Journal of the Nigerian Association of Mathematical Physics
Volume34,(March, 2016), pp199 – 208
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Comparison of the Invariant Solutions to the Multivariate Behrens-Fisher Problem

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Abstract

Four of numerous solutions to multivariate Behrens-Fisher problem were selected for comparison. The four selected invariant solutions were; Yao, Johansen, James and Krishnamoorthy and Yu. Data were simulated to compare the four solutions under different distributions (Multivariate Beta, Multivariate Gamma and Multivariate Normal), sample sizes (N = 20, 30, 50, 100, 200, 400 and 600), number of variables (p = 2, 3 and 5) and for equal and unequal sample sizes. The comparisons were done at three levels of significance (α = 0.01, 0.025 and 0.05) using power of the test and type I error rate. The results showed that James procedure is better than all other procedure when p = 2 with small sample sizes, because it has second highest power with the lowest type I error rate. But when number of variables p = 3 or 4 and with large sample sizes, all the four procedures' performances are the same.

Key words: Hotelling T^2, Multivariate Behrens-fisher problem, Power, Type I error rate

1.0 Introduction

The statistic used to test the hypothesis that two mean vectors are equal ($H_o: \mu_1 = \mu_2$) is Hotelling's T^2.

$$T^2 = \frac{n_1 n_2}{n_1+n_2}(\bar{x}_1 - \bar{x}_2)' {S_{pl}}^{-1}(\bar{x}_1 - \bar{x}_2) \qquad (1)$$

Where

$$S_{pl} = \frac{1}{n_1+n_2-2}[(n_1-1)S_1 + (n_2-1)S_2] \qquad (2)$$

and $\bar{x}_i$ and S_i are the sample mean vector and sample variance –covariance matrix of the *i*th sample.

Hotelling's T^2, has three basic assumptions that are fundamental to the statistical theory; independent, multivariate normality and equality of variance-covariance matrices [1, 2]. A statistical test procedure is said to be robust or insensitive if departures from these assumptions do not greatly affect the significance level or power of the test.

To use Hotelling's T^2, one must assume that the two samples are independent and that their variance-covariance matrices are equal($\Sigma_1 = \Sigma_2 = \Sigma$). When variance –covariance matrices are not homogeneous, the test statistic will not be distributed as a T^2. This predicament is known as the multivariate Behrens-Fisher problem[3].

The Behrens-Fisher Problem is the problem of interval estimation and hypothesis testing concerning the differences between the means of two normally distributed populations when the variances of the two populations are not equal. Multivariate Behrens-Fisher problem deal with testing the equality of two normal mean vectors under heteroscedasticity of dispersion matrices

The problem of comparing independent sample means arising from two populations with unequal variances has been studied for many years and there is a sizable literature. Historically, this problem has come to be known as the Behrens-fisher problem. The comparison of the means of two populations on the basis of two independent samples is one of the oldest problems in statistics. Indeed, it has been a testing ground for many methods of inference as well as for a variety of analytic approaches to practical problems.

Yao [4]test procedure's Type I error rate was lower than that of James in almost all cases. Subrahmaniam and Subrahmaniam[5, 6] compared the tests of Bennett, James and Yao and concluded that though Bennett's procedure achieves exact protection of the level of significance, its power was "extremely poor, particularly for unequal sample sizes". They also found that James' test had the highest power among the three solutions but had a higher Type I error rate than Yao's test, particularly when the smaller sample is associated with the larger variance-covariance matrix.

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Alginaand Tang [7] and Algina, Oshima and Tang [8] verified the results of [4]. Algina, Oshima and Tang [8] included Johansen's solution in their simulation and found its Type I error rate to be roughly equivalent to that of Yao's solution. It was shows that the Type I error rates for Johansen's test improve as the ratio of the smaller sample size to the number of variables increases.

Kim[9] reported that the Type I error rate for his test was almost always more conservative than that of Yao's test. He showed that there was no apparent difference in power (after adjusting for the Type I error rate) between the two procedures when the smaller sample was associated with the smaller variance-covariance matrix. However, Kim's procedure had higher power than Yao's (after adjusting for the Types I error rate) when the smaller sample was associated with the larger variance-covariance matrix.

De la Rey and Nel's[10] comparisons of Bennett's, James', Yao's and Nel and van der Merwe's and Yao's stand out as better solutions. James' test showed the highest power, but its significance levels were usually high, especially as the dimensionality (p) increased.

