

On the Application of Volatility Modeling Via Cholesky Decomposition

¹F. Ewere and ²R.C. Ike

^{1,2}Department of Mathematics, University of Benin, Benin City, Nigeria.

Abstract

Inflation variability over time has been the focus of both investors and financial analysts as the fluctuations in prices enunciate the stability of investing in a given stock or sector of the economy cum the manner in which volatility is measured, contributes to the problem of unexpected volatility. The aim of this paper is to demonstrate an approach to multivariate volatility modeling via the cholesky decomposition. An application using three firms from three major sub-sectors of the manufacturing sector of the Nigerian economy is presented. The volatility series of these sampled firms are indicators of the inflation variability of the economy within the period under consideration.

1.0 Introduction

Time-varying volatility is endemic in financial markets and this has been known for a long time. Despite the conceptual beauty of the trading realized volatilities and covariances, there are numerous practical issues such as measurement error [1], missing observations [2], the presence of discontinuous jump processes [3] to be potentially dealt with. The idea of Volatility Index (VIX) and financial instruments based on such an index was first developed in [4]. The volatility of a stock is simply the standard deviation of its returns. It is the statistical measure of the dispersions of returns of a given market index. Cholesky decomposition is an algorithm that takes a symmetric positive definite matrix and factorizes it as a product of a triangular matrix and its transpose.

2.0 Methodology

There are several ways of adopting cholesky decomposition in the modeling of volatility, but the method employed in this paper encompasses the following:

- (i) The recursive least squares method.
- (ii) The Exponentially Weighted Moving Average procedure (EWMA).
- (iii) Univariate GARCH (1,1) procedure.
- (iv) Inverses of lower triangular matrices with diagonal elements being unity.

2.1 The Log Returns of Prices

With the assumption of zero dividends, the net return of an asset (R_t) is given as:

$$R_t = \frac{P_t}{P_{t-1}} - 1 \quad (1)$$

Whereas the simple gross returns over k-periods is given as:

$$R_t + 1 = \frac{P_t}{P_{t-k}} \quad (2)$$

A continuously compounded returns is given in [5] as:

$$r_t = \ln(R_t + 1) \quad (3)$$

Corresponding author: F. Ewere, E-mail: friday.ewere@uniben.edu, Tel.: +2348058504893

where P_t = the price of the asset at time 't'.

P_{t-1} = the price of the asset at one period lag of 't' (ie at time 't-1').

2.2 Cholesky Decomposition

Given a symmetric positive definite matrix 'A', if L is a lower triangular matrix; then matrix A can be decomposed as:

$$A = LL^T \quad (4)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} \quad (5)$$

and the diagonal elements are given by:

$$l_{kk} = \sqrt{(a_{kk} - l_{kj}^2)} \quad (6)$$

The off diagonal elements are given by:

$$l_{ik} = \frac{1}{l_{kk}} \left(a_{ik} - \sum_{j=1}^{k-1} l_{ij} l_{kj} \right) \quad \forall i > k. \quad (7)$$

2.3 The Method of Recursive Least Squares

This method given in [6] is of the form:

$$Y_1 = a_{11}X_1 + a_{12}X_2 + U_1 \quad (8)$$

$$Y_2 = a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + b_{21}Y_1 + U_2 \quad (9)$$

$$Y_3 = a_{31}X_1 + a_{34}X_4 + b_{31}Y_1 + b_{32}Y_2 + U_3 \quad (10)$$

In matrix form, we have:

$$BY = AX + U \quad (11)$$

where

$$B = \begin{bmatrix} 1 & 0 & 0 \\ -b_{21} & 1 & 0 \\ -b_{31} & -b_{32} & 1 \end{bmatrix}, Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}, U = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \quad (12)$$

2.4 The Univariate GARCH (1,1) Process

A GARCH (q,p) process is given as:

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \dots + \alpha_q X_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \dots + \beta_p \sigma_{t-p}^2, \quad t \in \mathbb{Z} \quad (13)$$

where

$$\sigma_t^2 > 0, \quad \alpha_i \geq 0, \forall i=1,2,\dots,q, \quad \beta_j \geq 0, \forall j=1,2,\dots,p.$$

