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A Comparative Study of theType 1 Error Rates of Some Multivariate Tests of Normality

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Abstract

Several developed multivariate tests of normality including Skewness (S), Kurtosis (K), Mardia Skewness (MS), Mardia Skewness for small sample (MSS), Mardia Kurtosis (MK), Shapiro-Francia (SF), Shapiro-Wilk (SW), Royston (R), Henze-Zirkler (HZ), Doornik-Harsen (DH), Energy (E), Bontemps-Meddahi (BM), Gel-Gastwirth (GG) and Desgagne-Micheaux (DM) rarely lead to the same conclusion when applied to a set of multivariate data. Consequently, the Type 1 error rates of these tests were examined at five levels of dimension and eight levels of sample size through Monte Carlo study so as to identify the good ones that is recommendable for use. The results were compared at three levels of significance. A test is considered good if its estimated error rate approximates the true error rate and best if it has the highest number of times (Mode) it approximates the error rate when counted over the levels of significance. Results showed that the Type 1 error rates of all the tests of multivariate normality are seldomly the same at the levels of significance. The error rates of HZ, MSS, R and E; MS and E; and MK, MS, S, HZ and E are respectively good at 0.1, 0.05 and 0.01 levels of significance. Moreover, those of E and HZ are comparatively best therefore recommended to practitioners.

Key words: Multivariate Normality Test, Type 1 error rate, Level of Significance.

1.0 Introduction

Manymultivariate parametric statistical data analysis methods including Multivariate Analysis of Variance (MANOVA), Multivariate Analysis of Covariance (ANCOVA), Multivariate Regression Analysis, Seemingly Unrelated Regression (SURE) Analysis and Discriminant Analysis require that the error term of the model should be multivariate normal.Violation of this assumption often results to invalid inference(s) and misinterpretation of results[1]. Several multivariate normality tests including Skewness (S) and Kurtosis (K)[2], Mardia Skewness (MS), Mardia Skewness for small sample (MSS) and Kurtosis (MK)[3], Shapiro-Francia (SF)[4],Shapiro-Wilk (SW)[5], Royston (R)[6], Henze-Zirkler (HZ)[7], Doornik-Harsen (DH)[8], Energy (E)[9], Bontemps-Meddahi (BM)[10], Gel-Gastwirth (GG)[11] and Desgagne-Micheaux (DM)[12] have been developed.A major challenge associated with the use of these tests is that their results frequently lead to different conclusions. Consequently, this study was undertaken to determine the Type 1 error rates of the multivariate testsof normality, identify the good ones and recommend appropriately.

2.0 Reviews on Multivariate Normality Tests

Several procedures have been developed for assessing multivariate normality status of a data set. Some of these methods are discussed below.

2.1 Henze-Zirkler Test of Multivariate Normality

Henze and Zirkler[7] proposed a class of invariant consistent tests for testing multivariate normality. The Henze-Zirkler test is given as:

$$T_{n,\beta}=\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\left[\exp\left(-\frac{\beta^2}{2}\left\|Y_j-Y_k\right\|^2\right)\right]-2(1+\beta^2)^{-p/2}\sum_{j=1}^{n}\left[\exp\left(-\frac{\beta^2}{2(1+\beta^2)}\left\|Y_j\right\|^2\right)\right]+n(1+2\beta^2)^{-p/2} \qquad (1)$$

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where $\beta = \frac{1}{\sqrt{2}}\left(\frac{2p+1}{4}\right)^{1/(p+4)} n^{1/(p+4)}$, $\left\|Y_j - Y_k\right\|^2 = (X_j - X_k)^T S^{-1}(X_j - X_k)$

$\left\|Y_j\right\|^2 = (X_j - \overline{X}_n)^T S^{-1}(X_j - \overline{X}_n)^T$

$T_{n,\beta}$ is log normally distributedwith mean and variance defined as follows:

$$E\left[T_\beta\right] = 1 - \left(1 + 2\beta^2\right)^{-p/2}\left[\frac{1 + p\beta^2}{1 + 2\beta^2} + \frac{p(p+2)\beta^4}{2(1 + 2\beta^2)^2}\right]$$

$$Var\left[T_\beta\right] = 2\left(1 + 4\beta^2\right)^{-p/2} + 2(1 + 2\beta^2)^{-p}\left[\frac{1 + 2p\beta^4}{(1 + 2\beta^2)^2} + \frac{3p(p+2)\beta^8}{4(1 + 2\beta^2)^4}\right]$$

$$- 4w(\beta)^{-p/2}\left[1 + \frac{3p\beta^4}{2w(\beta)} + \frac{p(p+2)\beta^8}{2w(\beta^2)}\right]$$

where $w(\beta) = (1 + \beta^2)(1 + 3\beta^2)$.

2.2 Generalized Shapiro-Wilk Test of Multivariate Normality

The generalized Shapiro-Wilk test W_p is a modification of the Shapiro-Wilk test W [5]for a multivariate case. The test according to [13]is given as:

$$W_p = \frac{\left\{\sum_{i=1}^{h} a_{(i,n)}\left(U_{(n-i+1)} - U_{(i)}\right)\right\}^2}{(X_m - \overline{X})^T A^{-1}(X_j - \overline{X})^T} \tag{2}$$

where $A = \sum_{j=1}^{n}\left(X_j - \overline{X}\right)\left(X_j - \overline{X}\right)^T$, $U_j = (X_m - \overline{X})^T A^{-1}(X_j - \overline{X})$ $for\ j = 1,2,\cdots,n$, $h = \frac{n}{2}$ if n is even and $h = \frac{(n-1)}{2}$ if n is odd, $a_{(i,n)}(i = 1,2,\cdots,h)$.

The test could be compared with a quantile of the rank α of the Shapiro-Wilk distribution as thus; Reject the hypothesis if $W_p < W_p^\alpha$ where W_p^α is the critical value[13].

2.3 Mardia's Measures of Multivariate Kurtosis and Skewness

Mardia[3] extended the concepts of kurtosis and skewness from univariate case to the multivariate case. He also obtained the asymptotic distribution of the multivariate kurtosis and skewness parameters which are needed to test the null hypothesis of multivariate normality[14].

The author defined the multivariate kurtosis coefficient as follows:

$$k = \frac{1}{n}\sum_{i=1}^{n}\left[\left(X_i - \overline{X}_n\right)^T S^{-1}\left(X_i - \overline{X}_n\right)^T\right]^2$$

$$\frac{k - \left[p(p+2)(n-1)/(n+1)\right]}{\left[8p(p+2)/n\right]^{0.5}} \sim N(0,1) \tag{3}$$

Mardia*et al.*[15]defined the measure of multivariate skewness to be as follows:

$$s = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(X_i - \overline{X}_n\right)^T S^{-1}\left(X_i - \overline{X}_n\right)\right]^3$$

$$\frac{ns}{6} \sim \chi^2_{p(p+1)(p+2)/6} \qquad (4)$$

where $\chi^2_{p(p+1)(p+2)/6}$ is the chi-square distribution with $p(p+1)(p+2)/6$ degrees of freedom [14].

