# The Impact of Stability and Accuracy Analysis on Finite Difference Method in Option 

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#### Abstract

The finite difference method is one of the numerous numerical methods for price determination in the financial world. The method comprises of implicit, explicit and Crank Nicolson method. Each of these methods will be discussed extensively in conjunction with the stability and accurarcy problems regarding the application of the finite difference method

Our major goal is to obtain an accurate result with few computation as possible. The forward difference for time discretization is accurate to $\mathrm{O}(\Delta \boldsymbol{t})$ and the central difference for stock discretization is accurate to $O\left(\Delta S^{2}\right)$. However the finite difference method is accurate to $O\left(\Delta t, \Delta S^{2}\right)$. This paper will also reveal that the Crank Nicolson method is more accurate than the explicit and implicit method with an accurarcy of up to $O\left(\Delta t^{2}, \Delta S^{2}\right)$ which will be achieved by equating the central difference and the symmetric central difference at $\boldsymbol{f}_{n+\frac{1}{2} m \equiv f\left(t+\frac{\Delta t}{2}, s\right)}$.


### 1.0 Introduction

This research work will focus on the types of finite difference method which are; Implicit Finite Difference (IFD), Explicit Finite Difference (EFD) and Crank Nicolson method. These are closely related but differ in stability, accuracy and execution of speed [1]. The issue of stability and accuracy is usually referred to as the two major problems associated with the method of finite difference. Amongst these categories of finite difference method, some are more accurate and stable compared to others. For instance, the forward difference for time discrtization is accurate to $O(\Delta t)$ and the central difference for stock discrtization is accurate to $O\left(\Delta S^{2}\right)$.
Crank Nicolson method is the average of the implicit and the explicit method and it is more accurate than the implicit and the explicit method with an accurarcy of up to $O\left(\Delta t^{2} \Delta S^{2}\right)$. This will be illustrated in this paper.

### 1.1 The Explicit Finite Difference Method (EFD).

Given the value of an option at the maturitytime, it is possible to give an expression that gives us the next value $f_{m, n}$ explicitly in terms of the given values $f_{m-1, n+1,} f_{m, j+1}$ and $f_{m+1, n+1}$.
Let's consider the Black- Scholes PDE given as
$\frac{\partial f\left(S_{t}, t\right)}{\partial S_{t}} r S_{t}+\frac{\partial f\left(S_{t}, t\right)}{\partial t}+\frac{1}{2} S_{t}^{2} \delta^{2} \frac{\partial^{2} f\left(S_{t, t}\right)}{\partial S_{t}^{2}}=r f\left(S_{t, t}\right)$
We discretize the Black - Scholes PDE by taking the forward difference for time discretization and the central difference for the asset price discretization. This gives
$\frac{f_{n+1, m}-f_{n, m}}{\Delta t}+\frac{r m \Delta S}{2 \Delta S}\left(f_{n+1, m+1}-f_{n+1, m-1}\right)+\frac{\delta^{2} m^{2} \Delta S^{2}}{2 \Delta S^{2}}\left(f_{n+1, m-1}-2 f_{n+1, m}+f_{n+1, m+1}\right)=r f_{n, m}$
By rearranging, we have
$f_{n, m}=\frac{1}{1+r \Delta t}\left[\beta_{1 m} f_{n+1, m-1}+\beta_{2 m} f_{n+1, m}+\beta_{3 m} f_{n+1, m+1}\right]$ forn $=0,1, \ldots \mathrm{~N}-1$ and $\mathrm{m}=1,2, \ldots \mathrm{M}-1$
The forward difference for the discretization is equivalent to $O(\Delta t)$ and the central difference for asset discretization is equal to $O\left(\Delta S^{2}\right)$. Therefore the finite difference method is equivalent to $O\left(\Delta t, \Delta S^{2}\right)$. The weight in (1.2) are given by