Thus a GARCH (1,1) process is written as:

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \quad (14)$$

2.5 The EWMA Procedure

The EWMA forecast equation is given as:

$$\hat{Z}_t(1) = (1 - \lambda)Z_t + \lambda \hat{Z}_{t-1}(1) \quad (15)$$

Hence the infinite autoregressive representation of the EWMA is given in [7] as:

$$\hat{Z}_t(1) = (1 - \lambda) [Z_t + \lambda \hat{Z}_{t-1} + \lambda^2 \hat{Z}_{t-2} + \lambda^3 \hat{Z}_{t-3} \dots] \quad (16)$$

where Z_t = the observed data (sales or prices) at a given time 't'.

$\hat{Z}_t(1)$ = the one step ahead forecast of all the observations of time 't+1' made at time 't'.

2.6 A Summary of the Algorithm of the Multivariate Volatility Modeling Via Cholesky Decomposition:

Consider a k-dimensional innovation ' a_t ' with volatility matrix ' Σ_t ', and let ' F_{t-1} ' denote the information available at time ' $t-1$ '; cholesky decomposition performs linear orthogonal transformations via a system of multiple linear regressions.

$$\text{Let } b_{1t} = a_{1t} \quad (17).$$

$$\text{Consider the simple linear regression } a_{2t} = \beta_{21,t} b_{1t} + b_{2t} \quad (18)$$

$$\text{here } \beta_{21,t} = \frac{\text{cov}(a_{2t}, b_{1t} / F_{t-1})}{\text{var}(b_{1t})} \quad (19)$$

$$\begin{aligned} \text{In practice } \beta_{21,t} \text{ is estimated using the ordinary least squares method given the available data in } F_{t-1}. \text{ Based on the least} \\ \text{squares theorem, } b_{2t} \text{ is orthogonal to } b_{1t} = a_{1t} \text{ and } \text{var}(b_{2t} / F_{t-1}) = \text{var}(a_{2t} / F_{t-1}) - \beta_{21,t} \text{var}(a_{1t} / F_{t-1}) \\ \text{var}(b_{2t} / F_{t-1}) = \text{var}(a_{2t} / F_{t-1}) - \beta_{21,t} \text{var}(a_{1t} / F_{t-1}) \end{aligned} \quad (20)$$

Now, consider the multiple linear regression:

$$a_{1t} = \gamma_{31,t} b_{1t} + \gamma_{32,t} b_{2t} + b_{3t} \quad (21)$$

$$a_{1t} = \beta_{31,t} a_{1t} + \beta_{32,t} a_{2t} + b_{3t} \quad (22)$$

where, $\beta_{31,t}$ are linear functions of $\gamma_{3j,t}$ and $\beta_{21,t}$ for $j = 1, 2$. Again, via the least squares theorem, b_{3t} is orthogonal to both a_{1t} , a_{2t} and hence to b_{1t} and b_{2t} . Repeat the priori process of multiple linear regression until:

$$a_{kt} = \beta_{k1,t} a_{1t} + \beta_{k,k-1,t} a_{k-1t} + b_{kt} \quad (23)$$

Applying the least-squares theory, we can obtain ' $\beta_{kj,t}$ ' the conditional variance of b_{kt} given F_{t-1} . In addition, b_{kt} is orthogonal to a_{1t} and hence b_{it} for $i = 1, 2, \dots, k-1$

Apply the Recursive Least Squares (RLS) estimation to the equations (18) to (23) for the i^{th} linear regression, denote the estimates by $\hat{\beta}_{i,t}$, where, $i = 2, \dots, k$ and $\hat{\beta}_{i,t}$ is an $(i-1)$ dimensional vector. With $0 < \lambda \leq 1$, apply the EWMA procedure with $\lambda = 0.96$ to obtain smoothed estimates of β_i . Specifically, let $\hat{\mu}_i$ be the sample mean of $\{\hat{\beta}_{i,t}\}$ and

$$\hat{\beta}_{i,t}^* = \hat{\beta}_{i,t} - \hat{\mu}_i \quad (24)$$

be the deviation vector. Compute the smoothed estimates of $\hat{\beta}_{i,t}$ via

$$\tilde{\beta}_{i,t} = \tilde{\beta}_{i,t}^* + \hat{\mu}_i \quad (25)$$

Where

$$\tilde{\beta}_{i,t}^* = \lambda \tilde{\beta}_{i,t-1}^* + (1 - \lambda) \hat{\beta}_{i,t-1}^*, \quad \forall t = 2, 3, \dots$$