2.4 The Energy Test for Multivariate Normality

Szekely and Rizzo [16] proposed and defined a new test for multivariate normality known as energy test as follows. Let $X_1, X_2, \ldots, X_n$ be a sample from some p variate distribution. Then, the Energy test statistic is defined as follows:

$$E = n\left[\frac{2}{n}\sum_{i=1}^{n} \mathrm{E}\left\|X_i^* - Z\right\| - E\left\|Z - Z^T\right\| - \frac{1}{n^2}\sum_{i=1}^{n}\left\|X_i^* - X_j^*\right\|\right] \qquad (5)$$

where $X_i^*,\ i = 1,2,\ldots,n$ is the standardized sample, Z and Z^T are independent identically distributed $p-$ variate standard normal random vectors, and $\|\ \cdot\ \|$ denotes Euclidean norm.

2.5 Royston's H Test for Multivariate Normality

Let W_j denote the value of the Shapiro-Wilk statistic for the j th variable in a p variate distribution[6]. Then, defining

$$R_j = \left\{\Phi^{-1}\left[\frac{1}{2}\Phi\left\{-\left(\left(1 - W_j\right)^{\lambda} - \mu\right)/\sigma\right\}\right]\right\}^2 \qquad (6)$$

where $\lambda, \mu, and\ \sigma$ are calculated from polynomial approximations given in[17] and $\Phi(\cdot)$ denotes the standard normal cumulative distribution function (cdf). Thus, Royston's H statistic is defined as

$$H = \xi \sum_j \frac{R_j}{p} \sim \chi^2_{\xi} \qquad (7)$$

where $\xi = \dfrac{p}{\left[1 + (p-1)\bar{c}\right]}$

2.6 The Gel-Gastwirth Test

Gel-Gastwirth proposed a new test of normality which is a robust version of the Jarque-Bera (JB) test[11]. The statistic is defined as follows:

$$RJB = \frac{n}{6}\left(\frac{m_3}{J_n^3}\right)^2 + \frac{n}{64}\left(\frac{m_4}{J_n^4} - 3\right)^2 \sim \chi^2_2 \qquad (8)$$

where $J_n = \dfrac{\sqrt{\pi/2}}{n}\sum_{i=1}^{n}\left|x_i - M\right|$, M is the sample median

2.7 The Bontemps-Meddahi Test

Bontemps and Meddahi [10] proposed a family of normality tests based on moment. The general expression of the test family is given in[18] as:

$$BM_{3-d} = \sum_{k=3}^{d}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} H_k(z_i)\right)^2 \sim \chi^2_{d-2} \qquad (9)$$

where $z_i = \dfrac{(x_i - \bar{x})}{s}$ and $H_k(\cdot)$ represents the k th order normalized Hermite polynomial having the general expressionby the following recursive formulation[18]:

$$\forall i > 1,\ H_i(u) = \frac{1}{\sqrt{i}}\left[u \cdot H_{i-1}(u) - \sqrt{i-1} \cdot H_{i-2}(u)\right], \qquad H_0(u) = 1, \quad H_1(u) = u$$

2.8 Shapiro-Francia Test of multivariate Normality

Shapiro and Francia [4]proposed a new test statistic for testing multivariate normality assumption of a multivariate data.The test is as follows:

$$W_{SF} = \frac{(m'X)^2}{(m'm)\sum_{i=1}^{n}(X_{jn} - \overline{X})^2} \tag{10}$$

where X_{jn} is the jth order statistic from the sample, and m is the expectation vector of the order statistics for a standard normal sample [19].

2.9 Doornik-Harsen Multivariate Normality Test

Doornik and Hansen [8]proposed a test statistic that is based on Omnibus test for normality by making use of Skewness and Kurtosis based[20]. The Doornik-Hansen multivariate test is

$$E_p = Z_1'Z_1 + Z_2'Z_2 \sim \chi_\alpha^2(2p) \tag{11}$$

where $Z_1' = (z_{11}, \cdots, z_{1p})\ and\ Z_2' = (z_{21}, \cdots, z_{2p})$

2.10 Desgagne-Micheaux Multivariate Normality Test

Desgagné*et al.*[12] proposed a new multivariate normality tests based on the distribution defined below:

$$R_n = n\, r_n\left(\overline{X}_n, S_n\right)^T \left(J_0 - \frac{1}{2}\upsilon_0\upsilon_0^T\right)^{-1} r_n\left(\overline{X}_n, S_n\right) \sim \chi_3^2 \tag{12}$$

where $r_n\left(\overline{X}_n, S_n\right) = r_n(\mu,\sigma) - \frac{1}{2}(1 - T_n)\upsilon_0 + O_p(n^{-1/2})1_3$, $r_n(\mu,\sigma) = \frac{1}{n}\sum_{i=1}^{n} d_0(Y_i)$, $Y_i = \frac{X_i - \mu}{\sigma}$ $T_n = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$,

$$1_3 = (1 \quad 1 \quad 1)^T,\ J_0 = E_0[d_0(Y)d_0(Y)^T]\ d_0(Y) = \frac{\partial}{\partial\theta}\log g(y;\theta)\Big|_{\theta = (2 \quad 0 \quad 0)^T}$$

$$\upsilon_0 = -E_0[Y^2 d_0(Y)]$$

2.11 Skewness (S) and Kurtosis (K) Tests

Kankainen*et al.*[2]proposed tests of multinormality based on location vectors and scatter matrices. The test for skewness is:

$$U = (T_1 - T_2)^T C^{-1}(T_1 - T_2) \sim \chi_p^2 \tag{13}$$

where T_1 and T_2 are two separate location vectors, and C a scatter matrix.

while that of kurtosis:

$$W = \left[Tr\left(\left(C_1^{-1}C_2\right)^2\right) - \frac{1}{p}Tr^2\left(C_1^{-1}C_2\right)\right] + \frac{1}{p}\left[Tr\left(C_1^{-1}C_2\right) - p\right]^2 \sim \chi_{p(p+1)/2}^2 \tag{14}$$

$C_1\ and\ C_2$ are two separate scatter matrices equipped with the correction factor.

3.0 Methodology

The Type 1 error rates of the multivariate tests of normality are evaulated through Monte Carlo simulation study. The simulation parameters are: four levels of dimension, p = 2, 3, 4 and 5, three levels of significance, $\alpha = 0.1,\ 0.05\ and\ 0.01$, eight sample sizes (n): 10, 20, 30, 50, 75, 100, 150 and 300. We generated 1000 , Replications (R), multivariate normally distributed samples for specified values of p and n from R-programenvironment [21]. The generated data were subjected to the multivariate normality tests and the p-value associated with each test was documented for each of the replications. We defined

$$W_i = \begin{cases} 1, & if\ \ p - value < \alpha\ ,\ level\ of\ \ significance \\ 0, & Otherwise \end{cases} \tag{15}$$

$$i = 1,\ 2, \cdots, R$$

Let $W = \sum_{i=1}^{R} W_i$, then the empiricalType 1 error rate at α level of significance is given as:

$$T_\alpha = \frac{W}{R} \tag{16}$$

The procedure was repeated until all the parameters are utilized.A normality test is considered good if its empirical error rate approximates the true error rate and best if it has the highest number of times (Mode) it approximates the error rate when counted over the levels of significance.