2.0 Methodology

This study would only compare the solutions that are invariant to nonsingular transformations of the data namely: Yao, Johansen, James and Krishnamoorthy and Yu. Where $\tilde{S}_i = S_i/n_i,\ i = 1,2, \tilde{S} = \tilde{S}_1 + \tilde{S}_2,\ T^2 = (\bar{X}_1 - \bar{X}_2)'\tilde{S}^{-1}(\bar{X}_1 - \bar{X}_2)$

1. Yao

   Yao [4] invariant test. This is a multivariate extension of the Welch 'approximate degree of freedom' solution provided by Turkey. And is based on $T^2 \sim (vp/(v-p+1))F_{p,v-p+1}$ with the *d.f.v* given by

$$\frac{1}{v} = \frac{1}{(T^2)^2}\sum_{i=1}^{2}\frac{1}{n_i-1}\left[(\bar{x}_1 - \bar{x}_2)'\tilde{S}^{-1}\tilde{S}_i\tilde{S}^{-1}(\bar{x}_1 - \bar{x}_2)\right]^2 \qquad (3)$$

2. Johansen

   Johansen [11] invariant test. We use $T^2 \sim qF_{pv}$ where

$$q = p + 2D - 6D/[p(p-1) + 2],\ v = p(p+2)/3D \qquad (4)$$

$$D = \frac{1}{2}\sum_{i=1}^{2}\left\{tr\left[(I - \left(\tilde{S}_1^{-1} + \tilde{S}_2^{-1}\right)^{-1}\tilde{S}_i^{-1})^2\right] + tr\left[\left(I - \left(\tilde{S}_1^{-1} + \tilde{S}_2^{-1}\right)^{-1}\tilde{S}_i^{-1}\right)\right]^2\right\}/n_i \qquad (5)$$

3. James

   James[12] test for equality of mean vector involves a correction for χ^2 critical values. The statistic T^2 has approximate critical value as ${\chi^2}_{\alpha,p}(A + B\chi^2_{\alpha,p})$, where ${\chi^2}_{\alpha,p}$ is the upper α quantile of the chi-square distribution with p degrees of freedom. The values of A and B are given by

$$A = 1 + \frac{1}{2p}\sum_{i=1}^{2}\frac{1}{n_i-1}\left[tr\left(\tilde{S}^{-1}\tilde{S}_i\right)\right]^2 \qquad (6)$$

$$B = \frac{1}{2p(p+2)}\sum_{i=1}^{2}\frac{1}{n_i-1}\left\{tr\left[2\left(\tilde{S}^{-1}\tilde{S}_i\right)^2\right] + \left[tr\left(\tilde{S}^{-1}\tilde{S}_i\right)\right]^2\right\} \qquad (7)$$

4. Krishnamoorthy and Yu

   Krishnamoorthy and Yu [13, 14] modified Nel/ Van der Merwe invariant solution. We use the idea as before, namely,

   $T^2 \sim (vp/(v-p+1))F_{p,v-p+1}$with the *d.f.* v defined by

$$v_{KY} = (p + p^2)/C\left(\tilde{S}_1, \tilde{S}_2\right)$$

$$\left(\tilde{S}_1, \tilde{S}_2\right) = \frac{1}{n_1}\left\{tr\left[\left(\tilde{S}_1\tilde{S}^{-1}\right)^2\right] + \left[tr\left(\tilde{S}_1\tilde{S}^{-1}\right)\right]^2\right\} + \frac{1}{n_2}\left\{tr\left[\left(\tilde{S}_2\tilde{S}^{-1}\right)^2\right] + \left[tr\left(\tilde{S}_2\tilde{S}^{-1}\right)\right]^2\right\} \quad (8)$$

