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Application of Semidefinite Relaxation Technique in Nigeria Aviation System

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Abstract

This paper presents the use of semidefinite relaxation in air traffic schedule that optimally satisfies a list of sector capacity constraints and minimizes the total delay compared to the original schedule. The problem of optimally scheduling air traffic flow with sector capacity constraints was formulated using some methods following from linear programming through to the semidefinite relaxation. Using the software CVX toolbox in Matlab, the optimal solution for flight within Nigeria was obtained. The result shows that the delays were minimized and sector capacities were maximized. Thus, this work provides an approximate solution to the scheduling problem and a condition to check for its optimality.

Key words: Matlab, CVX toolbox, Semidefinite relaxation, Lagrangian relaxations, and air traffic schudule

1.0 Introduction

Semidefinite optimization is concerned with choosing a symmetric matrix to optimize a linear function subject to linear constraint and a further constraint that the matrix be positive semidefinite. Semidefinite programs (SDP) can be regarded as an extension of linear programming where the componentwise inequalities between vectors are replaced by matrix inequalities. SDP unifies several standard problems (e.g., linear and quadratic programming), and finds many applications in combinatorial optimization and engineering. There are efficient solution algorithms for SDP and it has been applied to optimal production problem, model predictive control (MPC) and minimax MPC [1-3].

Researchers have shown increasing concern within the past decade to develop optimized and automated systems for air traffic flow scheduling in order to manage congestion and delay in flight schedules. The primary purpose of air traffic scheduling worldwide is to prevent collisions, organize and expedite the flow of traffic, and provide information and other support for pilots[4]. The major portion of delay in Air Traffic Management Systems (ATMS) in US and Europe arises from the convective weather [5], while in Nigeria it is due to a lot of factors. Knowing the constraints contained in sector capacities, a mathematical technique makes air flight traffic scheduling easier, efficient and optimal [4,6]. Semidefinite relaxations and randomization techniques provide more efficient solutions to air flight traffic scheduling problem [7-9] than other optimization progammes [10-16] because of its interesting features [3, 8, 17-23]. In this work, we are going to simulate an example of air traffic scheduling using CVX. CVX is a mathematical toolbox which employs solvers such as SDPT3, Sedumi, Gurobi etc., to solve convex optimization problems in combinatorial problems and operation research efficiently in Matlab [24]. It turns Matlab into a modelling language, allowing constraints and objectives to be specified using standard Matlab expression syntax with the purpose of arriving at an approximate optimal solution for the flight scheduling problem which is stochastic in nature given that the weather constraint are non-deterministic in nature. Its ease of use in convex optimization makes this toolbox the appropriate tool for this scheduling problem.

2.0 Methodology

First of all a problem of minimizing total delay while satisfying capacity constraints using linear programming was formulated, then lifting this procedure a semi definite relaxation of the problem was formed and then rewritten as a Non-Convex Quadratic Constrained Quadratic Program (QCQP) (see [17,25]for details). In this paper, we will use the same problem formulation, semidefinite relaxation and langrange relaxation outlined in [11]. However, the work in this paper did not consider randomization which was considered in [11]. In this work, CVX solver was used and not SEDUMI as in [11].

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2.1 Semidefinite Relaxation and its Application to Scheduling Problem

The problem to be considered is of the form in (1) which can be rewritten as (2). However, (2) is a non-convex quadratic program that is computationally hard.

$$\min \sum_{i=1}^{n}\sum_{j=0}^{d} x_{ij} j$$
$$sub.to \sum_{i=1}^{n}\sum_{j=0}^{d} x_{ij} R^{(i,s+j)} \leq C$$
$$\sum_{j=0}^{d} x_{ij} = 1$$
$$x_{ij}{}^2 - x_{ij} = 0, i = 1, \dots, n; j = 0, \dots, d, \tag{1}$$

in the variable $x_{ij} \in \boldsymbol{R}^{n\times(d+1)}$.

$$\min \sum_{i=1}^{n}\sum_{j=0}^{d} x_{ij} j$$
$$sub.to \sum_{i=1}^{n}\sum_{j=0}^{d} x_{ij} R^{(i,s+j)} \leq C$$
$$\sum_{j=0}^{d} x_{ij} = 1, i = 1, \dots, n$$
$$X = vec(x)vec(x)^T$$
$$diag(X) = vec(x), \tag{2}$$

in the variables $x_{ij} \in \boldsymbol{R}^{n\times(d+1)}$ and $X \in \boldsymbol{S}^{n\times(d+1)}$

R is the flight route, *d* is the delay in units of time, *C* is the capacity matrix, *s* is the start time, and x_{ij}=1means that aircraft *i* will be delayed by *j* units of time. The sector capacity constraints are met by the first constraint and the objective is the sum of aircraft delay.

Convex optimization can be used to find bounds on the optimal value of a hard problem, and can also be used to find good (but not necessarily optimal) feasible points.