With the initial value of the smoothing being:

$$\tilde{\beta}_{i,1}^* = \hat{\beta}_{i,1}^* \quad (26)$$

$$\text{Let } \hat{b}_{i,t} = a_{i,t} \dots \text{for } i = 2, \dots, k \quad (27)$$

$$\text{Compute the residual series, given as: } \hat{b}_{i,t} = a_{i,t} + a'_{i,t} \tilde{\beta}_{i,t} \quad (28)$$

$$\text{where, } a'_{i,t} = (a_{1,t}, a_{2,t}, \dots, a_{i-1,t})'$$

Fit a univariate GARCH (1,1) model to each $\hat{b}_{i,t}$ series to obtain the conditional variance processes; $\hat{\sigma}_{b^2,i,t}$ for $i = 1, 2, \dots, k$

Compute the fitted volatility matrix Σ_t using $\tilde{\beta}_{i,t}$ and the conditional variance processes, where;
 $i = 1, 2, \dots, k$ See [8]

3.0 Dataset, Analysis and Results

The data used in this paper was obtained from the Nigerian Stock Exchange (NSE) daily official price list. The data consist of the average daily prices for the firms under study. These average daily prices were obtained by summing the daily high and low prices for each day and dividing by 2. The dataset is now made up of 2100 observations. The data was summed for the five working days (Monday to Friday) of each week and this reduced the observations to 420. This becomes the weekly prices of these firms. The sampled firms are:

- (1) Food, beverages and tobacco: PS Mandrides & Co PLC.
- (2) Health care: Pharma-Deko PLC.
- (3) Industrial/domestic products: Vitafoam Nigeria PLC.

The data used was for the period (2003-2011) and was analyzed using the MTS package in the R software.

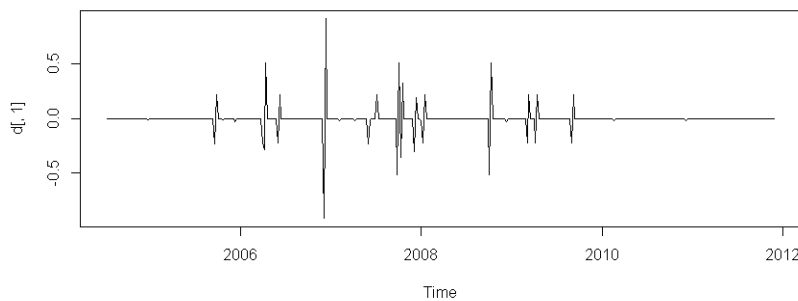


Figure 1: Time plots showing the log returns of stock prices of PS Mandrides & Co Plc.

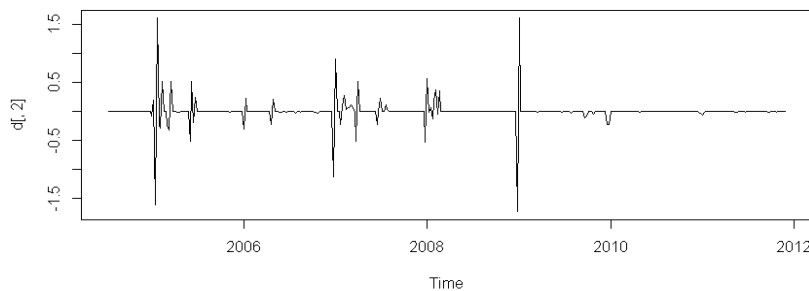


Figure 2: Time plots showing the log returns of stock prices of Pharma-Deko Plc.