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$$
\begin{align*}
& \beta_{1 m}=\frac{1}{2} \delta m^{2} \Delta t-\frac{1}{2} r m \Delta t \\
& \beta_{3 m}=\frac{1}{2} r m \delta t+\frac{1}{2} \delta^{2} m^{2} \Delta t \tag{1.4}
\end{align*}
$$

Where $\beta_{1 m}, \beta_{2 m}$, and $\beta_{3 m}$ are the risk neutral probability of the three asset prices of $S-\Delta S$ and $S+\Delta S$ respectively at $t+\Delta t$ and they sum to unity assuming that the expectedreturns on asset is also true in risk neutral world. However the explicit finite difference will be more accurate if the three probabilities are positive. To achieve this, we make use of the following conditions
$\delta^{2} m^{2} \Delta t<1$ and $r<\delta^{2} m$ [2].
The system of equation in (1.3) give rise to a tridiagonal system written as $A u+e=b$. The vector $e$ is a result of the boundary conditions at $M=0$ and M for all $n>0$. The system is given as;

This system can be written in the form $A f_{n+1, m}=f_{n, m}$ for $m=0,1, \ldots, M$ and we ignore the error term as the boundary condition will take care of them[1]. The vector of asset price $f_{n+1, m}$ is known at time $T$ from our initial boundary condition, we can move backward by solving for $f_{n, m}$
$(m=0,1 \ldots, M)$ using the matrix A which comprises of the probabilities, $\beta_{k m}(k=1,2,3)$ that are known. The backward iteration lead us to the value of the option obtained at time zero.
The iteration in finding the solution leads to finding errors. Solving the difference is to solve to obtain the numerical solutionif the errors are magnified at each iteration, the system is stable otherwise it is unstable. There are two major problems associated with the use of finite difference meshes, these are stability and accuracy of the method but our major concern is to obtain an accurate solution with a few iterations, hence stability and accuracy are important.

### 1.2 Stability Analysis

Truncation error in the stock price discretization and in the time discretization are the two major sources of error. The implication of truncation erroris that numerical scheme solves a problem that is not exactly the same as the problem we are trying to solve.
Remark I : Numerical scheme have the following characteristics;
(a) Convergency: the solution to a Finite Difference Equation (FDE) approaches the true solution to the PDE as both grid, interval and time step sizes are reduced.
(b) Consistency : consistency exists if the difference between PDE and FDE vanishes as the mesh interval and time step size approaches zero. The error vanishes so that
$\lim _{\Delta t \rightarrow 0}(P D E-F D E)=0$. This also reveals how well the FDE approximate the PDE and it is a necessary condition for stability.
(c) Stability : for a stable numerical scheme, the errors from any source will not grow unboundedly with time.

The above three characteristics are linked togetherby Lax equivalent theorem which we state without proof.
THEOREM (Lax equivalent theorem):This states that given a properly posed linear initial value problem and a consistent finite difference scheme, stability is the necessary and sufficient condition for convergence[3].

### 1.3 A Necessary and Sufficient Condition For Stability

Let $f_{n+1}=A f_{n}$ be a system of equations where $A$ is the matrix and $f_{n+1}$ and $f_{n}$ are the column vector as represented in (1.5)

$$
\begin{align*}
& f_{n}=A f_{n-1} \\
&=A^{2} f_{n-2} \\
&= A^{n} f_{0} \text { for } \mathrm{n}=1,2, \ldots . ., \mathrm{N} \tag{1.6}
\end{align*}
$$

Where $f_{0}$ is the vector of initial value $f_{0}$ to $f_{0}^{*}$. The exact solution at the $n^{\text {th }}$ term row will then be
$f_{n}^{*}=A^{n} f_{0}^{*}$
(1.7)