Rewriting (2) as a semidefinite relaxation of the air traffic schedule we have:

$$\min \sum_{i=1}^{n}\sum_{j=0}^{d} x_{ij} j$$
$$sub.to \sum_{i=1}^{n}\sum_{j=0}^{d} x_{ij} R^{(i,s+j)} \leq C$$
$$\sum_{j=0}^{d} x_{ij} = 1, i = 1, \dots, n$$
$$\begin{bmatrix} X & vec(x) \\ vec(x)^T & 1 \end{bmatrix} \geq 0$$
$$diag(X) = vec(x), \tag{3}$$

where the problem variables in the semidefinite relaxation are $x_{ij} \in \boldsymbol{R}^{n\times(d+1)}$ and $X \in \boldsymbol{S}^{n\times(d+1)}$, this can be solved efficiently. The objective of this program is to find a lower bound on the global solution. An important structure of (3) is that "*diag*(*X*) = *vec*(*x*)". This gives a condition that makes the semidefinite relaxation technique tighter than the Lagrangian technique. The dual of (3) is a Maximum Conic Eigenvalue minimization problem for which first order methods such as Spectral Bundle Methods and the Interior Point Methods can be used to solve increasingly large-scale problems. The Spectral bundle method solves large-scale Eigenvalue minimization problems to return an optimal (*x*,*X*). In this paper, we would employ the Spectral bundle methods because it is efficient for both general and special classes of SDP's and the interior point method is restricted by the fact that their algorithms are second-order methods and need to store and factorize a large and often dense matrix.

2.2 Lagrangian Relaxation and its Application to Scheduling Problem

Lagrangian relaxation is a tool to find upper bounds on a given (arbitrary) maximization problem (see [26] for details). To obtain a convex relaxation of problem (3) we use Lagrangian relaxation which uses weak duality and the convexity of duals to get bounds of the problem (see [22] for details).

Using a linear program relaxation of (4) to represent the scheduling problem and its simplified Lagrangian function of (5), we get the dual function presented in (6).

$$\min \mathrm{C}^{\mathrm{T}} * \mathrm{x}$$
$$sub.to\ (A * x) - b \le 0, \quad 0 \le x \le 1 \tag{4}$$
$$L(x,\lambda) = C^T x + \lambda^T((A*x) - b)\ ,\ x \in \{0,1\} \tag{5}$$

Minimizing over x, the dual function is given as:

$$g(\lambda) = \min L(x,\lambda) = \begin{cases} -b^T\lambda & if\ (A^T * \lambda) + C = 0 \\ +\infty\ otherwise \end{cases} \tag{6}$$

L is the Langrangian function and λ is the eigenvalue.

2.3 Eigenvalue Optimization

Eigenvalue optimization is a field in its own right and has many practical applications. Several basic problems in eigenvalue optimization may be formulated as semidefinite programs [26].

Considering the semidefinite relaxation to the scheduling problem, its equivalent is given as:

$$\min \sum_{i=1}^{n} \sum_{i=0}^{d} (C_i, X)$$
$$such\ that\ AX - b \le 0, X \ge 0, \tag{7}$$

where $C_i = j$ (total delay), A is the sub-symmetrical matrix and b is the capacity of each sector (C).

As usual the $X \ge 0$ means that X is in S^n, the cone is symmetric and is made up of positive semidefinite matrices.

$$\min f(y),$$
$$f(y) = a\lambda_{max}(A^* y - C) - < b, y >,$$
$$y \in \boldsymbol{R}^m$$
$$where\ A^* : \boldsymbol{R}^m \to \boldsymbol{S}^n : y \to \sum_{i=1}^{m} y_i A_i \tag{8}$$

A^* denotes the adjoint of A. The function f is amenable to minimization by the classical sub gradient bundle methods of convex programming. The resulting algorithm is called spectral bundle method and has been very successful in solving large scale SDP relaxations from combinatorial optimization [27].

3.0 Numerical Example

Let us take a scheduling problem to fix ideas. Suppose Benin City airport has schedules as shown in the table.1 and there are three flights at a given time and in airspace there are six sectors where each sector has a capacity of one. Each sector has a capacity of one. This airport has a finite capacity and can handle only so many aircraft per hour. One aircraft can land or depart from the runway at a given time, and aircraft's landing and takeoff times are separated by a certain time to avoid collision.

Table 1: Schedule of flight from Benin City airport

Flight Number	Destination
1	Benin to Abuja
2	Benin to Kaduna
3	Benin to Kano

3.1 Simulation and Results

A solution where flight 1 starts from sector 1 and ends in sector 2 is given by [1 0 0 0 0 0;0 1 0 0 0 0].

While a solution where flight 2 starts from sector 3 and ends in sector 4 is given by [0 0 1 0 0 0;0 0 0 1 0 0]. The solution where flight 3 starts from sector 5 and ends in sector 6 is [0 0 0 0 1 0; 0 0 0 0 0 1], while the solution where flight 1 and 2 leaves at the same time is given by [1 0 1 0 0 0], and a solution where flight 1 and 3 leaves at the same time is [1 0 0 0 1 0].

