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Optimal Controllability of Functional Differential Systems of Sobolev Type in Banach Spaces

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Abstract

In this work, the Functional Differential Systems of Sobolev Type in Banach Spaces of a particular form is presented for the investigation of Existence of an Optimal Controlin such asystem.The solution of the system is given by an Integral Equation

We state the following very important concepts that provide criteria for determining the optimal control for the system - reachable set, attainable set, target set and controllability grammian. Optimal control problem is stated and proved. Necessary and Sufficient Conditions for the existence of an optimal control of the system are established respectively.

Key words: Optimal Control, Sobolev System, Banach Spaces, Functional Differential Systems

1.0 Introduction

The problem of controllability of linear and non linear systems represented by ordinary differential systems in finite dimensional spaces has been extensively studied. Several Researchers [1-4] have extended the concept of Controllability of linear and non linear systems represented by ordinary differential systems in finite dimensional spaces to infinite dimensional systems in Banach Spaces with bounded operators. Triggiani [5] established sufficient conditions for controllability of linear and nonlinear systems in Banach spaces.

Balachandran, and co-workers [6,7] have studied the Controllability and Local Null controllability of Nonlinear Integrodifferential Systems and Functional Differential Systems in Banach Spaces and established that the Controllability problems in Banach Spaces can be converted into one of a fixed-point problem for a single – valued function.

The purpose of this paper is to investigate the existence of **Optimal Control of Functional Differential Systems of Sobolev type in BanachSpaces.**(i.e,theSobolev type partial functional differential systems in Banach Spaces). The equation considered here serves as an abstract formulation of Sobolev type partial functional differential equations which arise in many physical phenomena [8-10].

Consider a nonlinear partial functional differential system of the form.

$$(Tx(t))' + A\,x(t) = Bu(t) + f(t,x_t),\ t > 0 \qquad (1.1)$$
$$x\,(t) = \phi\,(t), \qquad -s \le t \le 0$$

where the state x(.) takes values in a Banach space X and the control function u(.) is given in a Banach space R^n and the control function u(.) is given in Lebesque Space of square integrable (equivalent class of) functions from $J = [t_0, t_1] to\ R^n\ will\ be\ denoted\ L_2(J, R^n)$.

$L^1([t_0,t_1],R^n)$ *denotes the space of integrable functions from* $J = [t_0,t_1]$ *to* R^n.
The control function $u(.)$ *is given in* $L_2(J,U)$, *the Banach space of admissible control functions with* U *a Banach space.* B *is a bounded linear operator from* U *into* Y, *a Banach space. The nonlinear operator* $f: JxC \rightarrow Y$ *is continuous.*
Here $J = [t_0,t_1]$ *and for a continuous function* $x: J^* = [-h,t_1] \rightarrow X$ *is the element of* $C = C([-h,0]:X)$ *defined by* $x_t(s) = x(t+s), h < s < 0$.
The domain $D(G)$ *of* G *becomes a Banach space with norm.*
$||x||_{D(G)} = ||Gx||_y, x\epsilon\, D(G)$ *and* $C(G) = C([-h,0]:D(G)$.

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2.0 Preliminaries

Definition 2.1

The Sobolev spaces are special subspaces of $Lp(\Omega)$ spaces (Lp-spaces over the open subspace of R) on which differential operators are continuous which makes the study of partial differential equations in such spaces convenient.

$$\text{The set of functions } f\epsilon Lp(\Omega) \text{such that } f \text{ and its partial derivatives of order one,}$$
$$\text{namely}: \boldsymbol{\delta_1 f, \delta_2 f, \delta_3 f, \dots, \delta_n f} \epsilon\, \boldsymbol{Lp\ (\Omega)}, \qquad \text{is a vector space for } 1 \le p \le \infty.$$

Definition2.2: *We denote Sobolev space of order* $1, p$ *by* $w^{\,1,p}(\Omega)$.

The vector space $\boldsymbol{w^{\,1,p}(\Omega)}$ ***given by***

$$\boldsymbol{w^{\,1,p}}(\boldsymbol{\Omega}) = \{\boldsymbol{f}: \boldsymbol{f}, \boldsymbol{\delta_1 f}, \ \boldsymbol{\delta_2 f}, \ \boldsymbol{\delta_3 f}, \dots, \ \boldsymbol{\delta_n f} \epsilon\, \boldsymbol{Lp}\ (\boldsymbol{\Omega})\}\ , \text{is a Sobolev space of order } 1, p.$$

Definition 2.3:(Norms on Sobolev of order 1, p).