Let the perturbation or error vector $e$ be defined by $e=f_{0}^{*}-f_{n}$
Applying (1.6) and (1.7) we have we have
$e_{n}=f_{n}^{*}-f_{n}=A^{n}\left(f_{0}^{*}-f_{0}\right)=A^{n} e_{0}$ for $\mathrm{n}=1,2, \ldots, \mathrm{~N}$
Hence for compartible matrix and vector norms[3]
$\left\|e_{n}\right\| \leq\left\|A^{n}\right\|\left\|e_{0}\right\|$
According to Lax and Richtmyer in [3] the difference scheme is said to be stable when there exist a positive number in independent of $\mathrm{n}, \Delta t$ and $\Delta s$ such that
$\left\|A^{n}\right\| \leq L$, for $\mathrm{n}=1,2, \ldots \ldots ., \mathrm{N}$.
This limits the amplification of any perturbation and hence for any arbitrary initial rounding errors
$\left\|e_{n}\right\| \leq L\left\|e_{0}\right\|$
Since $\left\|A^{n}\right\|=\left\|A . A^{n-1}\right\| \leq\|A\|\left\|A^{n-1}\right\| \leq \cdots \leq\left\|A^{n}\right\|$ then the Lax- Richtmyer's definition of stability is satisfied when $\|\mathrm{A}\| \leq 1$
Equation (1.9) is the necessary and sufficient condition for the difference equation to be stable [1]. Since the spectral radius $\rho(A)$ ssatifies

$$
\begin{equation*}
\rho(A) \leq\|A\| \tag{1.10}
\end{equation*}
$$

It follows automatically from (1.9) that $\rho(A) \leq 1$
$\|\mathrm{A}\|_{2}=\rho(A)=\max \left|\lambda_{i}\right|$
Where $\lambda_{i}$ is an eigen value of matrix $A$.
The other method used in the analysis of stability is the use of eigen values of the tridiagonal system, the eigen values of $N \times N$ matrix.


Are $\lambda_{n}=y+2[\sqrt{x z}] \cos \frac{n \pi}{N+1}$, for $\mathrm{n}=1,2, \ldots \ldots . \mathrm{N}$ where xy may be real or complex [4].

### 2.0 The Implicit Finite Difference Method

In this case, we still express $f_{n+1, m}$ implicitly in terms of the unknowns $f_{n, m-1}, f_{n, m}$ and $f_{n, m-1}$ and discretize the BlackScholes PDE given in (1.1) using the forward difference for time and central difference for the stock price to have
$\frac{f_{n+1, m}-f_{n, m}}{\Delta t}+r m \Delta s\left[\frac{f_{n, m+1}-f_{n, m-1}}{2 \Delta s}\right]+\frac{1}{2} \delta^{2} m^{2} \Delta s^{2}\left[\frac{f_{n, m+1}-2 f_{n, m}+f_{n, m-1}}{\Delta s^{2}}\right]=r f_{n+1, m}$
By rearranging we get
$f_{n+1, m}=\frac{1}{1-r \Delta t}\left[\alpha_{1 m} f_{n, m-1}+\alpha_{2 m} f_{n, m}+\alpha_{3 m} f_{n+1, m}\right]$
For $\mathrm{n}=0,1, \ldots \ldots ., N-1$ and $\mathrm{m}=1,2, \ldots \ldots, M-1$
Similar to the explicit method, the implicit method is accurate to $O\left(\Delta t, \Delta s^{2}\right)$. The parameter $\alpha_{k m}$ for $k=1,2,3 \ldots .$. are given as
$\alpha_{2 m}=1+\delta^{2} m^{2} \Delta t$,

$$
\begin{aligned}
& \alpha_{1 m}=\frac{1}{2} r m \Delta t-\frac{1}{2} \delta^{2} m^{2} \Delta t \\
& \alpha_{3 m}=-\frac{1}{2} r m \Delta t-\frac{1}{2} \delta^{2} m^{2} \Delta t
\end{aligned}
$$