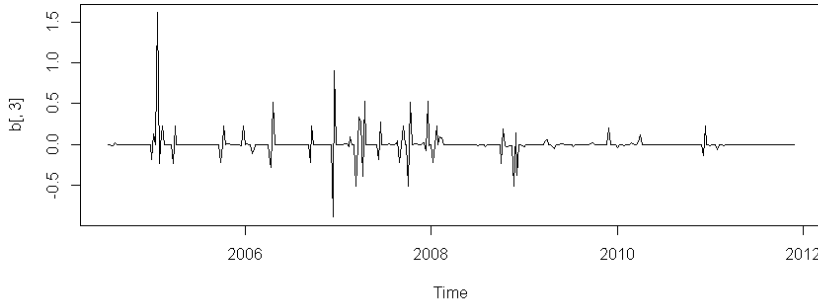


Figure 3: Time plots showing the log returns of stock prices of Vitafoam Nigeria Plc.

Data for the first 28-weeks were omitted from the plots because they were used to initiate the Recursive Least Squares (RLS) estimation.

The results for the Cholesky decomposition using the R software is as shown below:

Test Results

Sample means: 0.007959 0.007031 0.011854

Estimation of the first component

Estimate (alpha0, alpha1, beta1): 0.005445 0.456294 0

s.e. : 0.000403 0.124913 NaN

t-value : 13.5164 3.652902 NaN

Component 2: Estimation Results (residual series):

Estimate (alpha0, alpha1, beta1): 0.01167 1 0.212302

s.e. : 0.001372 0.196438 0.051465

t-value : 8.504057 5.090671 4.125158

Component 3: Estimation Results (residual series):

Estimate (alpha0, alpha1, beta1): 0.002181 1 0.395427

s.e. : 4e-04 0.114795 0.047184

t-value : 5.449933 8.711169 8.380464

For the recursive least squares estimation, the first twenty-eight (28) observations were used to compute the initial values so that the volatility estimation used three hundred & ninety two (392) observations; {that is from August 2004 to December 2011}. The mean equation consists only of sample means and the univariate GARCH (1,1) models are used for the transformed residual series “ b_t ”. Specifically, the three univariate GARCH (1,1) models are:

$$\sigma_{1t}^2 = 0.005445 + 0.456294 b_{1,t-1}^2 \quad (29)$$

$$\sigma_{2t}^2 = 0.01167 + b_{2,t-1}^2 + 0.212302 \sigma_{2,t-1}^2 \quad (30)$$

$$\sigma_{3t}^2 = 0.002181 + b_{3,t-1}^2 + 0.395427 \sigma_{3,t-1}^2 \quad (31)$$

where all the estimates are significant at the 5% level. Let

$$\hat{a}_t = Z_t - \hat{\mu} \quad (32)$$

where all but one of the estimates are significant at the 5% level and the sample means are: (0.007959, 0.007031, 0.011854).

Also let us denote the volatility matrix as Σ_t . We apply four tests for the conditional heteroscedasticity in the standardized residuals at the 5% level; with 5-lags and 10-lags respectively. The tests collectively called “Portmanteau test”, are:

- 1) $Q_k^*(m)$: Ljung-Box Statistics
- 2) $Q_k^r(m)$: robust version of the $Q_k^*(m)$ with 5% tail trimming.
- 3) $Q^*(m)$: based on the transformed scalar residuals e_t .
- 4) $Q_R(m)$: rank test based on the rank of e_t .

These tests are carried out using the MTS package in R.

Test results: (Tests for heteroscedasticity)

Test results: {AT LAG=5}

$Q(m)$ of ϵ_t :

Test and p-value: 0.1310221 0.9996846

Rank-based test:

Test and p-value: 70.49702 8.071321e-14

$Q_k(m)$ of $\epsilon_{t|t}$:

Test and p-value: 57.40352 0.1016099

Robust $Q_k(m)$:

Test and p-value: 114.592 5.455859e-08

Test results: {AT LAG=10}

$Q(m)$ of ϵ_t :

Test and p-value: 0.2959987 0.9999995

Rank-based test:

Test and p-value: 71.85167 1.944001e-11

$Q_k(m)$ of $\epsilon_{t|t}$:

Test and p-value: 60.86488 0.992073

Robust $Q_k(m)$:

Test and p-value: 141.7093 0.0004159671

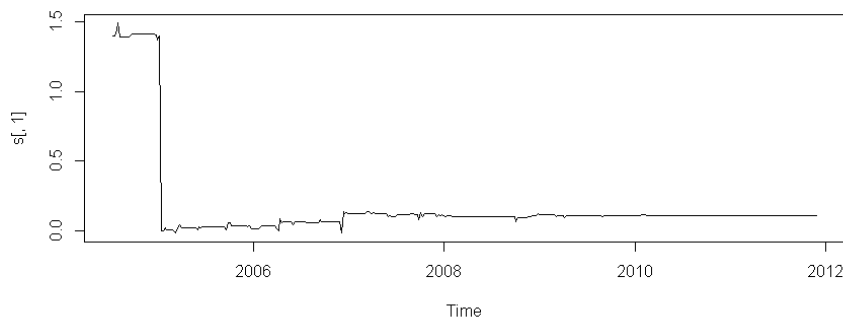


Figure 4: Time plots showing the RLS estimate for beta u

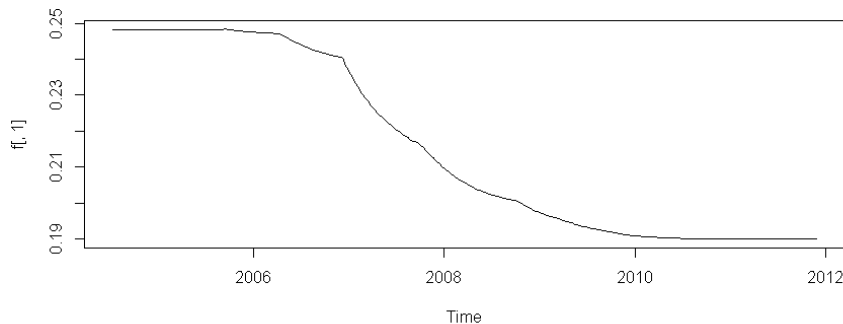


Figure 5: Time plots showing the smoothed version of the RLS estimate for beta u.

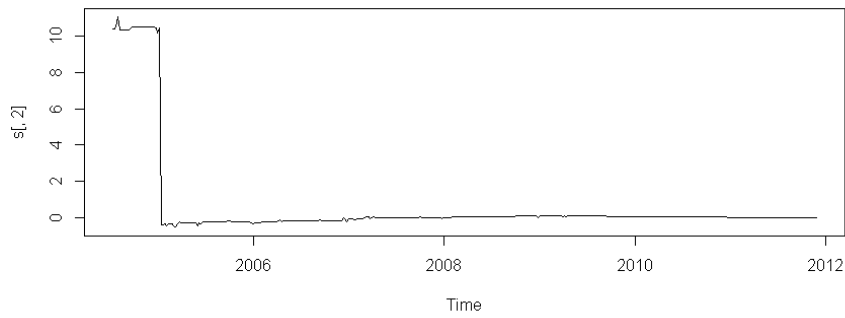


Figure 6: Time plots showing the RLS estimate for beta.beta

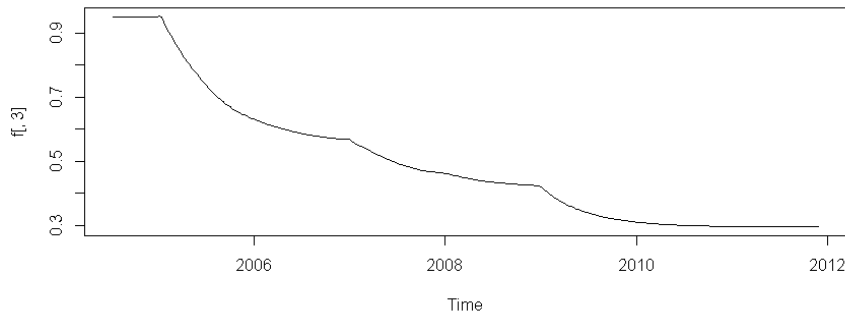


Figure 7: Time plots showing the smoothed version of the RLS estimate for beta. beta.