For $f\ \epsilon\ \boldsymbol{w^{\,1,p}(\Omega)}$ *and* $1 \le P \le \infty$, *we define*

$$\|f\|_{1,p} = \left(\left[\int_\Omega |f|^p + \sum_{I=1}^{n} |\delta_i f|\right] dx\right)^{1/p}$$

$$= \left(\|f\|_p{}^p + \sum_{I=1}^{n} \|\delta_i f\|_p{}^p\right)^{1/p} \qquad (1.3)$$

$$\|f\|_\infty = \max\left(\|f\|_\infty, \|\delta_1 f\|_\infty, \|\delta_2 f\|_\infty, \dots \|\delta_n f\|_\infty\right) \quad (1.4)$$

It follows from (1.3) and (1.4) that $W^{1,P}(\Omega)$ is a normed vector space, and

$$||f||_{1,p} = ||f||_p + \left||\delta_1 f|\right|_p + \|\delta_2 f\|_p \ \dots + \|\delta_n f\|_p \qquad (1.5)$$

is equivalent norm on $W^{1,P}(\Omega)$.

Definition 2.4:(Sobolev Space of order m,p given by $W^{m,p}(\Omega)$).

By SobolevSpace of order m, p, *where* $m > 0$ *is an integer and* $1 \le p \le \infty$,
we mean the vector space $W^{m,p}(\Omega)$. suchthat

$$W^{m,p}(\Omega) = \{f: f\epsilon\, L_p(\Omega) : D_W{}^\alpha f\ , \delta^\alpha f\ \epsilon\, L_p(\Omega)\}$$

Where f *is a set of functions in* $L_p(\Omega)$, *and* $\delta^\alpha f$ *is generalized partial derivative of* f *of order* $|\alpha|, 0 \le |\alpha| \le m$, $D_W{}^\alpha f$ *are the* ***distributional or weak derivatives***.

The associated norms are

$$\|f\|_{m,p} = \left(\sum_{0\le|\alpha|\le m} \|f\|_p{}^p\right)^p, 1 \le p \le \infty$$

$$\|f\|_{m,p} = \max_{0\le|\alpha|\le m} \|f\|_\infty$$

Proposition 2.1

$$\text{The differential operator } \delta^\alpha: W^{m,p}(\Omega) \to L_p(\Omega) \text{ is continuous.}$$

Proof

Let $f\ \epsilon\ W^{m,p}(\Omega), \delta^\alpha$ is linear and bounded, since

$$\|\boldsymbol{\delta^\alpha f}\|_{\boldsymbol{p}} = \left(\int_{\boldsymbol{\Omega}} |\boldsymbol{\delta^\alpha f}|^{\boldsymbol{p}} \boldsymbol{dx}\right)^{\boldsymbol{1/p}} \le \left(\sum_{0\le|\alpha|\le m} \int_\Omega |\delta^\alpha f|^p\, dx\right)^{1/p} = \|f\|_{m,p}$$

$$\text{Therefore, } \delta^\alpha, |\alpha| \le m \text{ is continuous on } W^{m,p}(\Omega)\ .$$

Definition 2.5: (Distributional derivative or weak derivative)

The concept of distributional (weak) derivative has been used in physics before Schwartz gave the mathematical structure of these concepts. There are several ways to generalize the idea of usual derivatives. The definition of distributional (weak) derivative given by Schwartz came from the following simple idea.

Let f: R → R be a real-valued differentiable function and ψ be any function in

$$C_0^\infty(R) = \{\psi : \psi\epsilon\, C_m(\Omega) \text{and supp } \psi \subset\subset \Omega\},$$
$$\text{where supp } \psi \subset\subset \Omega \text{ means } \psi \text{ has compact support in } \Omega.$$
$$\text{Since } \psi \text{ has compact support, we can choose some open interval}(a, b)$$