The system of equation can be expressed as a tridiagonal system as,

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Which can be written as $A f_{n, m}=f_{n+1, m}$ for $m=0,1 \ldots \ldots ., \mathrm{M}$.
Suppose $f_{n, m}=f_{n}$, then we solve for the $f_{n}$ given matrix A and column vector $f_{n+1}$ which implies that $f_{n}=A^{-1} f_{n+1}$. Matrix A has $\alpha_{2 m}=1+\delta^{2} m^{2} \Delta t$ in the diagonal which is positive. We can solve the above system by finding the inverse matrix $A^{-1}$.
Applying the boundary conditions together with (2.2), we observe some changes in the element of matrix A with $\alpha_{20}, \alpha_{2 m}=$ 1 and $\alpha_{30}, \alpha_{1 m}=0$. Recall that our initial condition gives values for the $N^{t h}$ time step, and we solve for $f_{n}$ at $t_{n}$ in terms of $f_{n+1}$ at $t_{n+1}$. This is done by setting the right hand side of the system to our initial condition inorder to produce a solution for time step $N-1$. By repeatedly iteration in such a manner, we obtain the value of $f$ at any time step $0,1, \ldots . ., N-1$. The difference between the explicit method and implicit method is that the implicit method allows the usage of large numbers of s-mesh point without having to take small time step, also the implicit method although unconditionally stable, more efficient than the explicit method.

### 2.1 The Stability Issue of Explicit Method

The matrix A in (1.5) is use to analyse stability of the explicit finite difference method where $\beta_{k m}$ for $\mathrm{k}=1,2,3$, are given by (1.4). If $u_{n}$ is the $n^{\text {th }}$ eigenvalue of A , then
$\|\mathrm{A}\|_{2}=\rho(A)=\max _{n}\left|u_{n}\right|$
The eigenvalue $u_{n}$ are given by
$u_{n}=\beta_{2 m}+2\left[\beta_{1 m} \beta_{3 m}\right]^{1 / 2} \cos \frac{n \pi}{N} \quad$ for $n=1,2, \cdots \cdots, N-1$
By substitution the values of $\beta$ 's we have
$u_{n}=1-\delta^{2} m^{2} \Delta t+\delta^{2} m^{2} \Delta t\left[1-\frac{r^{2}}{\delta^{4} m^{2}}\right]^{1 / 2}\left[1-\sin ^{2} \frac{n \pi}{2 N}\right]$ for $n=1,2, \cdots \cdots, N$
By applying binomial expansion on the square root part, ignoring some terms and rearranging, we get
$u_{n} \approx 1-2 \delta^{2} m^{2} \Delta t+\sin ^{2} \frac{n \pi}{2 N}$, therefore the equation are stable when
$\|A\|_{2}=\max \left|1-2 \delta^{2} m^{2} \Delta t \sin ^{2} \frac{n \pi}{2 N}\right| \leq 1$, that is
$-1 \leq 1-2 \delta^{2} m^{2} \Delta t \sin ^{2} \frac{2 \pi}{2 N} \leq 1$ for $n=1,2, \cdots \cdots, N-1$
As $\Delta t \rightarrow 0, N \rightarrow \infty$ and $\sin ^{2}\left(\frac{(N-1) \pi}{2 N}\right) \rightarrow 1$,
Hence $0 \leq \delta^{2} m^{2} \Delta t \leq 1$
Therefore by Lax's equivalence theorem, the scheme is stable, convergent and consistent by (2.7).

### 2.2 Change of Variable of the Explicit Method

In this instance, we will apply the boundary condition and considered the differential equation given as
$\frac{\partial g}{\partial y}+\left(r-\frac{\delta^{2}}{2}\right) \frac{\partial g}{\partial y}+\frac{\delta^{2} \partial^{2} g}{2 \partial y^{2}}-r g=0$
In deriving the explicit finite difference method for change of variable, let's discretize the stock price with the central difference scheme and time by forward difference and substitute into (2.8) to get
$\frac{g(t+\Delta t, y)-g(t, y)}{\Delta t}+\frac{\left(r-\frac{\delta^{2}}{2}\right)}{2 \Delta y}[g(t+\Delta t, y+\Delta y)-g(t+\Delta t, y-\Delta y)]+\frac{\partial^{2}}{2 \Delta y^{2}}[g(t+\Delta t, y-\Delta y)-2 g(t+\Delta t, y)+$
$g(t+\Delta t, y+\Delta y)]=r g(t, y)$
By rearranging, we gets
$g_{m, n}=\frac{1}{1+r \Delta t}\left(\beta_{1 g n+1, m-1}^{*}+\beta_{2 g n+1, m}^{*}+\beta_{3 g n+1, m+1}^{*}\right)(2.10)$
Where