Given a maximum delay of 1 unit of time and using semidefinite relaxation, we find an approximate lower bound on the optimal solution of the scheduling problem. Recall that for an aircraft flying across an airspace composed of m sectors with capacities given by $C \in \boldsymbol{R}^m$. We decompose a particular day into 'T' periods, so that a particular flight route starting at time 's' can be represented by a matrix $R^{(i,s)} \in \boldsymbol{R}^{m \times T}$ such that:

$R_{jt}^{(i,s)} = 1$, if aircraft i is in sector j at time t
$R_{jt}^{(i,s)} = 0$, if not.
Also recall that our problem variables are $x \in \boldsymbol{R}^n$ and $X \in \boldsymbol{S}^n$ from the semidefinite relaxation of (3) where $n = N(d+1)$, N = Number of flights and d = delay (units of time). Therefore $n = 3(1+1) = 6$. T = day into periods of 3 minutes. Therefore, the matrix A when solved using eigenvalue optimization is $\boldsymbol{R}^{m\times T}$ which is $R^{6\text{x}3}$. The sector 'm' = 6 and each sector capacity is one (1). We can formulate the problem of minimizing total delay while satisfying capacity constraints as the linear sector capacity constraints is represented by (9).

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x \le 1 \tag{9}$$

3.1.1 Procedure 1

Applying the Lagrangian relaxation of the scheduling problem using the linear programming relaxation (4) which is easier to represent in CVX and given the following parameters:
$c = [0\ 1\ 0\ 1\ 0\ 1]$, where c is the objective vector which is the delay we are trying to minimize; $b = [1]$, and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Lagrangian relaxation gives a lower bound of

$$x = \begin{bmatrix} 0.0191 \\ 0.0000 \\ 0.0191 \\ 0.0000 \\ 0.0862 \\ 0.0000 \end{bmatrix}$$

Since there is no method of checking the validity of the obtained lower bound, we use the eigenvalue minimization which is equivalent to the semidefinite relaxation.

3.1.2 Procedure 2

Now, we employ the eigenvalue minimization of (7) which is an approximation of the semidefinite relaxation (3). Also, c = [0 1 0 1 0 1] is the objective vector which is the delay to be minimized; b = [1] and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using all the parameters and simulating with CVX toolbox in Matlab and using the parameters c, b and A we have:

$$x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \quad X = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, the fact that '$X = vec(x)*vec(x)^T$' and '$diag(X) = vec(x)$' shows that the relaxation is tight and that the solution x is approximately optimal on the global solution.

Increasing maximum delay given from 1 minute through 4 minutes, we find that for each trial we have the information presented in Table 2.

Table 2: Maximum Delays, CPU-time and optimal values for semidefinite and Lagrangian relaxations.

Maximum Delay(mins)	Total CPU time (secs)		Optimal value	
	Semidefinite	Lagrangian	Semidefinite	Lagrangian
1	0.91	1.5313	2.89421e-09	1.5986e-10
2	0.88	1.5156	1.11234e-09	6.0503e-11
3	0.87	1.4531	1.04159e-09	5.4182e-11
4	0.84	1.4375	1.06052e-09	1.5979e-11

The semidefinite relaxation has their lower bounds on their optimal value which was proven to be tight, also the Lagrangian relaxation have their lower bounds on their own optimal values which was not proven to be tight. The values gotten using the semidefinite relaxation is superior to that of the Lagrangian relaxation because the CPU time values from semidefinite is smaller than that of the Lagrangian relaxation. Now using these values from Table 2, a graph is presented in Figure 1 showing CPU time against maximum delay. It can be observed that while increasing the delay time from 1 to 4 minutes, the CPU time decreases.

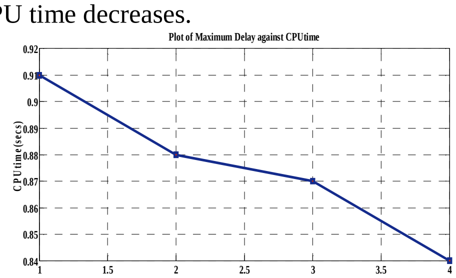

Figure 1: Plot of Maximum Delay against CPU-time

4.0 Conclusion

In this paper, the scheduling problem with sector constraints and delay was presented. This helps to prevent collisions, organize and expedite the flow of traffic, and provide information and necessary support to pilots. In conclusion, the equations representing the scheduling problem were implemented by the CVX which has the provision to solve for large problems as it calls the appropriate solver that can execute the spectral bundle method in its designated steps. This work provides an approximate solution to the scheduling problem which is dynamic in nature as it depends on time using the semidefinite relaxation technique which proved to be an efficient technique because it provides a condition to check for its optimality. This technique is applicable to bus scheduling, loading processes in bottling companies, scheduling of courier services and any other type of decision making in the industries.

5.0 References

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