where $-\infty < a < \infty$ *such that* $\psi \subset (a, b)$.. Then

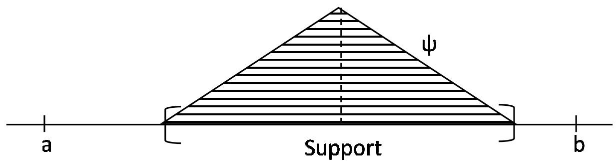

Fig 1

$$\int_R f'\psi dx = \int_R f'(x)\psi(x)dx$$

$$= [\psi(x)f(x)]_a{}^b - \int_a^b \psi'(x)f(x)dx$$

$$= \psi(b)f(b) - \psi(a)f(a) - \int_a^b \psi'(x)f(x)dx$$

$= \quad - \int_a^b \psi'(x)f(x)dx \quad , \quad$ since $\psi(a) = \psi(b) = 0.$

$$\text{So,} \int_R f^{1\prime}\psi dx = -\int_R f(x)\ \psi'(x)dx \quad \text{Or} \int_R f'\psi dx = -\int_R f(x)\ \psi' dx\,.$$

Similarly, if f has higher derivatives, then we have

(1) $\int_R f'\psi dx = (-1)^n \int_R f\psi^n dx\ , \forall\ \psi \epsilon\, D(R).$

(2) *if* $f \in C^{|\alpha|}(\Omega)$, *then using integration by parts method, we have the following*

$$\int_\Omega D^\alpha f\psi dx = (-1)^{|\alpha|}\int_\Omega f D^\alpha \psi dx\ , \forall\ \ \psi \in D(\Omega).$$

Definition 2.5: (The Distributional Derivative) (or the weak derivative).
Let f, g ϵ $L_1(\Omega)$. Then g is called a **Distributional derivative or weak derivative** of f of order α, denoted by $D_w^\alpha f$ *if*

$$\int_\Omega g\psi dx = (-1)^{|\alpha|}\int_\Omega f D^\alpha \psi dx\ , \forall \psi \epsilon\, D(\Omega).$$

i.e. iff

$$D_W^\alpha f = g \qquad \text{and} \int g\psi dx = (-1)^{|\alpha|}\int f D_W^\alpha \psi dx.$$

Theorem 2.1

$W^{m,p}(\Omega)$ *is a Banach space.*

Proof

Let $\{U_n\}$ *be a Cauchy sequence in* $W^{m,p}(\Omega)$ *then from the norm given above, we have that each sequence* $\{D^\alpha U_n\}$ *is a Cauchy sequence in* $L_p(\Omega)$ *for all* $1 \le |\alpha| \le m$.
Since $L_p(\Omega)$ *is a Banach space, so there exists the limit of each of these Cauchy sequences.*
That is to say that there exist functions u *and* u_α*'s in* $L_p(\Omega)$ *such that*

$$U_n \to u \text{ and } D^\alpha U_n \to U_\alpha \text{ in } L_p(\Omega).$$

For any $\psi \in D(\Omega)$, *we have*

$$|Tu_n(\psi) - T_u(\psi)| \le \int_\Omega |U_n(x) - u(x)||\psi(x)|dx$$

$$\le ||U_n - u||\ ||\psi||q\ (\text{by holder's inequality}).$$

Where q is the conjugate of P, defined as

$$\frac{1}{p} + \frac{1}{q} = 1, if\ \ 1 < p < \infty, n\ q = 1\ if\ p = \infty\ and\ q = \infty, if\ p = 1.$$

Since $U_n \to u$, *so we have*

$$TU_n(\psi) \to Tu(\psi)\ for\ all\ \psi \in D\ (\Omega)$$

Similarly,

$$TD^\alpha U_n \to Tu_\alpha(\psi), \forall\ \psi \epsilon D(\Omega)$$

$$TU_n(\psi) = \lim_{n\to\infty} TD^\alpha U_n = \lim_{n\to\infty} (-1)^{|\alpha|} TU_n(D^\alpha\psi) = (-1)^{|\alpha|} TU(D^\alpha\psi).$$

$By\ definition\ of\ distributional\ (weak) derivative, we\ have\ U_\alpha = D^\alpha U,$

$for\ | \le |\alpha| \le m.$

$$So\ u\epsilon W^{m,p}(\Omega) and\ we\ have\ \lim_{n\to\infty} \|U_n - U\|_{m,p} = 0$$

$$Hence, W^{m,p}(\Omega)\ is\ complete\ and\ hence\ a\ Banach\ space.$$

Definition 2.6

The system (1.1) is said to be controllable on the interval $J = [t_0, t_1]$ *if for every continuous initial function* ϕ *defined on* $[-h, 0]$ *and every* $x_1 \epsilon X$ *there exists a control* $u \in L_2(J, U)$ *– square integrable functions with unlimited control such that the solution* $x(.)$ *of system (1.1) satisfies* $x(t_1) = x_1$.