$$
\begin{gather*}
\beta_{1}^{*}=\frac{1}{2}\left[\frac{\delta}{\Delta y}\right]^{2} \Delta t-\frac{1}{2} \frac{\left(r-\frac{\delta^{2}}{2}\right)}{\Delta t} \Delta t \\
\beta_{2}^{*}=1-\left[\frac{\delta}{\Delta y}\right]^{2} \Delta t \tag{2.11}
\end{gather*}
$$

$\beta_{3}^{*}=\left[\frac{\delta}{\Delta y}\right] \Delta t+\frac{1}{2} \frac{\left(r-\frac{\delta^{2}}{2}\right)}{\Delta y} \Delta t$
Are the new weight.

### 2.3 Stability Issue of the Change of Variable

To analyse the stability of the explicit FDM, we use the matrix method. Now consider $\beta_{k}^{*}$ for $(k=1,2,3)$ the new weight that makesup the matrix under consideration, the parameter in (2.11) will help us in carrying out the analysis if
$1-\left[\frac{\delta^{2}}{\Delta y}\right]^{2} \Delta t \geq 0$ then
$\left[\frac{\delta^{2}}{\Delta y}\right] \Delta t \leq 1$ and $\|A\|_{\infty}=\beta_{1}^{*}+\beta_{2}^{*}+\beta_{3}^{*}=1$
When $1-\left[1-\left[\frac{\delta^{2}}{\Delta y}\right]^{2}\right] \Delta t \leq 0$, then $\left|1-\left[\frac{\delta^{2}}{\Delta y}\right]^{2} \Delta t\right|=\left[\frac{\delta^{2}}{\Delta y}\right] \Delta t-1$ and $\|A\|_{\infty}=2\left[\frac{\delta^{2}}{\Delta y}\right] \Delta t-1>1$
Therefore by Lax's equivalence theorem, the scheme is stable, convergent and consisted for

$$
0 \leq\left[\frac{\delta^{2}}{\Delta y}\right]^{2} \Delta t \leq 1
$$

### 2.4 The Stability Issue of the Implicit Method

Given the eigenvalue
$u_{n}=\alpha_{2 m}+2\left[\alpha_{1 m} \alpha_{3 m}\right]^{1 / 2} \cos \frac{n \pi}{N}$ for $n=1,2, \cdots \cdots, N-1$ (2.12)
Integrating the value of $\alpha^{\prime} s$ we have
$\alpha_{n}=1+\delta^{2} m^{2} \Delta t+\delta^{2} m^{2} \Delta t\left[1-\frac{r^{2}}{\delta^{4} m^{2}}\right]^{1 / 2}\left[1-2 \sin ^{2} \frac{n \pi}{2 N}\right]$
Applying the binomial expansion on the square root part and rearranging,
$\alpha_{n} \approx 1+2 \delta^{2} m^{2} \Delta t-2 \delta^{2} m^{2} \Delta t \sin ^{2} \frac{n \pi}{2 N}$.
The equation are stable when
$\|\mathrm{A}\|_{1}=\max \left|1+2 \delta^{2} m^{2} \Delta t-2 \delta^{2} m^{2} \Delta t \sin ^{2} \frac{n \pi}{2 N}\right| \leq 1$
That is,
$-1 \leq 1+2 \delta^{2} m^{2} \Delta t-2 \delta^{2} m^{2} \Delta t \sin ^{2} \frac{n \pi}{2 N}$ for $n=1,2, \cdots \cdots, N-1$
As $\Delta t \rightarrow 0, N \rightarrow \infty$ and $\sin ^{2} \frac{(N-1) \pi}{2 N} \rightarrow 1$, (2.14) reduces to $|1|$.
Alternatively, $1+\delta^{2} m^{2} \Delta t \geq 0$ and $\|\mathrm{A}\|_{\infty}=1$
Therefore by Lax's theorem, the scheme is unconditionally stable, convergent and consistent.