2.1 Testing Equality of Variance-Covariance

For a test of equality of variance-covariance matrices, we used the statistic

$$M = (N-g)log|S| - \sum_{i=1}^{g} v_i log|S_i|$$

S is the pooled-within estimate of the variance –covariance matrix and g denotes the number of groups (populations)

$$S = \frac{1}{N-g}\sum_{i=1}^{g} v_i S_i$$

Where

$$N = \sum_{i=1}^{g} N_i$$

Anderson [15] and Kullback[16] used this statistic to test equality of variance-covariance and however, multiplying M by 1 – C, where;

$$C = \frac{2p^2 + 3p - 1}{6(p+1)(g-1)}\left(\sum_{i=1}^{g} \frac{1}{v_i} - \frac{1}{N-g}\right)$$

$$\chi_B^2 = (1 - C)M$$

More rapidly approximates a chi-square distribution with degrees of freedom

$$v = \frac{p(p-1)(g-1)}{2}$$

H_o is rejected at the significance level α if $\chi_B^2 > \chi_{\alpha(v)}^2$

3.0 Simulation

A simulation using R statistics was conducted in order to estimate the Type I error rate and power for each of the previously discussed approximate solutions (James, Johanson, Yao and Krishnamoorthy). The distributions considered were; Multivariate Normal, Multivariate Beta (Dirichlet) and Multivariate Gamma. Small (N = 20, 30), medium (N = 50, 80, 100) and large (N = 150, 400, 600) samples were also considered and the dimensionality (p) used were p = 2, 3 and 4. For each of the above combinations, an n_1 x p data matrix X_1 and n_2 x p data matrix X_2 were replicated 1,000. The comparison criteria; type I error rate and power of the test were therefore obtained and the results were presented in Tables 1 to 6.

Table 1: Power of the test for Multivariate Beta

			α=0.01				α=0.025				α=0.05			
			Jam	John	Yao	Kris	Jam	John	Yao	Kris	Jam	John	Yao	kris
Unequal sample($n_1 \neq n_2$)	P=2	20,30	0.317	0.662	0.147	0.147	0.432	0.746	0.233	0.233	0.526	0.799	0.320	0.320
		50,60	0.603	0.877	0.314	0.314	0.709	0.922	0.429	0.429	0.788	0.950	0.533	0.533
		100,120	0.875	0.983	0.595	0.594	0.924	0.992	0.697	0.696	0.947	0.995	0.769	0.769
		400,600	1.000	1.000	0.997	0.997	1.000	1.000	0.999	0.999	1.000	1.000	0.999	0.999
	P=3	20,30	0.481	0.924	0.132	0.132	0.591	0.951	0.212	0.211	0.690	0.971	0.300	0.299
		50,60	0.796	0.990	0.287	0.286	0.866	0.994	0.405	0.405	0.906	0.996	0.502	0.502
		100,120	0.963	0.999	0.545	0.544	0.977	0.999	0.653	0.652	0.987	0.999	0.739	0.739
		400,600	0.999	1.000	0.981	0.981	1.000	1.000	0.990	0.990	0.999	1.000	0.993	0.993
	P=4	20,30	0.689	0.995	0.141	0.140	0.784	0.997	0.222	0.222	0.837	0.999	0.314	0.315
		50,60	0.962	1.000	0.357	0.356	0.977	1.000	0.477	0.476	0.988	1.000	0.584	0.583
		100,120	0.999	1.000	0.693	0.692	0.999	1.000	0.786	0.785	0.999	1.000	0.853	0.852
		400,600	1.000	1.000	0.999	0.999	1.000	1.000	0.999	0.999	1.000	1.000	1.000	1.000
Equal sample($n_1 = n_2$)	P=2	30,30	0.337	0.681	0.157	0.157	0.481	0.796	0.261	0.261	0.556	0.822	0.342	0.343
		80,80	0.773	0.958	0.457	0.456	0.824	0.964	0.557	0.556	0.886	0.979	0.657	0.656
		150,150	0.964	0.998	0.773	0.773	0.969	0.997	0.813	0.812	0.999	0.999	0.887	0.887
		600,600	1.000	1.000	0.999	0.999	1.000	1.000	0.999	0.999	1.000	1.000	0.999	0.999
	P=3	30,30	0.550	0.945	0.154	0.155	0.663	0.969	0.240	0.241	0.747	0.978	0.334	0.336
		50,50	0.913	0.999	0.418	0.417	0.946	0.999	0.532	0.532	0.963	0.999	0.632	0.631
		150,150	0.985	0.999	0.679	0.679	0.992	0.999	0.772	0.772	0.995	0.999	0.845	0.845
		600,600	1.000	1.000	0.989	0.989	0.999	1.000	0.994	0.994	1.000	1.000	0.997	0.997
	P=4	30,30	0.797	0.999	0.184	0.184	0.865	0.999	0.279	0.280	0.909	0.999	0.381	0.382
		80,80	0.994	1.000	0.533	0.532	0.997	1.000	0.661	0.660	0.999	1.000	0.745	0.745
		150,150	1.000	1.000	0.841	0.841	1.000	1.000	0.906	0.906	1.000	1.000	0.941	0.941
		600,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 2: Type I error rate of the test for Multivariate Beta