The solution of the system (1.1) is given by the integral equation (see Krishnan Balachandram and Jerald P. Dauer [1].

$$x(t) = G^{-1}T(t)G\,\phi(0) + \int_0^t G^{-1}T(t-s)f(s, x_s)ds + \int_0^t G^{-1}T(t-s)Bu(s)ds, t > 0 \qquad (2.1)$$

$$x(t) = \phi(t), \quad -h \le t \le 0$$

We state the following very important concepts that provide criteria for determining the optimal control for the system (1.1).

Definition 2.7: (Reachable Set)

Consider the system (1.1) given as

$$(G\,x(t))' + Ax(t) = Bu(t) + f(t, x_t), t > 0 \qquad (1.1)$$

$$x(t) = \phi(t), \quad -s \le t \le 0.$$

with all its basic assumptions.

Let the solution $x(t)$ *be given as*

$$\boldsymbol{x(t) = G^{-1}\,T(t)G\,\phi(0) + \int_0^t G^{-1}\,T(t-s)f(s, x_s)ds + \int_0^t G^{-1}\,T(t-s)Bu(s)ds}, \quad \mathbf{t > 0}$$

$x(t) = \phi(t), -h \le t \le 0.$

where G is a fundamental matrix.

We define the reachable set of system (1.1) as

$$R(t_1, t_0) = \left\{\int_0^t \mathbf{G^{-1}\,T(t-s)Bu(s)ds: u\epsilon U and\ |U| \le 1,\ t > 0}\right\}$$

Where U is the set of constraint admissible controls.

Definition 2.8 (Attainable set)

Attainable set is the set of all possible solutions of a given control system. In the case of the system. In the case of the system (1.1), for instance, it is given as

$$A(t_1, t_0) = \left\{x(t) = G^{-1}T(t)G\phi(0) + \int_0^{t_1} G^{-1}T(t-s)\,f(s, x_s)f(s, x_s)ds\right\} + \left\{\int_0^{t_1} G^{-1}\,T(t-s)B(s)u(s)ds:\ u\epsilon U \text{ and } |U| \le 1.\right\}$$

Definition 2.9 (Target set)

The target set for the system (1.1) denoted by $G(t_1, t_0)$ *is given as*

$$G(t_1, t_0) = \{x(t_1, x_0, u): t_1 > \tau > t_0\ for\ fixed\ time\ \tau\ and\ u\epsilon U\}$$

$$where\ U = \{u \in L_2(t_0, t_1], X^n): |U_j| \le 1, \quad j = 1, 2, \ldots, m\}.$$

Definition 2.10: (Controllability Grammian Or Map)

The controllability grammian for the system (1.1) is given as

$$W(t_1, t_0) = \int_0^{t_1} z(t, s)z^T(t, s)ds,$$

where $z(t, s) = G^{-1}T(t-s)B(t)$ *and T denotes matrix transpose.*

Definition 2.11: (Properness)

The system (1.1) is proper in X on interval $J = [-1]$, *if spam* $R(t_1, t_0) = X$ *and if*

$$C^T[G^{-1}T(t-s)B(t)] = 0\ a.e, t_1 > 0 \Rightarrow C = 0, C\epsilon X.$$

3.0 Main Results

The optimal control problem can best be understood in the context of a game of pursuit [9, 10]. The emphasis here is the search for a control energy that can steer the state of the system of interest to the target set (which can be a moving point function or a compact set function) in minimum time. In other words, the optimal control problem is stated as follows:

$$\boldsymbol{If\ t^* = infimum\,\{t:\ A(t_1, 0) \cap G\,(t_1, 0) \neq \phi\ for\ t \epsilon\,[t_0, t_1], t_1 > t_0 = 0\}},$$

Then there exists an admissible control $u^* \epsilon U$ such that the solution of the system of interest with this admissible control $u^*$is steered to the target. The theorem that follows illustrates this ascertain.

$\boldsymbol{Theorem\ 3.1}$ ($\boldsymbol{Necessary\ Conditions}$).