### 2.5 Change of Variable of the Implicit Method

By substituing the finite difference approximationfor the asset price and time into (2.8) we have,
$\frac{g(t+\Delta t, y)-g(t, y)}{\Delta t}+\frac{r-\delta^{2} / 2}{2 \Delta y}[g(t, y+\Delta y)-g(t, y-\Delta y)]+\frac{\delta^{2}}{2 \Delta y^{2}}[g(t, y-\Delta y)-2 g(t, y)+g(t, y+\Delta y)]=r g(t+\Delta t, y)$
(2.15)

We rearrange to get
$g_{n+1, m}=\frac{1}{1-r \Delta t}\left[\alpha_{1}^{*} g_{n, m-1}+\alpha_{2}^{*} g_{n, m}+\alpha_{3}^{*} g_{n, m+1}\right]$
Where $\alpha_{k}^{*}$ for $k=1,2,3$ are given by

$$
\begin{gather*}
\alpha_{1}^{*}=\frac{1 / 2\left(r-\frac{\delta^{2}}{2}\right)}{\Delta y} \Delta t-\frac{1}{2}\left(\frac{\delta}{\Delta y}\right)^{2} \Delta t \\
\alpha_{2}^{*}=1+\left(\frac{\delta}{\Delta y}\right)^{2} \Delta t \tag{2.17}
\end{gather*}
$$

$\alpha_{3}^{*}=\frac{-1 / 2\left(r-\frac{\delta^{2}}{2}\right)}{\Delta y} \Delta t-\frac{1}{2}\left(\frac{\delta}{\Delta y}\right)^{2} \Delta t$
The implicit method is generally better but a bit more difficult to implement than the explicit finite method.

### 3.0 Crank Nicolson Method

The Crank Nicolson method is the average of the explicit and implicit finite difference method. This is achieved by using equation (1.3) and (2.2) which are the explicit and implicit methods equations respectively. Taking the average of the two equations we have,
$\frac{f_{n+1, m}-f_{n, m}}{\Delta t}+\frac{r m \Delta s}{4 \Delta s}\left[f_{n+1, m+1}-f_{n, m+1}-f_{n, m-1}\right]+\frac{\delta^{2} m^{2} \Delta s^{2}}{4 \Delta s^{2}}\left[f_{n, m-1}-2 f_{n, m+1}+f_{n+1, m-1}-2 f_{n+1, m}+f_{n+1, m+1}\right]\left[\frac{1}{2}\left(r f_{n, m}+\right.\right.$ $\left.\left.r f_{n+1, m}\right)\right]$

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By rearranging we have

$$
\begin{array}{r}
\quad\left(\frac{1}{4} \delta^{2} m^{2} \Delta t-\frac{1}{4} r m \Delta t\right) f_{n, m+1}+\left(1+\frac{1}{2} r \Delta t+\frac{1}{2} \delta^{2} m^{2} \Delta t\right) f_{n, m}+\left(-\frac{1}{4} \delta^{2} m^{2} \Delta t-\frac{1}{4} r m \Delta t\right) f_{n, m+1} \\
=\left(\frac{1}{4} \delta^{2} m^{2} \Delta t-\frac{1}{4} r m \Delta t\right) f_{n+1, m+1}+\left(1-\frac{1}{2} r \Delta t-\frac{1}{2} \delta^{2} m^{2} \Delta t\right) f_{n+1, m}+\left(\frac{1}{4} r m \Delta t+\frac{1}{4} \delta^{2} m^{2} \Delta t\right) f_{n+1, m+1}(3.2)
\end{array}
$$

