'})}{\rho} \tag{43}$$

And the coefficient $\Theta_{\mu n}$ derived in detail by [12] is

$$\Theta_{\mu n}=\begin{cases}\dfrac{2}{\pi r\Gamma_{pq}\left[J_{p+1}\left(\Gamma_{pq}\right)\right]^2}\rho\neq 0\\ \dfrac{1}{\pi r\Gamma_{pq}\left[J_{p+1}\left(\Gamma_{pq}\right)\right]^2}\rho=0\end{cases} \tag{44}$$

And finally by [7] we can write (35) as follows:

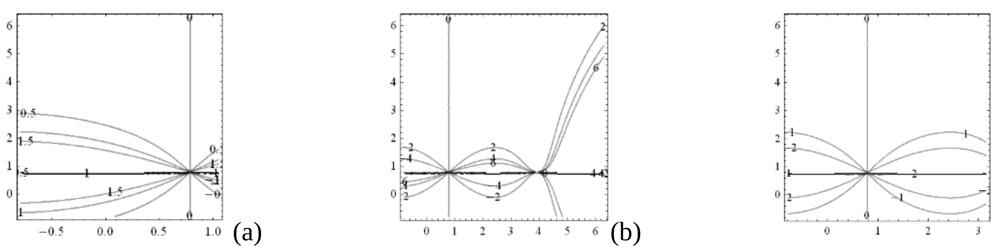
$$G_{\Omega}\left(\xi,\xi^{'}\right)=\sum_{n=1}^{+\infty}\frac{2}{\pi r\Gamma_{0n}[J_1(\Gamma_{0n})]^2}J_0\left(\frac{\Gamma_{0n}}{r}\rho^{'}\right)J_0\left(\frac{\Gamma_{0n}}{r}\rho\right)\frac{Sinh\left(\frac{\Gamma_{0n}}{r}z\right)Sinh\left(\frac{\Gamma_{0n}}{r}(l-z)\right)}{Sinh\left(\frac{\Gamma_{0n}}{r}l\right)}+ \tag{45}$$

$$\sum_{\mu=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{2}{\pi r\Gamma_{0n}[J_1(\Gamma_{0n})]^2}J_0\left(\frac{\Gamma_{0n}}{r}\rho^{'}\right)J_0\left(\frac{\Gamma_{0n}}{r}\rho\right)\frac{Sinh\left(\frac{\Gamma_{0n}}{r}z\right)Sinh\left(\frac{\Gamma_{0n}}{r}(l-z)\right)}{Sinh\left(\frac{\Gamma_{0n}}{r}l\right)}Cos\left(\mu\left(\phi-\phi^{'}\right)\right)$$

$$(46)$$

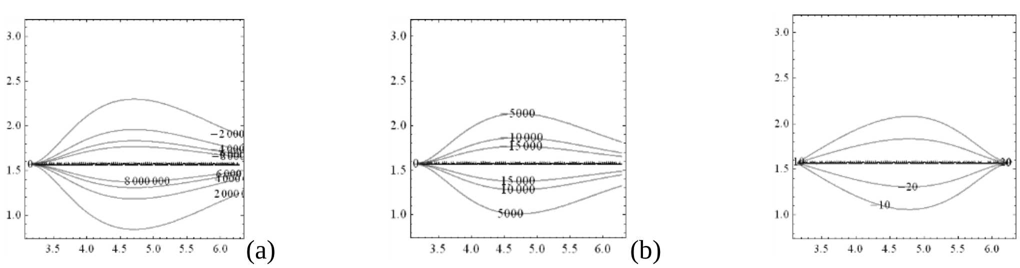
$$=\sum_{\mu=1}^{+\infty}\sum_{n=1}^{+\infty}\frac{2}{\pi r\Gamma_{0n}[J_1(\Gamma_{0n})]^2}J_0\left(\frac{\Gamma_{0n}}{r}\rho^{'}\right)J_0\left(\frac{\Gamma_{0n}}{r}\rho\right)\frac{Sinh\left(\frac{\Gamma_{0n}}{r}z\right)Sinh\left(\frac{\Gamma_{0n}}{r}(l-z)\right)}{Sinh\left(\frac{\Gamma_{0n}}{r}l\right)}\left(1+Cos\left(\mu\left(\phi-\phi^{'}\right)\right)\right),r<\rho\neq 0$$

Taking the coefficient of (45) as the strength of the vortex flow and in a similar way we obtained the contour plots in figure1, we then obtain the contour plots of the solution in (46), characterizing it as the stream function of the vortex flow in the interior region of a closed Cylinde



(a) (b) (c)

Figure:3. Contour plots of the Green's function, showing vortex flow in the interior region of a closed Cylinder at various points on l where $0<z<z^{'}<l$



(a) (b) (c)

Figure: 4. Contour plots of the stream function, showing vortex flow in the interior region of a closed Cylinder with $\phi=\pi,4\pi,6\pi$, at various points on l where $0<z<z^{'}<l$

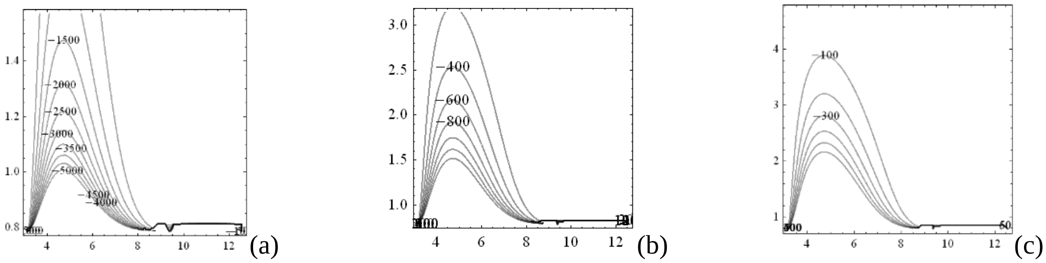


Figure: 5. Contour plots of the stream function, showing vortex flow in the interior region of a closed Cylinder with the angle the point of flow make with the $xy-plane$ as $\pi/2, \pi, 3\pi/2$ at various points on l where $0 < z < z' < l$

4. CONCLUSION

Vortex flow is a helical motion or circular spiral in a fluid (such as gas). In general a region in a fluid in which the flow is revolving around an axis line which may be straight or curve is called vortex flow and its plural is called vortices. Vortices form in a stirred fluid and may be observed in nature like smoke rings, whirlpool in the wake of boat or the winds surrounding a tornado or dust devil. Vortices are the major component of turbulent flow. This type of flow (vortex) on the interior region of a closed Cylinder is our interest in this paper. Thus we first obtained the Laplacian for a Cylinder and presented its solution in a form of Green's function for the interior region of a closed Cylinder and then characterized the solution as the stream function for such flow (vortex) on the interior region of the closed Cylinder. The contour plots shown in figure 1, 3, 4 and 5 describe the nature of the vortex flow when the angle of rotation is taking at various points on the length l of the Cylinder.

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