			α=0.01				α=0.025				α=0.05			
			Jam	John	Yao	Kris	Jam	John	Yao	Kris	Jam	John	Yao	kris
Unequal sample($n_1 \neq n_2$)	P=2	20,30	0.168	0.873	0.167	0.165	0.309	0.920	0.304	0.307	0.444	0.940	0.439	0.444
		50,60	0.586	0.986	0.586	0.586	0.735	0.989	0.735	0.736	0.846	0.997	0.846	0.846
		100,120	0.934	1.000	0.934	0.934	0.972	1.000	0.975	0.975	0.987	1.000	0.987	0.987
		400,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	P=3	20,30	0.133	0.994	0.325	0.316	0.487	0.996	0.477	0.480	0.633	1.000	0.628	0.625
		50,60	0.820	1.000	0.818	0.817	0.902	1.000	0.903	0.902	0.952	0.999	0.952	0.952
		100,120	0.992	1.000	0.991	0.992	0.995	1.000	0.996	0.995	0.999	1.000	0.999	0.999
		400,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	P=4	20,30	0.617	1.000	0.592	0.593	0.768	1.000	0.753	0.758	0.827	1.000	0.816	0.820
		50,60	0.991	1.000	0.991	0.991	0.993	1.000	0.993	0.993	1.000	1.000	1.000	1.000
		100,120	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		400,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Equal sample($n_1=n_2$)	P=2	30,30	0.190	0.900	0.180	0.190	0.370	0.940	0.370	0.370	0.430	0.970	0.440	0.430
		80,80	0.810	1.000	0.800	0.810	0.860	1.000	0.860	0.860	0.940	0.990	0.940	0.940
		150,150	0.990	1.000	0.990	0.990	0.990	1.000	0.990	0.990	1.000	1.000	1.000	1.000
		600,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	P=3	30,30	0.434	0.998	0.429	0.430	0.611	1.000	0.599	0.607	0.739	0.999	0.731	0.738
		80,80	0.965	1.000	0.965	0.965	0.989	1.000	0.991	0.989	0.991	1.000	0.991	0.991
		150,150	0.997	1.000	0.997	0.997	0.999	1.000	0.999	0.999	0.999	1.000	0.999	0.999
		600,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	P=4	30,30	0.795	1.000	0.788	0.785	0.880	1.000	0.870	0.876	0.939	1.000	0.935	0.93
		80,80	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		150,15	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		600,60	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 3: Power of the test for Multivariate Gamma