For simplicity (2.19) can be written as
$\varphi_{1 m} f_{n, m-1}+\varphi_{2 m} f_{n, m}+\varphi_{3 m} f_{n, m+1}=X_{1 m} f_{n+1, m-1}+X_{2 m} f_{n+1, m}+X_{3 m} f_{n+1, m+1}(3.3)$
For $n=0,1, \cdots \cdots, N-1$ and $m=1,2, \cdots \cdots, M-1$, such that the parameter $\varphi_{k m}$, and $X_{k m}$ for $k=1,2,3$ are given as;

$$
\begin{align*}
\varphi_{1 m} & =\frac{1}{4} r m \Delta t-\frac{1}{4} \delta^{2} m^{2} \Delta t \\
\varphi_{2 m} & =1+\frac{1}{2} r \Delta t+\frac{1}{2} \delta^{2} m^{2} \Delta t \\
\varphi_{3 m} & =-\frac{1}{4} \delta^{2} m^{2} \Delta t-\frac{1}{4} r m \Delta t \\
X_{1 m} & =\frac{1}{4} \delta^{2} m^{2} \Delta t-\frac{1}{4} r m \Delta t \\
X_{2 m} & =1-\frac{1}{2} r \Delta t-\frac{1}{2} \delta^{2} m^{2} \Delta t \tag{3.4}
\end{align*}
$$

$X_{3 m}=\frac{1}{4} r m \Delta t+\frac{1}{4} \delta^{2} m^{2} \Delta t$
The system in (2.20) can be express as a matrix of the form $B f_{n}=C f_{n+1}$. This result into a tridiagonal system given by


The elements of vector $f_{n+1}$ are known at maturity time $T$, hence the system can be expressed as $f_{n}=B^{-1} C f_{n+1}$. By repeated iteration from T to time zero, we have the value of $f$ as the price of the option. The diagonal entries of matrix B is $\varphi_{2 m}=1+$ $\frac{r \Delta t}{2}+\delta^{2} m^{2} \frac{\Delta t}{2}$. This imply that the matrix is non singular since the diagonal entries are non zero.
The boundary conditions and (2.20) result in some entries change in the tridiagonal matrix B and C . For $\mathrm{B}, \varphi_{20}, \varphi_{2 m}=1$ and $\varphi_{30}, \varphi_{1 m}=0$ and for $\mathrm{C}, X_{20}, X_{2 m}=1$ and $X_{30}, X_{1 m}=0$.

### 3.1 Crank Nicolson Method Accuracy Test

The Crank Nicolson method is more accurate than the explicit and the implicit method with an accuracy of up to $O\left(\Delta t^{2}, \Delta s^{2}\right)$. This will be illustrated by equating the central difference and the symmetric central difference at $f_{n+\frac{1}{2}, m} \equiv$ $f\left(t+\frac{\Delta t}{2}, s\right)$.
By expanding $f_{n+1, m}$ using Taylor's series at $f_{n+\frac{1}{2}, m}$ we have
$f_{n+1, m}=f_{n+\frac{1}{2}, m}+\frac{1}{2} \frac{\partial f}{\partial t} \Delta t+O\left(\Delta t^{2}\right)$
Also expanding $f_{n, m}$ at $f_{n+\frac{1}{2} m}$ gives
$f_{n, m}=f_{n+\frac{1}{2}, m}-\frac{1}{2} \frac{\partial f}{\partial t} \Delta t+O\left(\Delta t^{2}\right)$