Table 1: The True Level of Significance and Their Preferred Interval

True levels of significance	Preferred interval
0.1	0.095 – 0.14
0.05	0.045 - 0.054
0.01	0.005 - 0.014

Source: [22]

4.0 Results and Discussion

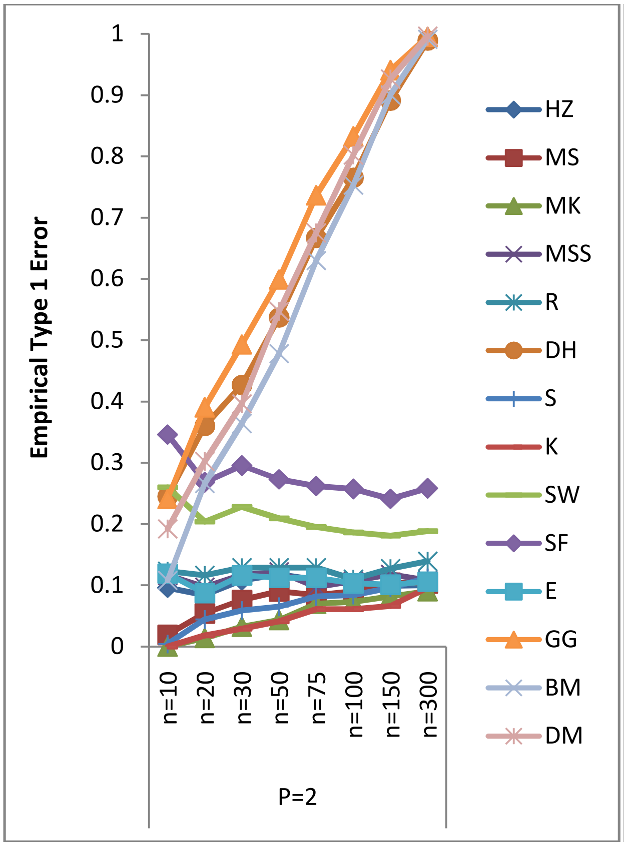
The results of the Type 1 error of the multivariate normality tests for three levels of significance are presented and discussed as follows:

4.1 Results at 0.1 Level of Significance

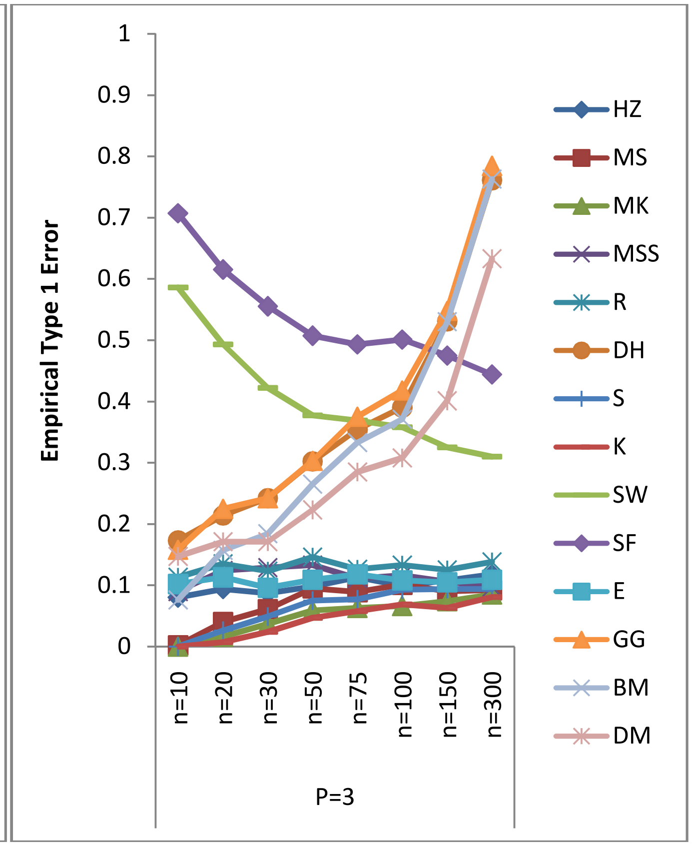
Table 2 summarizes the empirical Type 1 error rates of the tests at 0.1 level of significance. The results show that the Type 1 errors of DH, BM, GG, DM, SW and SF are comparatively far from the true level of significance while those of K, S, MK, MS, HZ, MSS, E and R revolve around it. Thus, the latter ones are good.

The error rates when p=2 and p=3 are respectively presented in Figures 1a and 1b. From the figures, it is observed that the Type 1 errors of DM, DH, BH and GG tests increases as the sample size increases and that of SF and SW tests are high when sample size is small; however, they reduced and tend to the true level of significance as sample size increases. Besides, K, MK, S, and MS tests approximate the true error rate at very large sample sizes. In addition, E, HZ, R and MSSgenerally revolve around the true level of significance.

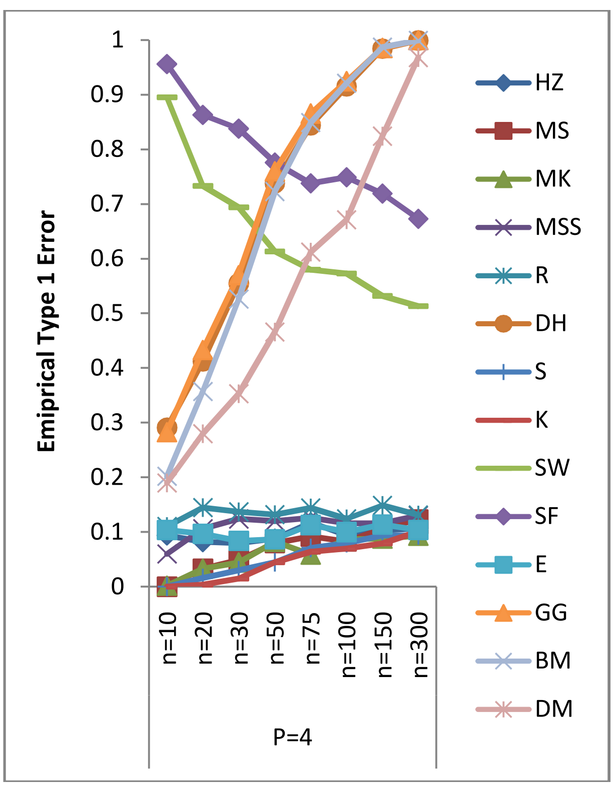
The error whenp=4 and p=5 are respectively presented in Figures 1c and 1d. From the graphs, it was observed that the Type 1 error rates of DM, BM, DH and GG tests increase with increase in sample size and that at small sample size, their Type 1 errors are far above the level of significance. Also, the SF and SW tests have very high Type 1 error for small sample size but reduce gradually as sample size increases. However, the errors are still far away from the true level of significance at large sample sizes. From the Figures, it can also be seen that the Type 1 error rates of K, S, MK and MS are far below 0.1 but tend to converge to the true rate as sample increases. The results further show that E and HZ tests have Type 1 errors that converge to 0.1 at all sample sizes while R and MSS tests Type 1 error rates also revolve around 0.1 level of significance.