			α=0.01				α=0.025				α=0.05			
			Jam	John	Yao	Kris	Jam	John	Yao	Kris	Jam	John	Yao	kris
Unequal sample($n_1 \neq n_2$)	P=2	20,30	0.096	0.267	0.049	0.050	0.159	0.355	0.092	0.093	0.223	0.424	0.142	0.142
		50,60	0.122	0.285	0.058	0.058	0.185	0.366	0.101	0.101	0.258	0.444	0.158	0.158
		100,120	0.164	0.344	0.075	0.075	0.244	0.443	0.129	0.129	0.317	0.515	0.187	0.187
		400,600	0.424	0.682	0.205	0.205	0.533	0.761	0.301	0.301	0.605	0.807	0.381	0.381
	P=3	20,30	0.245	0.683	0.071	0.070	0.339	0.749	0.126	0.124	0.437	0.810	0.191	0.190
		50,60	0.441	0.830	0.126	0.125	0.545	0.878	0.203	0.202	0.647	0.913	0.289	0.289
		100,120	0.711	0.961	0.249	0.248	0.777	0.972	0.339	0.339	0.843	0.984	0.447	0.446
		400,600	0.999	1.000	0.875	0.875	0.999	1.000	0.923	0.923	0.999	1.000	0.949	0.949
	P=4	20,30	0.219	0.795	0.044	0.044	0.330	0.856	0.089	0.088	0.414	0.888	0.141	0.140
		50,60	0.405	0.890	0.075	0.075	0.507	0.924	0.128	0.128	0.586	0.942	0.193	0.193
		100,120	0.632	0.971	0.129	0.129	0.724	0.984	0.208	0.208	0.779	0.986	0.289	0.289
		400,600	0.997	1.000	0.633	0.632	0.999	1.000	0.722	0.721	0.999	1.000	0.798	0.798
Equal sample($n_1=n_2$)	P=2	30,30	0.092	0.252	0.048	0.048	0.157	0.348	0.090	0.091	0.227	0.426	0.144	0.144
		80,80	0.131	0.312	0.064	0.064	0.207	0.393	0.112	0.112	0.286	0.484	0.171	0.171
		150,150	0.192	0.392	0.086	0.086	0.275	0.486	0.145	0.145	0.348	0.548	0.207	0.207
		600,600	0.470	0.734	0.232	0.232	0.574	0.798	0.328	0.328	0.649	0.841	0.419	0.419
	P=3	30,30	0.285	0.699	0.079	0.079	0.385	0.776	0.138	0.138	0.480	0.826	0.206	0.206
		80,80	0.587	0.916	0.185	0.185	0.683	0.944	0.271	0.271	0.534	0.959	0.362	0.362
		150,150	0.833	0.987	0.344	0.344	0.881	0.991	0.446	0.446	0.915	0.993	0.548	0.548
		600,60	0.999	1.000	0.981	0.981	1.000	1.000	0.966	0.966	1.000	1.000	0.981	0.981
	P=4	30,30	0.249	0.804	0.048	0.049	0.346	0.850	0.091	0.091	0.449	0.888	0.149	0.149
		80,80	0.501	0.936	0.093	0.094	0.625	0.964	0.166	0.167	0.701	0.969	0.241	0.242
		150,150	0.749	0.992	0.172	0.172	0.817	0.993	0.261	0.262	0.874	0.996	0.359	0.359
		600,600	0.999	1.000	0.723	0.723	0.999	1.000	0.819	0.819	0.999	1.000	0.869	0.869

Table 4: Type I error rate of the test for Multivariate Gamma

			α=0.01				α=0.025				α=0.05			
			Jam	John	Yao	Kris	Jam	John	Yao	Kris	Jam	John	Yao	kris
Unequal sample($n_1 \neq n_2$)	P=2	20,30	0.016	0.392	0.016	0.016	0.034	0.493	0.031	0.032	0.057	0.538	0.057	0.057
		50,60	0.023	0.427	0.023	0.023	0.045	0.510	0.045	0.045	0.079	0.579	0.079	0.079
		100,120	0.052	0.519	0.052	0.052	0.096	0.623	0.096	0.096	0.138	0.666	0.138	0.138
		400,600	0.323	0.870	0.323	0.323	0.458	0.907	0.458	0.458	0.554	0.935	0.554	0.554
	P=3	20,30	0.099	0.910	0.098	0.095	0.170	0.929	0.168	0.165	0.258	0.958	0.259	0.255
		50,60	0.306	0.978	0.306	0.306	0.418	0.981	0.420	0.418	0.571	0.985	0.572	0.570
		100,120	0.690	1.000	0.692	0.689	0.780	0.999	0.781	0.780	0.856	0.999	0.856	0.856
		400,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	P=4	20,30	0.058	0.986	0.055	0.054	0.136	0.983	0.130	0.127	0.192	0.987	0.189	0.185
		50,60	0.212	0.995	0.211	0.211	0.302	0.993	0.297	0.298	0.395	0.999	0.395	0.395
		100,120	0.514	0.999	0.514	0.514	0.642	1.000	0.642	0.642	0.714	1.000	0.714	0.714
		400,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Equal sample($n_1 = n_2$)	P=2	30,30	0.012	0.379	0.012	0.012	0.031	0.048	0.031	0.031	0.064	0.535	0.063	0.063
		80,80	0.028	0.478	0.028	0.028	0.061	0.554	0.061	0.061	0.101	0.643	0.101	0.101
		150,150	0.063	0.573	0.062	0.063	0.116	0.673	0.116	0.116	0.192	0.709	0.192	0.192
		600,600	0.385	0.912	0.385	0.385	0.546	0.932	0.546	0.546	0.616	0.953	0.616	0.616
	P=3	30,30	0.140	0.917	0.135	0.138	0.209	0.936	0.206	0.208	0.308	0.958	0.303	0.304
		50,50	0.486	0.994	0.486	0.485	0.640	0.997	0.640	0.639	0.730	0.995	0.732	0.730
		150,150	0.882	0.999	0.862	0.862	0.907	1.000	0.907	0.907	0.954	1.000	0.954	0.954
		600,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	P=4	30,30	0.083	0.987	0.078	0.081	0.127	0.984	0.122	0.123	0.226	0.988	0.218	0.223
		80,80	0.323	0.998	0.323	0.323	0.462	0.998	0.460	0.462	0.594	0.999	0.592	0.593
		150,150	0.701	1.000	0.700	0.700	0.798	1.000	0.798	0.798	0.888	1.000	0.888	0.888
		600,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 5: Power of the test for Multivariate Normal