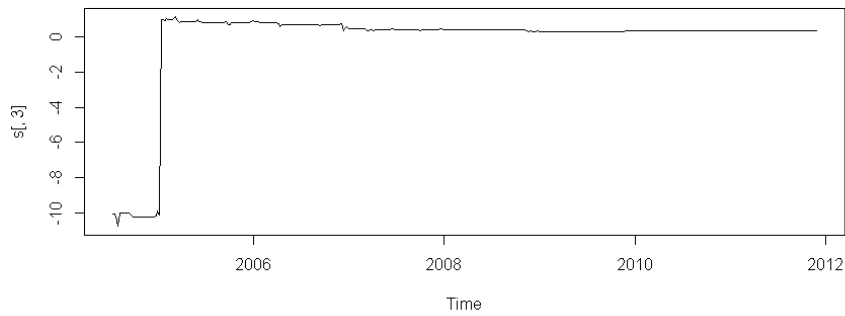


Figure 8: Time plots showing the RLS estimate for beta bb.

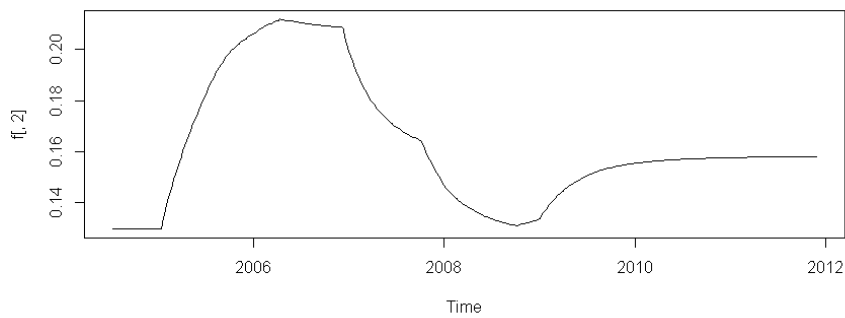


Figure 9: Time plots showing the smoothed version of the RLS estimate for beta bb.

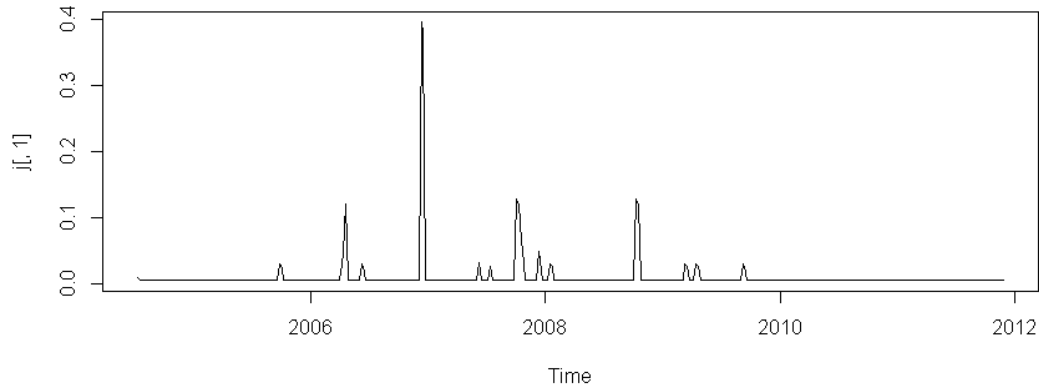


Figure 10: Time plots showing the volatility series of PS Mandrides & Co Plc.