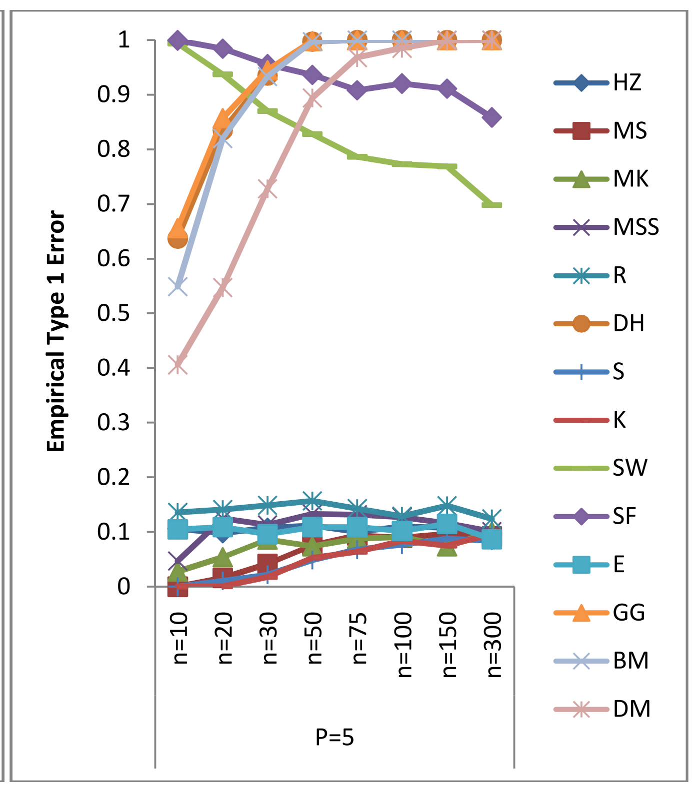
(a) $p = 2,\ \alpha = 0.1$



(b) $p = 3,\ \alpha = 0.1$



(c) $p = 4,\ \alpha = 0.1$



(d) $p = 5,\ \alpha = 0.1$

Figure 1: Comparison of Type 1 Error rate of Different Normality Tests at $\alpha = 0.1$

Table 2: Comparison of Type 1 Error rates of Multivariate Normality tests at 0.1 Level of Significance

n	p / Test	HZ	MS	MK	MSS	R	DH	S	K	SW	SF	E	GG	BM	DM
10	2	0.096	0.019	0	0.116	0.123	0.245	0.004	0	0.26	0.346	0.119	0.241	0.108	0.192
	3	0.08	0.002	0	0.09	0.113	0.173	0	0	0.586	0.707	0.102	0.158	0.076	0.147
	4	0.094	0	0.002	0.06	0.11	0.291	0	0	0.895	0.956	0.104	0.283	0.202	0.19
	5	0.106	0	0.028	0.047	0.136	0.637	0	0	0.993	0.999	0.105	0.656	0.549	0.406
20	2	0.084	0.054	0.014	0.097	0.116	0.36	0.044	0.018	0.204	0.268	0.087	0.39	0.266	0.302
	3	0.094	0.04	0.016	0.124	0.135	0.214	0.026	0.007	0.493	0.615	0.113	0.225	0.157	0.171
	4	0.082	0.033	0.033	0.106	0.145	0.412	0.016	0.004	0.733	0.863	0.097	0.433	0.357	0.28
	5	0.098	0.016	0.054	0.125	0.141	0.835	0.011	0.001	0.937	0.984	0.109	0.857	0.82	0.547
30	2	0.108	0.076	0.032	0.117	0.129	0.427	0.059	0.029	0.228	0.295	0.116	0.493	0.364	0.396
	3	0.087	0.061	0.037	0.129	0.123	0.242	0.049	0.024	0.422	0.555	0.096	0.242	0.184	0.171
	4	0.08	0.049	0.044	0.124	0.137	0.555	0.03	0.015	0.694	0.838	0.084	0.57	0.526	0.353
	5	0.108	0.042	0.086	0.113	0.149	0.935	0.022	0.018	0.87	0.955	0.096	0.945	0.933	0.728
50	2	0.118	0.09	0.043	0.122	0.129	0.537	0.065	0.041	0.209	0.273	0.112	0.599	0.478	0.547
	3	0.098	0.095	0.059	0.133	0.146	0.302	0.075	0.047	0.377	0.507	0.109	0.303	0.265	0.223
	4	0.088	0.08	0.082	0.12	0.132	0.738	0.045	0.045	0.613	0.776	0.087	0.76	0.723	0.466
	5	0.112	0.076	0.074	0.133	0.157	0.997	0.049	0.053	0.828	0.936	0.109	0.998	0.997	0.894
75	2	0.109	0.084	0.07	0.097	0.129	0.667	0.082	0.061	0.195	0.262	0.112	0.736	0.63	0.675
	3	0.113	0.089	0.063	0.111	0.126	0.354	0.077	0.058	0.369	0.493	0.118	0.375	0.333	0.285
	4	0.117	0.092	0.059	0.126	0.144	0.844	0.071	0.064	0.58	0.738	0.113	0.865	0.849	0.612
	5	0.1	0.094	0.088	0.132	0.143	1	0.068	0.064	0.786	0.908	0.109	1	1	0.968
100	2	0.101	0.09	0.073	0.107	0.11	0.765	0.084	0.061	0.186	0.257	0.103	0.832	0.753	0.802
	3	0.099	0.101	0.066	0.116	0.133	0.39	0.093	0.069	0.358	0.501	0.108	0.418	0.371	0.308
	4	0.094	0.083	0.087	0.116	0.124	0.915	0.08	0.07	0.573	0.749	0.1	0.924	0.921	0.671
	5	0.11	0.09	0.091	0.127	0.129	1	0.077	0.084	0.773	0.92	0.102	1	1	0.985
150	2	0.099	0.104	0.082	0.118	0.127	0.891	0.096	0.066	0.181	0.241	0.101	0.94	0.9	0.926
	3	0.108	0.09	0.074	0.106	0.125	0.531	0.094	0.063	0.325	0.475	0.105	0.548	0.53	0.401
	4	0.103	0.107	0.088	0.116	0.149	0.984	0.094	0.079	0.532	0.719	0.114	0.985	0.987	0.824
	5	0.108	0.097	0.074	0.116	0.148	1	0.088	0.075	0.769	0.911	0.115	1	1	0.999
300	2	0.1	0.103	0.09	0.108	0.139	0.989	0.112	0.096	0.188	0.258	0.106	0.995	0.991	0.996
	3	0.118	0.092	0.085	0.097	0.138	0.761	0.094	0.081	0.31	0.444	0.109	0.784	0.763	0.633
	4	0.107	0.123	0.093	0.13	0.131	0.999	0.094	0.101	0.513	0.673	0.104	1	0.999	0.968
	5	0.094	0.092	0.101	0.101	0.124	1	0.084	0.097	0.698	0.858	0.087	1	1	1

Source: Simulation Result

4.2 Results at 0.05 Level of Significance

Table 3 shows the of empirical Type 1 errors of the tests. It was observed that the Type 1 errors of DM, BM, GG, DH, SW and SF are high above 0.05 level of significance while that of K, S, MK, MS, HZ, MSS, E and R are closer to 0.05 level of significance.

The error rates when p=2 and p=3 are respectively presented in Figures 2a and 2b. The results revealed that the Type 1 error rates of BM, DM, DH, and GG are twice more than 0.05 level of significance and as sample size increases the error rates increases. At all sample sizes, Type 1 error rates of SW and SF are more than twice 0.05. The Type 1 error rates of K, MK, S and MS are less than 0.05 at all sample sizes but tend to converge to the true error at large sample sizes. Furthermore, the Type 1 error rates of HZ, E, R and MSS are equivalent to 0.05 at almost all sample sizes.