Consider the system (1.1) *as a different game of pursuit*

$$(Gx(t))' + Ax(t) = B\,u(t) + f\,(t, x_t), t > 0 \qquad (1.1)$$
$$x(t) = \phi\,(t), -s \leq t \leq 0$$

with its basic assumptions.

Suppose $\boldsymbol{A(t_1, 0)}$*and* $\boldsymbol{G(t_1, 0)}$*are compact set functions then there exists an admissible control* $u \in U$ *such that the state of the weapon for the pursuit of the target satisfies the system* (1.1)*if and only if*

$$\boldsymbol{A(t_1, 0) \cap G(t_1, 0) \neq \phi}.$$

Proof

Let $\{u_n\}$ *be a sequence in* U. *Since the constraint control set* U *is compact (closed and convex), then the sequence* $\{u_n\}$ *has a limit* u, *as* n *tends to infinite that is,*

$$\lim_{n\to\infty} u_n = u$$

Suppose the state $z(t)$*of the weapon for pursuit of the target satisfies the system* (1.1)*on the time interval* $[\boldsymbol{t_0}, \boldsymbol{t_1}]$, *then* $z(t) \epsilon\, G([\boldsymbol{t_1}, \boldsymbol{t_0}])$, *for* $t \in [\boldsymbol{t_0}, \boldsymbol{t_1}]$.

We need to show that there exists a solution $x(t, u)$*(state)of the system suchthat* $x(t, u) \epsilon\, A(t_1, t_0)$ *for* $t \in [\boldsymbol{t_0}, \boldsymbol{t_1}]$ *and that* $z(t) = x(t, u)$ *for some* $u \epsilon U$.

Now, $x\,(t_1, x_0, u^n) \in A(t_1, t_0)$ *, and from*

$$x\,(t, x_0, u^n) = G^{-1}(t)G\phi(0)\int_0^t G^{-1}\mathrm{T}\,(t-s)f(s, x_s)ds$$
$$+\int_0^t G^{-1}\mathrm{T}\,(t-s)Bu^n(s)ds, \quad t > 0 \qquad (3.1)$$

Taking limit on both sides of (3.1), we have

$$\underset{n\to\infty}{\text{limit}}\, x\,(t, x_0, u^n) = G^{-1}(t)G\,\phi(0)\int_0^t G^{-1}\mathrm{T}\,(t-s)f(s, x_s)ds$$
$$+\int_0^t G^{-1}\mathrm{T}\,(t-s)B(s)\lim_{n\to\infty} u^n(s)ds\,, t > 0$$

Since $A(t_1, t_0)$ *is compact and* $\lim_{x\to\infty} x\,(t, x_0, u^n) = x\,(t, x_0, u)$.

Thus, there exists a control $u \epsilon U$ *such that* $x(t_1, x_0, u) = z(t_1)$, *for* $t_1 > 0$.

Since $z(t_1) \in G(t_1, t_0)$ *and also, is in* $A(t_1, t_0)$, *it follows that*

$$A(t_1, t_0) \cap G(t_1, t_0) \neq \phi\ for\ t \in [(t_0, t_1)].$$

Conversely,

Suppose that the intersection condition holds, that is $A(t_1, t_0) \cap G(t_1, t_0) \neq \phi$, $t \in [t_0, t_1]$, *then* $Z(t) \epsilon\, A\,(t_1, t_0)$*such that* $Z(t) \epsilon\, G(t_1, t_0)$.

This implies that $Z(t) = x\,(t, x_0, u)$. *and hence establishes that the state of the weapon at pursuit of the target satisfies the system* (1.1). *this completes the proof.*

Remark 3.1

The above stated and proved theorem in other words states that in any game of pursuit described by a functional differential systems of Sobolev type in Banach Spaces, it is always possible to obtain the control energy function to steer the system state to the target in finite time. The next theorem is, therefore, a consequence of this understanding and provides sufficient conditions for the existence of the control function that is capable of steering the state of the system (1.1) to the target set in minimum time.

Theorem 3.2 (sufficient conditions)

Consider the system (1.1), that is

$$(G\,x(t))' + Ax(t) = Bu(t) + f(t, x_t), t > 0 \qquad (1.1)$$
$$x\,(t) = \phi(t), -s \leq t \leq 0.$$

with its basic assumptions.