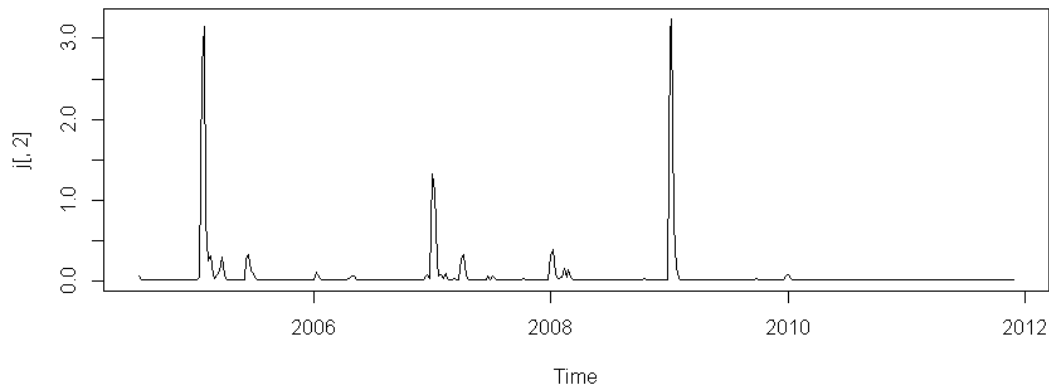


Figure 11: Time plots showing the volatility series of Pharma-Deko Plc.

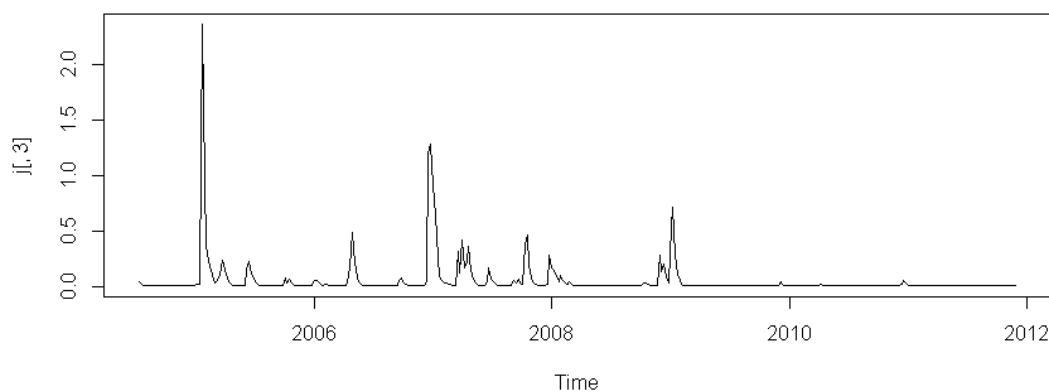


Figure 12: Time plots showing the volatility series of Vitafoam Nigeria Plc.

4.0 Discussion

Figures 1-3 shows the plots of the log returns of stock prices of the three firms under study. Figures 4, 6 and 8 gives the plots of the RLS estimates for the volatility test. The plots of the smoothed versions of the RLS estimates are provided in Figures 5, 7 and 9. Figure 10 which shows the volatility series of PS Mandrides and Co Plc. indicates that volatility fluctuated rapidly to a very high level at the year 2007, and subsequently at moderate levels till the end of the year 2009. Afterwards, it became constant till 2011. Thus within this period (2009-2011), the stock prices rate of PS Mandrides and Co. Plc was less volatile. Figure 11 showing the volatility series of Pharma-Deko Plc indicates that the asset experienced periods of high volatility at years 2005 and 2009. However, there were moderate fluctuations in volatility at other periods and it became constant between the periods 2010 and 2011. Figure 12 which displays the volatility series of Vitafoam Nigeria Plc shows that after the rapid increase in volatility between the periods of 2004-2005, moderate fluctuations was experienced at other periods within the scope of the study. With five (5)-lags and ten (10)-lags respectively, two out of the four portmanteau tests failed to detect conditional heteroscedasticity as indicated by their p-values.

5.0 Conclusion

A multivariate volatility modeling approach via cholesky decomposition has been presented. An application using data of stock prices of three firms: PS Mandrides and Co Plc, Pharma-Deko Plc and Vitafoam Nigeria Plc in the manufacturing sector for the periods 2003 to 2011 has been provided. The time plots of the volatility series of these firms' shows that stock prices were less volatile within the periods 2009-2011. This is an indication of stability in returns from investments in these firms within the periods.

6.0 References

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