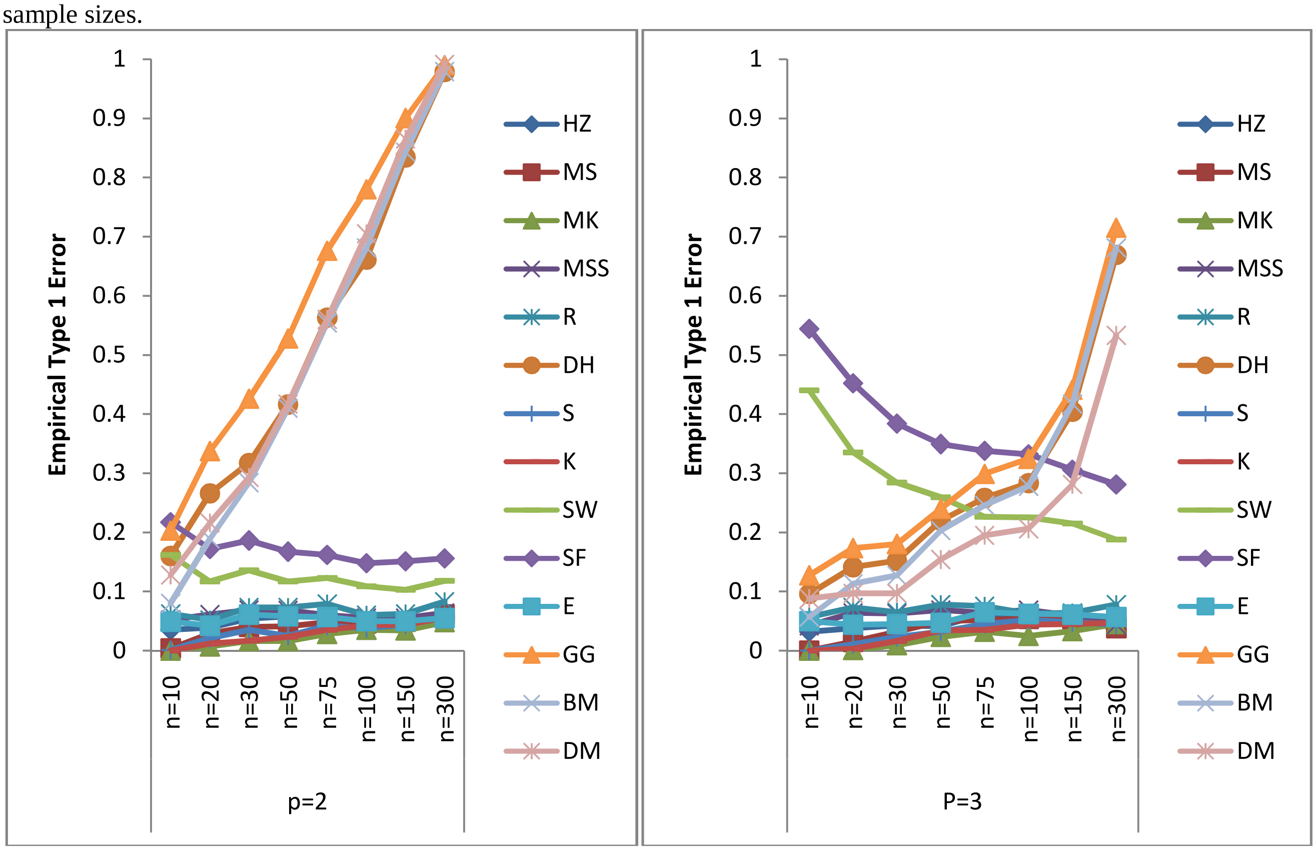
The error rates when p=4 and p=5 are presented in Figures 2c and 2d respectively. It was observed that the Type 1 error rates of DM, BM, DH, and GG increase with increase in sample size while the Type 1 error rates of SW and SF reduce as sample size increases. S, MK, K and MS have low Type 1 error rates at small sample sizes but converge to 0.05 when the sample size is at least 50. Furthermore, Type 1 error rates of HZ, E, MSS and R are in the neighborhood of 0.05 level of significance at all sample sizes.

4.3 Results at 0.01 Level of Significance

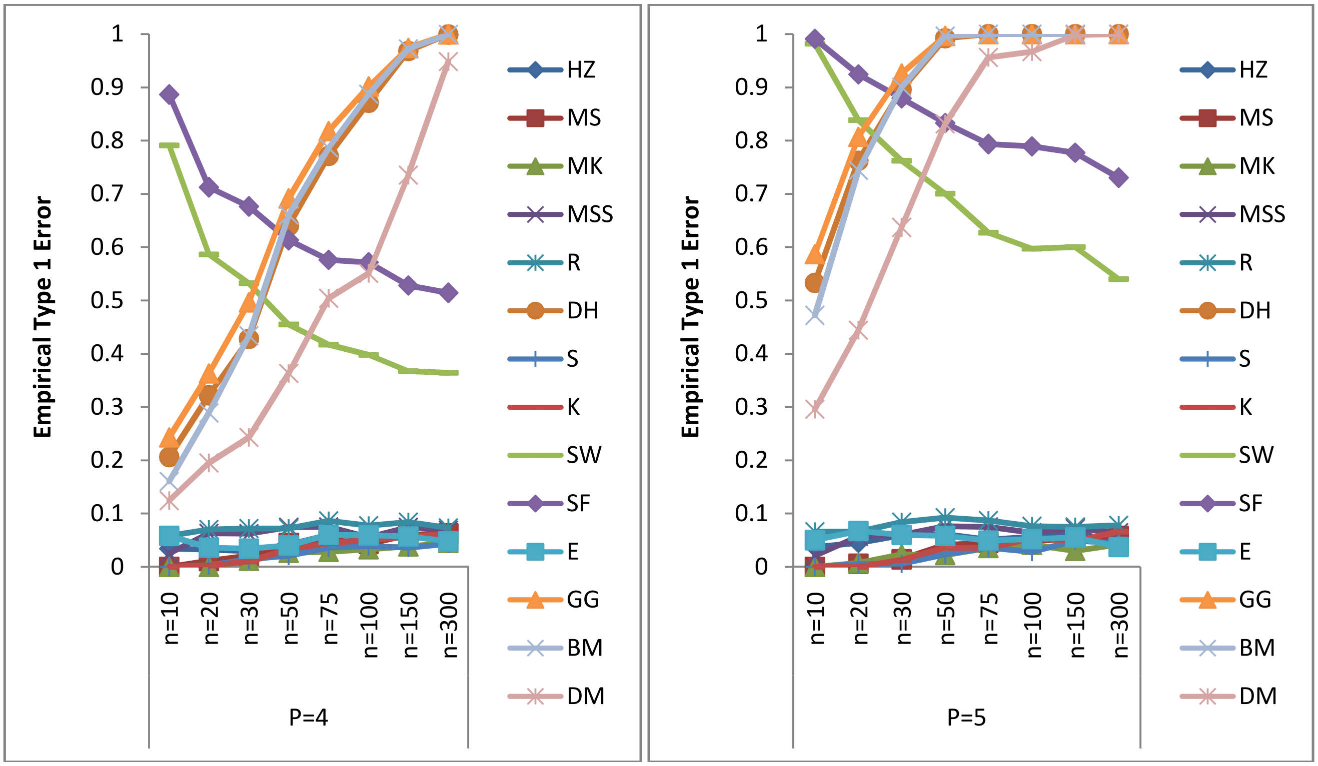
The results from Table 4 show that the Type 1 errors of BM, DM, DH, SF, SW and GG are greater than 0.01. However, the Type 1 error rates of MSS, MK, K, S, MS, R, E and HZ are closer to 0.01.