			α=0.01				α=0.025				α=0.05			
			Jam	John	Yao	Kris	Jam	John	Yao	Kris	Jam	John	Yao	kris
Unequal sample($n_1 \neq n_2$)	P=2	20,30	0.118	0.349	0.059	0.059	0.195	0.455	0.110	0.110	0.260	0.522	0.110	0.111
		50,60	0.172	0.395	0.079	0.079	0.250	0.476	0.134	0.134	0.331	0.554	0.200	0.200
		100,120	0.269	0.514	0.124	0.124	0.333	0.571	0.177	0.177	0.418	0.640	0.251	0.251
		400,600	0.701	0.909	0.402	0.402	0.780	0.937	0.512	0.512	0.835	0.955	0.603	0.603
	P=3	20,30	0.294	0.754	0.079	0.081	0.409	0.823	0.139	0.147	0.503	0.860	0.212	0.221
		50,60	0.589	0.919	0.181	0.187	0.696	0.953	0.269	0.276	0.769	0.968	0.373	0.379
		100,120	0.864	0.993	0.375	0.382	0.905	0.994	0.491	0.497	0.942	0.998	0.589	0.594
		400,600	1.000	1.000	0.969	0.970	1.000	1.000	0.985	0.985	1.000	1.000	0.991	0.991
	P=4	20,30	0.747	0.996	0.173	0.174	0.816	0.997	0.263	0.264	0.879	0.998	0.374	0.375
		50,60	0.974	1.000	0.430	0.432	0.984	1.000	0.547	0.548	0.992	1.000	0.657	0.658
		100,120	0.999	1.000	0.813	0.813	1.000	1.000	0.877	0.877	1.000	1.000	0.923	0.923
		400,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Equal sample($n_1 = n_2$)	P=2	30,30	0.123	0.352	0.061	0.061	0.181	0.418	0.102	0.102	0.266	0.506	0.167	0.167
		80,80	0.209	0.430	0.094	0.094	0.296	0.535	0.158	0.158	0.379	0.602	0.227	0.277
		150,150	0.318	0.571	0.148	0.148	0.414	0.663	0.224	0.224	0.479	0.698	0.289	0.289
		600,600	0.769	0.938	0.471	0.471	0.833	0.959	0.576	0.576	0.882	0.974	0.669	0.669
	P=3	30,30	0.375	0.806	0.103	0.107	0.497	0.863	0.178	0.184	0.593	0.902	0.257	0.262
		50,50	0.776	0.982	0.285	0.291	0.840	0.986	0.393	0.399	0.891	0.993	0.501	0.506
		150,150	0.957	0.999	0.542	0.547	0.975	0.999	0661	0.665	0.984	0.999	0.734	0.737
		600,600	1.000	1.000	0.999	0.999	1.000	1.000	0.999	0.999	1.000	1.000	0.997	0.997
	P=4	30,30	0.802	0.998	0.204	0.209	0.863	0.999	0.302	0.307	0.905	0.999	0.400	0.404
		80,80	0.997	1.000	0.619	0.621	0.998	1.000	0.729	0.731	0.999	1.000	0.796	0.797
		150,150	1.000	1.000	0.916	0.916	1.000	1.000	0.952	0.953	1.000	1.000	0.973	0.973
		600,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 6: Type I error rate of the test for Multivariate Gamma