Taking average of (2.23) and (2.24), we have
$\frac{1}{2}\left(f_{n, m}+f_{n+1, m}\right)=f_{n+\frac{1}{2}, m}+O\left(\Delta t^{2}\right)$.
The subscripts $m$ was arbitrary and we can write this for subscripts $m-1, m$ and $m+1$ as follows;
$f_{n+\frac{1}{2}, m-1}-2 f_{n+\frac{1}{2}, m}+f_{n+\frac{1}{2}, m+1}=\frac{1}{2}\left(f_{n, m}-1-2 f_{n, m}+f_{n+1, m+1}\right)+O\left(\Delta t^{2}\right)$
Note that, the RHS of (3.8) is an average of two symmetric central difference centered at mesh points $n$ and $n+1$
Dividing by $\Delta s^{2}$ we obtain the equality
$\frac{\partial^{2} f(t+1 / 2 \Delta t, S)}{\partial S^{2}}=\frac{1}{2}\left(\frac{\partial^{2} f(t, S)}{\partial S^{2}}+\frac{\partial^{2} f(t+\Delta t, S)}{\partial S^{2}}\right)+O\left(\Delta t^{2}, \Delta S^{2}\right)$
Which is the second order partial derivative defined by symmetric central difference approximation. The subscripts m is arbitrary and we derived the central difference approximation as follows;
$f_{n+\frac{1}{2}, m+1}-f_{n+\frac{1}{2}, m-1}=\frac{1}{2}\left(f_{n, m+1}-f_{n, m}\right)+\frac{1}{2}\left(f_{n+1, m+1}-f_{n+1, m-1}\right)+O\left(\Delta t^{2}\right)$
Again if we divide the equation with $2 \Delta s$, we have the equality
$\frac{\partial f(t+1 / 2 \Delta t, s)}{\partial s}=\frac{1}{2}\left(\frac{\partial f(t, s)}{\partial s}+\frac{\partial f(t+\Delta t, s)}{\partial s}\right)+O\left(\Delta t^{2}, \Delta S^{2}\right)$
Which is the first order partial derivative defined by symmetric central difference approximation. We now substract (2.24) from (2.23) to get the approximation of $\frac{\partial f}{\partial t}$ centered at $(t+1 / 2 \Delta t, S)$.
$\frac{\partial f(t+1 / 2 \Delta t, s)}{\partial s}=\frac{f_{n+1, m}-f_{n, m}}{\Delta t^{2}}+O\left(\Delta t^{2}\right)(3.12)$
The Crank Nicolson method has a leading truncation error of order $\mathrm{O}\left(\Delta t^{2}, \Delta S^{2}\right)$ [5].

### 4.0 Summary and Conclusion

The finite difference method is one of the various numerical analytical techniques for option price valuation. This method is further divided into implicit, explicit and the crank nicolson method. In application, the method is known to have two major problems which are stability and accuracy.
We have carefully examined the stability issues of the finite difference method as it affects the three types of the finite difference method. It was obvious that though all categories of the FDM are accurate and stable but some are more accurate and stable than others. Applying various stability analysis and accuracy test, taking into consideration the boundary condition, it became evidents that the necessary and sufficient condition given in (1.9) is required for the difference equation to be stable. Also by Lax's equivalence theorem, the numerical scheme referred to in (2.7) is stable,convergent and consistent Under change of variable of the Explicit method, we considered the differential equation in (2.8) with the application of the boundary condition. The discretization of the stock price with central difference scheme and time by foward difference and using (2.8), we obtained equation (2.10) with parameter $\beta_{k}^{*}$, where $k=1,2,3$ in (2.11), as the weight.
Also in the case of the Implicit method, if we substitute the finite difference approximation for the asset price and time into (2.8), we have (2.15) with parameter $\alpha_{k}^{*}$ in (2.16) as the weight. Finally the Crank Nicolson accuracy test was shown by equating the central difference and the symmetric central difference at $f_{n+1 / 2, m}=f\left(t+\frac{\Delta t}{2}, S\right)$. The Crank Nicolson method have a leading truncation error of order $\mathrm{O}\left(\Delta t^{2}, \Delta S^{2}\right)$.
In conclusion therefore, this study has been able to reveal that, though the issues of stability and accuracy affect the three different types of the FDM, some are more accurate when compared to others. For instance, the implicit method is generally better and more accurate than the Explicit method while the Crank Nicolson method is more accurate than the Explicit and Implicit method with an accuracy of up to $\mathrm{O}\left(\Delta t^{2}, \Delta S^{2}\right)$ and the finite difference method is accurate to $\mathrm{O}\left(\Delta t, \Delta S^{2}\right)$.

### 5.0 References

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