From Figures 3(a-b) at p=2 and p=3, the result revealed that the Type 1 error rates of BM, DM, DH, and GG increase as sample size increases. The Type 1 error rates of SW and SF are higher than 0.01 for small sample size but approach 0.01 as sample size increases. The result of Type 1 error rates of K, MK, S, MS, HZ, E, R and MSS revolve around 0.01 at all sample sizes.

It is seen from Figures 3(c-d) when p=4 and p=5 that the Type 1 error rates of DM, BM, DH, and GG increase with increase in sample sizes while the Type 1 error rates of SW and SF reduce as sample size increases but does not approximate 0.01. The Type 1 error rates of S, MK, K and MS are lower than 0.01, level of significance, for small sample sizes but approaches 0.01 when the sample size is at least 50. Furthermore, Type 1 error rates of E, R, MSS and HZ revolve around 0.01 at all sample sizes.



(a) $p = 2$, $\alpha = 0.05$ (b) $p = 3$, $\alpha = 0.05$



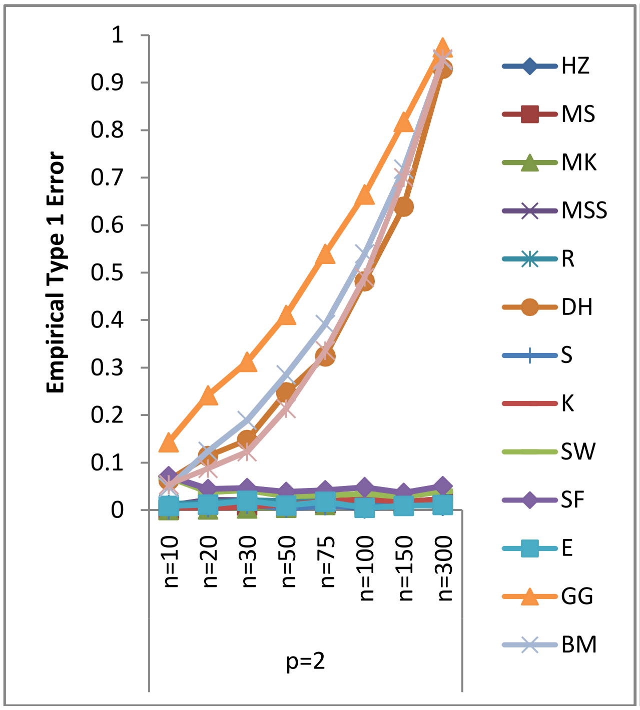
(c) $p = 4,\ \alpha = 0.05$ (d) $p = 5,\ \alpha = 0.05$

Figure 2: Comparison of Type 1 Error rate of Different Normality Tests at $\alpha = 0.05$

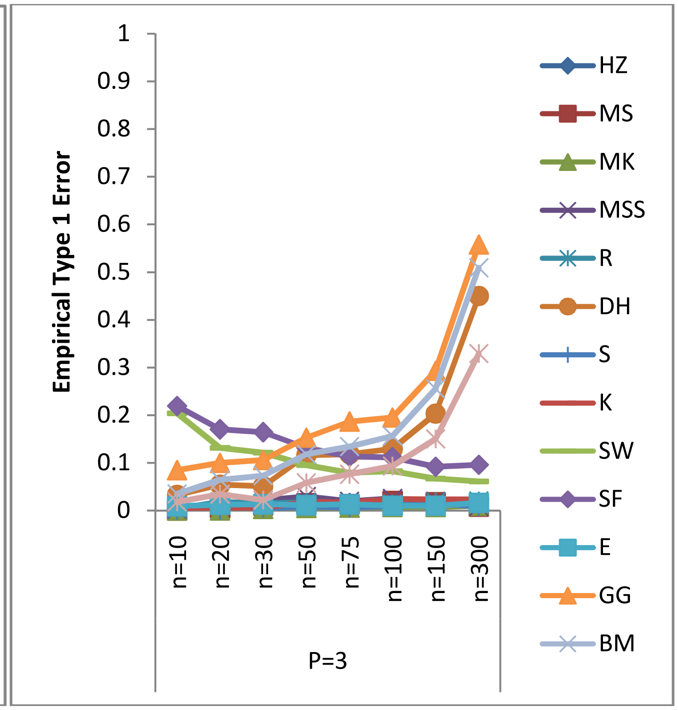
Table 3: Comparison of Type 1 Error rates of Multivariate Normality tests at 0.05 Level of Significance

n	p/ Test	HZ	MS	MK	MSS	R	DH	S	K	SW	SF	E	GG	BM	DM
10	2	0.035	0.004	0	0.052	0.062	0.16	0.001	0	0.162	0.217	0.049	0.203	0.08	0.128
	3	0.032	0	0	0.042	0.057	0.096	0	0	0.44	0.544	0.05	0.127	0.057	0.088
	4	0.034	0	0	0.025	0.058	0.206	0	0	0.791	0.886	0.058	0.243	0.16	0.124
	5	0.037	0	0	0.021	0.066	0.533	0	0	0.982	0.991	0.05	0.587	0.472	0.296
20	2	0.039	0.029	0.007	0.061	0.052	0.266	0.021	0.012	0.117	0.172	0.042	0.337	0.189	0.216
	3	0.038	0.015	0.001	0.064	0.073	0.141	0.011	0.003	0.335	0.452	0.044	0.173	0.113	0.097
	4	0.032	0.011	0	0.063	0.07	0.322	0.003	0.002	0.586	0.712	0.036	0.363	0.289	0.195
	5	0.045	0.006	0.007	0.057	0.066	0.762	0.005	0	0.838	0.924	0.067	0.807	0.744	0.444
30	2	0.054	0.04	0.016	0.069	0.073	0.317	0.035	0.017	0.136	0.186	0.061	0.426	0.284	0.293
	3	0.043	0.032	0.009	0.063	0.065	0.152	0.023	0.016	0.284	0.384	0.045	0.18	0.127	0.097
	4	0.029	0.021	0.011	0.062	0.072	0.428	0.014	0.01	0.532	0.676	0.034	0.496	0.434	0.243
	5	0.059	0.014	0.023	0.06	0.084	0.896	0.006	0.013	0.762	0.879	0.06	0.926	0.9	0.637
50	2	0.058	0.041	0.016	0.068	0.073	0.416	0.025	0.023	0.117	0.167	0.058	0.528	0.41	0.417
	3	0.041	0.05	0.023	0.069	0.078	0.22	0.032	0.034	0.259	0.349	0.047	0.24	0.203	0.154
	4	0.041	0.045	0.026	0.074	0.072	0.64	0.021	0.033	0.455	0.613	0.04	0.691	0.66	0.363
	5	0.061	0.041	0.022	0.076	0.092	0.992	0.023	0.036	0.7	0.833	0.058	0.996	0.996	0.831
75	2	0.057	0.048	0.028	0.06	0.079	0.563	0.042	0.035	0.123	0.162	0.057	0.676	0.554	0.56
	3	0.063	0.052	0.032	0.064	0.075	0.258	0.045	0.035	0.226	0.338	0.065	0.299	0.246	0.195
	4	0.06	0.051	0.028	0.075	0.086	0.771	0.036	0.044	0.417	0.576	0.06	0.818	0.786	0.504
	5	0.051	0.045	0.035	0.075	0.087	1	0.036	0.035	0.627	0.793	0.049	1	1	0.956
100	2	0.05	0.046	0.035	0.055	0.06	0.661	0.038	0.043	0.109	0.148	0.05	0.78	0.681	0.705
	3	0.057	0.054	0.025	0.069	0.063	0.283	0.052	0.044	0.225	0.332	0.062	0.324	0.278	0.206
	4	0.057	0.043	0.033	0.055	0.077	0.871	0.039	0.05	0.398	0.571	0.059	0.901	0.887	0.551
	5	0.056	0.048	0.041	0.064	0.076	1	0.028	0.053	0.597	0.789	0.052	1	1	0.967
150	2	0.044	0.049	0.034	0.055	0.062	0.834	0.044	0.044	0.103	0.151	0.05	0.9	0.843	0.865
	3	0.055	0.053	0.033	0.059	0.064	0.404	0.048	0.045	0.215	0.306	0.061	0.441	0.417	0.281
	4	0.058	0.06	0.038	0.074	0.084	0.968	0.036	0.056	0.367	0.528	0.057	0.974	0.972	0.735
	5	0.064	0.053	0.03	0.072	0.075	1	0.049	0.049	0.6	0.777	0.055	1	1	0.998
300	2	0.049	0.061	0.048	0.064	0.083	0.978	0.056	0.056	0.118	0.156	0.055	0.99	0.979	0.991
	3	0.055	0.038	0.045	0.044	0.078	0.669	0.044	0.047	0.188	0.281	0.057	0.715	0.681	0.533
	4	0.053	0.063	0.045	0.069	0.073	0.999	0.043	0.062	0.364	0.514	0.048	0.999	0.999	0.948
	5	0.046	0.058	0.042	0.064	0.078	1	0.044	0.066	0.54	0.73	0.037	1	1	1