			α=0.01				α=0.025				α=0.05			
			Jam	John	Yao	Kris	Jam	John	Yao	Kris	Jam	John	Yao	kris
Unequal sample($n_1 \neq n_2$)	P=2	20,30	0.020	0.500	0.020	0.020	0.059	0.606	0.057	0.059	0.096	0.663	0.057	0.059
		50,60	0.050	0.575	0.050	0.050	0.113	0.645	0.113	0.113	0.167	0.690	0.167	0.167
		100,120	0.134	0.706	0.134	0.134	0.188	0.744	0.187	0.187	0.270	0.799	0.270	0.270
		400,600	0.710	0.985	0.710	0.710	0.818	0.989	0.818	0.818	0.885	0.995	0.885	0.885
	P=3	20,30	0.135	0.953	0.122	0.134	0.249	0.969	0.213	0.244	0.345	0.973	0.323	0.343
		50,60	0.509	0.995	0.484	0.509	0.670	0.995	0.657	0.670	0.750	0.999	0.743	0.749
		100,120	0.900	1.000	0.888	0.900	0.928	1.000	0.927	0.928	0.975	1.000	0.974	0.975
		400,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.969	0.970
	P=4	20,30	0.694	1.000	0.681	0.682	0.785	1.000	0.772	0.777	0.893	1.000	0.884	0.890
		50,60	0.998	1.000	0.988	0.988	0.993	1.000	0.991	0.993	0.998	1.000	0.998	0.998
		100,120	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		400,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
Equal sample($n_1 = n_2$)	P=2	30,30	0.025	0.516	0.023	0.023	0.052	0.564	0.052	0.052	0.120	0.639	0.120	0.120
		80,80	0.075	0.621	0.075	0.075	0.141	0.731	0.139	0.140	0.229	0.750	0.229	0.229
		150,150	0.196	0.770	0.196	0.196	0.275	0.836	0.275	0.275	0.388	0.845	0.388	0.388
		600,600	0.798	0.996	0.798	0.798	0.878	0.999	0.878	0.878	0.934	0.996	0.934	0.934
	P=3	30,30	0.216	0.958	0.192	0.210	0.360	0.975	0.339	0.356	0.472	0.982	0.460	0.469
		80,80	0.771	1.000	0.765	0.770	0.867	0.999	0.864	0.865	0.925	1.000	0.921	0.925
		150,150	0.984	1.000	0.984	0.984	0.991	1.000	0.991	0.991	0.996	1.000	0.996	0.996
		600,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	P=4	30,30	0.788	1.000	0.769	0.783	0.863	0.999	0.302	0.306	0.905	0.999	0.400	0.404
		80,80	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		150,150	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		600,600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

4.0 Discussion of Results and Conclusions

For all values of p for Multivariate Beta (Dirichlet Distribution),Johanson procedure has the highest power and is followed by James throughout the three significance levels: 0.01, 0.025 and 0.05 for the two cases (n_1=n_2 and $n_1 \neq n_2$). When the sample size increases to 400:600 and 600:600, the four procedures have the same or roughly equivalent power. When p = 4, the power of James and Johanson are the same for the following sample sizes 100:120, 400:600, 150:150 and 600:600 (Table 1).
Considering type I error rate in Multivariate Beta Distribution (Dirichlet Distribution), Johanson procedure has highest type I error rate while the remaining three procedures performed almost alike for all values of p and significance levels $(\alpha)$for small sample sizes.When p =4, the four procedures performed the same as the sample size increases (100:120, 400:600, 80:80, 150:150 and 600:600) for all α levels considered for both cases (n_1=n_2 and $n_1 \neq n_2$), (Table 2).
For Multivariate Gamma Distribution, Johanson procedure has the highest power, followed by James while Yao and Krishnamoorthybehaviours are almost the same for all values of p and α. But when p=3 or 4 and large sample sizes(400:600 and 600:600), James and Johanson are the same and better than Yao and Krishnamoorthy in the two cases (Table 3).
Type I error rate in Multivariate Gamma, Johanson procedure has the highest while others (James, Yao and krishnamoorthy) are almost the same for all values of p and α. But when p= 3 or 4 and for large sample sizes (400:600 and 600:600), the performance of the four procedures are the same in the two cases (Table 4)
In Multivariate Normal, Johanson procedure has the highest power and is followed by James while other two have almost the same values of power in the two cases (equal and unequal sample sizes) for all values of p and significance levels(α). But when p = 4 and for medium sample sizes (100:120, 150:150), James and Johanson behave alike while for large sample sizes (400:600 and 600:600), the performance of the four procedures are the same (Table 5)
For type I error rate of Multivariate Normal, the Johanson procedure has highest type I error rate while the remaining three procedures performed almost the same for all values of p and significance levels (α). But when p = 3 or 4 the four procedures performed the same at medium and large sample sizes(100:120, 400:600, 80:80, 150:150 and 600:600) (Table 6).
In the three distributions considered, the power of the four procedures behaves alike; the power increases with dimension (p) and sample sizes. Though the powers obtained for the four procedures in Multivariate Beta are higher than that of Multivariate Normal with Multivariate Gamma having the least power. Among the four procedures across the three distributions, James and Johanson have the highest power in that order. Therefore, chance of committing type I error rate reduces in James and Johanson procedures as the sample sizes and number of variables (p) increases in the three distributions.
Type I error rate of James, Yao, and krishnamoorthy in the three distributions are almost the same and are always smaller than that of Johanson for all values of p and α.When the power equal to one, it shows that we did not commit type II error at all, that is, we reject a false null hypothesis where necessary. Also when type I error rate equal to one, it shows that throughout the simulation, type I error is committed,that is, an incorrect rejection of a true null hypothesis. Therefore, Johanson is not the best procedure in the three distributions which means whether the assumption of normality holds or not, it is not the best, because Johanson has the highest power and the highest type I error rate.
James procedure is better and preferable toall other procedures, because it has second highest power with a low type I error rate.