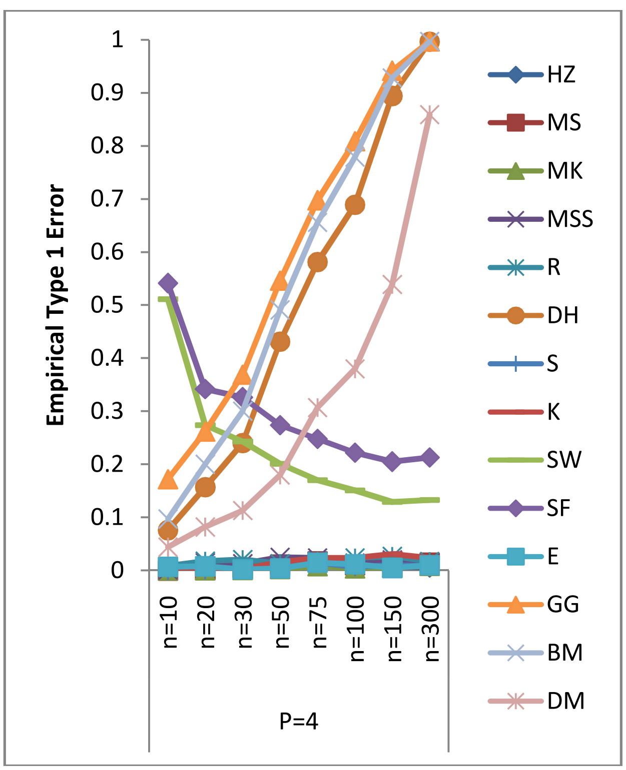
Source: Simulation Result



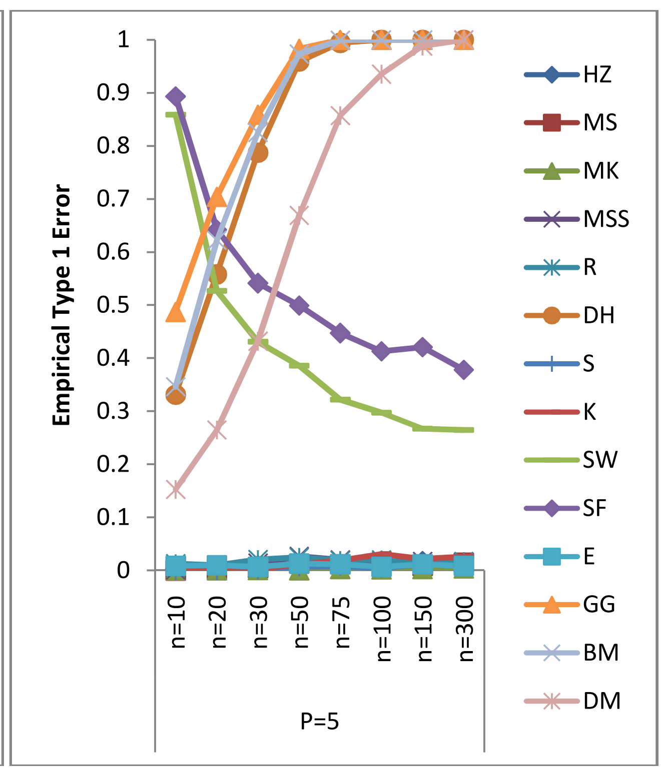
(a) $p = 2,\ \alpha = 0.01$



(b) $p = 3,\ \alpha = 0.01$



(c) $p = 4,\ \alpha = 0.01$



(d) $p = 4,\ \alpha = 0.01$

Figure 3: Comparison of Type 1 Error rate of Different Normality Tests at $\alpha = 0.01$

Table 4: Comparison of Type 1 Error rates of Multivariate Normality tests at 0.01 Level of Significance