5.0 References

[1] Fisher, R. A. (1935).The Fiducial Argument in Statistical Inference.*Annals of Eugenics,* 6, 391 – 398
[2] Fisher, R. A. (1939).The Comparison of Samples with Possibly Unequal Variances, *Annals of Eugenics,* 9.174 – 180.
[3] Kramer, C.Y. and Jensen, D.R.(1969). Fundamentals of Multivariate Analysis, Part II. Inference about TwoTreatments, *Journal of Quality of Technology* 1,189 – 204.
[4] Yao, Y. (1965). An approximation degrees of freedom solution to the multivariate Behrens – Fisher problem, *Biometrika* 52, 139 – 147.
[5] Subrahmaniam, K. and Subrahmaniam, K. (1973). "On the Multivariate Behrens-Fisher Problem," *Biometrika,* 60, 107 – 111.
[6] Subrahmaniam, K. and Subrahmaniam, K. (1975)."On the Confidence RegionComparison of Some Solutions for the Multivariate Behrens-Fisher Problem," *Communications in Statistics*,4, 57 – 67.
[7] Algina, J., and Tang, K. L. (1988)."Type I Error Rates for Yao's and James' Tests of Equality of Mean Vectors Variance-Covariance Heterscedasticity,"*Journal of Educational Statistics,* 13, 281 - 290.
[8] Algina, J., Oshima, T. C., and Tang, K. L. (1991). "Robustness of Yoa's, James'and Johansen's Tests underVariance-Covariance Heteroscedasticity and Nonnormality,"*Journal of Educational Statistics*, 16, 125-139.
[9] Kim, S. (1992). "A Practical Solution to the Multivariate Behrens – Fisher Problem", *Biometrika*, 79, 171 – 176.
[10] De la Rey, N., and Nel, D. G. (1993). "A Comparison of the Significance Levels and Power Functions of Several Solutions to the Multivariate Behrens-Fisher Problem," *South African Statistical Journal,* 27, 129 – 148.

[11] Johansen, S. (1980). The Welch – James Approximation to the Distribution of the Residual Sum of Squaresin a Weighted Linear Regression *Biometrika* 67, 85 – 95.
[12] James,G.S (1954).”Tests of Linear Hypothesis in Univariate and Multivariate Analysis When the Ratios ofthe Population Variance are Unknown,” *Biometrika*, 41, 19 – 43.
[13] Krishnamoorthy, K. and Yu, J. (2004). Modified Nel and Van der Merwe test for the multivariate Behrens –Fisher problem, *Statistica and probability letters* 66, 161 – 169
[14] Krishnamoorthy, K. and Yu, J.(2012). Multivariate Behrens-Fisher Problem with Missing Data.*Journal of Multivariate Anal.*,105, 141 -150.
[15] Anderson, T.W. (2003). An Introduction to Multivariate Statistical Analysis. Third edition. Wiley-Interscience
[16] Kullback, S. (1959).*Information Theory and Statistics*. John Wiley and Sons