n	p/ Test	HZ	MS	MK	MSS	R	DH	S	K	SW	SF	E	GG	BM	DM
10	2	0.002	0.001	0	0.01	0.013	0.063	0	0	0.069	0.072	0.009	0.143	0.048	0.054
	3	0.004	0	0	0.003	0.004	0.033	0	0	0.204	0.219	0.009	0.084	0.035	0.018
	4	0.003	0	0	0.001	0.009	0.076	0	0	0.511	0.541	0.007	0.172	0.097	0.044
	5	0.004	0	0	0.001	0.013	0.331	0	0	0.859	0.893	0.009	0.487	0.346	0.153
20	2	0.008	0.012	0.002	0.022	0.016	0.114	0.005	0.003	0.038	0.045	0.012	0.242	0.124	0.088
	3	0.007	0.005	0	0.012	0.018	0.054	0	0.001	0.131	0.17	0.012	0.1	0.064	0.034
	4	0.005	0.001	0	0.015	0.018	0.157	0.001	0	0.274	0.342	0.008	0.262	0.2	0.082
	5	0.007	0.003	0	0.009	0.01	0.558	0	0	0.527	0.642	0.01	0.704	0.622	0.265
30	2	0.012	0.01	0.004	0.021	0.021	0.148	0.012	0.008	0.042	0.047	0.02	0.312	0.189	0.123
	3	0.011	0.011	0.003	0.021	0.015	0.051	0.004	0.006	0.121	0.164	0.013	0.106	0.073	0.022
	4	0.002	0.005	0.001	0.013	0.021	0.24	0.002	0.003	0.243	0.326	0.002	0.369	0.3	0.113
	5	0.01	0.005	0.001	0.014	0.021	0.787	0.001	0.003	0.431	0.541	0.007	0.859	0.825	0.432
50	2	0.009	0.006	0.005	0.012	0.019	0.248	0.004	0.012	0.029	0.039	0.01	0.411	0.285	0.214
	3	0.011	0.015	0.005	0.029	0.017	0.116	0.008	0.018	0.095	0.131	0.011	0.153	0.118	0.058
	4	0.006	0.011	0.003	0.025	0.013	0.431	0.005	0.016	0.201	0.274	0.004	0.546	0.491	0.18
	5	0.013	0.013	0	0.027	0.025	0.959	0.007	0.012	0.386	0.499	0.013	0.983	0.973	0.669
75	2	0.012	0.015	0.011	0.016	0.02	0.324	0.009	0.019	0.031	0.042	0.018	0.54	0.391	0.336
	3	0.015	0.013	0.006	0.019	0.011	0.118	0.006	0.017	0.079	0.113	0.013	0.186	0.134	0.076
	4	0.014	0.017	0.008	0.024	0.018	0.581	0.015	0.024	0.17	0.248	0.015	0.697	0.656	0.307
	5	0.011	0.012	0.003	0.02	0.019	0.994	0.008	0.019	0.322	0.447	0.012	1	0.999	0.857
100	2	0.004	0.014	0.011	0.019	0.013	0.482	0.008	0.023	0.035	0.048	0.005	0.664	0.54	0.49
	3	0.012	0.02	0.007	0.025	0.014	0.129	0.009	0.024	0.082	0.111	0.01	0.195	0.156	0.093
	4	0.013	0.01	0.004	0.014	0.024	0.689	0.007	0.024	0.151	0.222	0.012	0.809	0.779	0.38
	5	0.013	0.012	0.002	0.017	0.02	1	0.004	0.031	0.297	0.413	0.007	1	1	0.935
150	2	0.011	0.016	0.013	0.017	0.012	0.639	0.01	0.018	0.028	0.037	0.009	0.817	0.718	0.7
	3	0.011	0.017	0.007	0.019	0.018	0.203	0.015	0.023	0.067	0.092	0.01	0.293	0.255	0.15
	4	0.007	0.016	0.006	0.02	0.027	0.894	0.01	0.031	0.129	0.205	0.005	0.941	0.928	0.539
	5	0.018	0.009	0.003	0.017	0.011	1	0.013	0.022	0.267	0.421	0.012	1	1	0.988
300	2	0.009	0.021	0.013	0.024	0.023	0.929	0.011	0.023	0.04	0.051	0.011	0.974	0.947	0.95
	3	0.014	0.008	0.01	0.009	0.02	0.45	0.008	0.023	0.061	0.096	0.016	0.558	0.509	0.329
	4	0.006	0.015	0.009	0.015	0.017	0.996	0.009	0.024	0.133	0.213	0.01	0.997	0.997	0.859
	5	0.006	0.015	0.004	0.015	0.017	1	0.009	0.026	0.265	0.378	0.009	1	1	1

Source: Simulation Result

4.4 Overall Type 1 Error Rating of Multivariate Normality Tests

Having further counted the results of the Type 1 error rates of the multivariate normality testsover the levels of significance and dimensions, it was observed that the Type 1 error rates of HZ, MS, MSS, R and E tests are relatively good while the Type 1 error rates of MK, DH, S, K, SW, SF, GG, BM and DM tests arecomparatively bad.Further examination of the results revealed that the Type 1 error rates of R, MS, MSS, HZ and E, in this order, perform well.Table 5 summarizes the results while Figure 4 displays them graphically.

Table 5: Total Number of Times Type 1 Error Rate Approximates True Level of Significance

	Sample Size									
Tests	10	20	30	50	75	100	150	300	Total	Rank
HZ	2	6	6	7	8	7	7	8	51	2
MS	0	2	4	6	6	7	7	3	35	4.5
MK	0	0	0	2	3	2	3	5	15	7
MSS	3	6	6	5	4	5	4	7	40	3
R	7	4	3	3	3	6	4	5	35	4.5
DH	0	0	0	0	0	0	0	0	0	12
S	0	1	1	3	4	4	6	4	23	6
K	0	0	2	2	0	2	2	4	12	8
SW	0	0	0	0	0	0	0	0	0	12
SF	0	0	0	0	0	0	0	0	0	12
E	11	7	6	7	7	10	9	8	65	1
GG	0	0	0	0	0	0	0	0	0	12
BM	1	0	0	0	0	0	0	0	1	9
DM	0	0	0	0	0	0	0	0	0	12

Source: Counted from Tables 2, 3 and 4

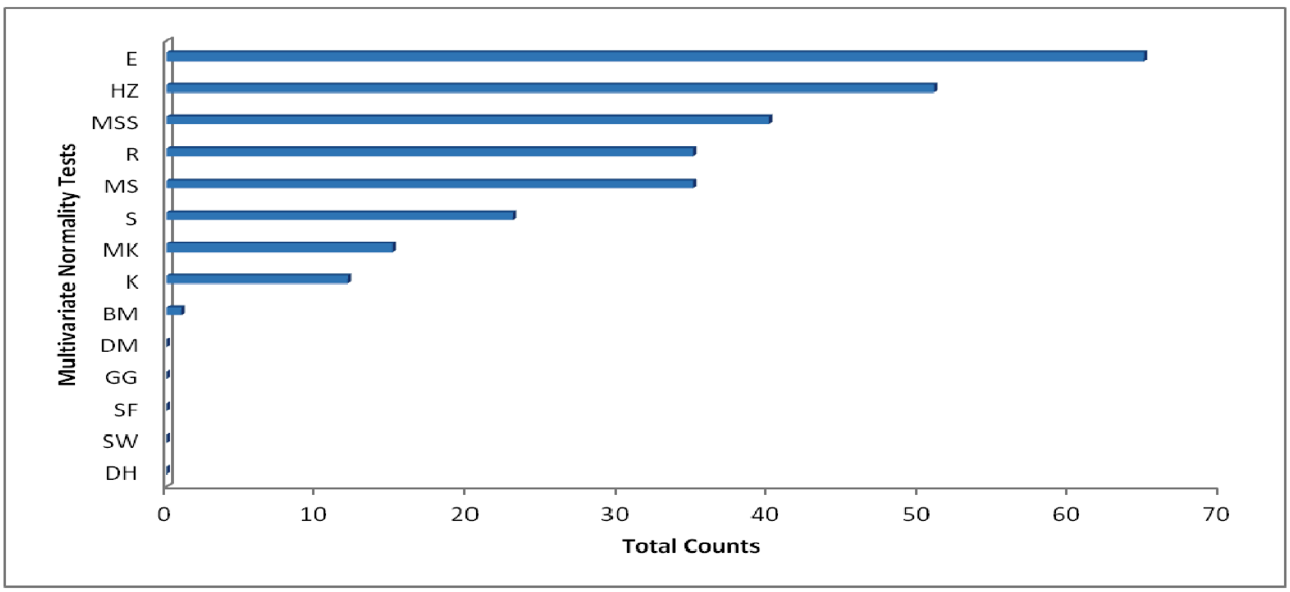


Figure 4: Bar Chart Showing Summary of Number of Times Type 1 Error Rates Approximate True Level of Significance

5.0 Conclusion

The Type 1 error rates of the multivariate tests of normality have been compared and the useful ones have been identified.The error rates of HZ, MSS, R and E; MS and E; and MK, MS, S, HZ and E are respectively good at 0.1, 0.05 and 0.01 levels of significance. Moreover, those of E and HZ are best.Consequently,the use of HZ and E multivariate tests of normalityare therefore recommended to users of statistics.

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