Journal of the Nigerian Association of Mathematical Physics
Volume 34, **(***March, 2016***), pp 57 – 64**

Orthogonal Diagonalization of Symmetric Rhotrices

Muhammad Aminu

Department of Mathematical Sciences, Abubakar Tafawa Balewa University, Bauchi, P.M.B 0248, Bauchi, Nigeria.

Abstract

A rhotrix R_n is said to be symmetric if $R_n = R_n^T$, such rhotrices are always diagonalizable. We present in this paper, a special way of diagonalizing such rhotrices called orthogonal diagonalization.

Keywords: Rhotrix, eigenvalue, eigenvector, diagonalization, orthogonal, symmetric

1.0 Introduction

Mathematical arrays that are in some way between two-dimensional vectors and (2×2)-dimensional matrices and matrix-tertions and noitrets were discussed in [1], as a result of this Ajibade in [2] introduced an object which lies in some ways between (2×2)-dimensional

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and (3× 3)-dimensional $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ matrices, and he called such an object a rhotrix. Algebra of rhotrices where initially introduced in [2] by Ajibade. Let R and Q be two rhotrices such that

$$R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle \text{and} Q = \left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & k & \end{matrix} \right\rangle \qquad (1)$$

Ajibade [2] defined the addition of these two rhotrices R and Q as:

$$R + Q = \left\langle \begin{matrix} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{matrix} \right\rangle,$$

and their multiplication as:

$$R \circ Q = \left\langle \begin{matrix} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{matrix} \right\rangle$$

Another multiplication method for rhotrices called row-column multiplicationwas introduced by Sani [3] in an effort to answer some questions raised by Ajibade. Using the rhotrices R and Q as defined in (1), Sani [3] illustrated the row-column multiplication of rhotrices as:

Corresponding author: Muhammad Aminu, E-mail: muhammadaminu47@gmail.com, Tel.: +2347064712475

$$R \circ Q = \left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bj+ek & \end{array} \right\rangle$$

A generalization of the row-column multiplication method for n-dimensional rhotrices was given by Sani [4]. That is: given n-dimensional rhotrices $R_n = \left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle$ and $Q_n = \left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle$ the multiplication of R_n and Q_n is as follows:

$$R_n \circ Q_n = \left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle \circ \left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle = \left\langle \sum_{i_2 j_1 = 1}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1 = 1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle,$$

$t = (n+1)/2$.

The method of converting a rhotrix to a special matrix called 'coupled matrix' was suggested by Sani [5]. The system $R_n x = b$ for which R_n is an n-dimensional rhotrix, x the unknown n-dimensional rhotrix vector and b the right-hand-side rhotrix vector was introduced by Aminu in [6], and a discussion was provided for the necessary and sufficient condition for the solvability of systems of the form $R_n x = b$. If a system is solvable it was shown how a solution can be found. Sharma and Kumar in [7] introduced the Hadamard rhotrices and developed balanced incomplete block designs (BIBD) using Hadamard rhotrices. Rhotrix diagonalization problem (RDP) was first introduced by Usaini and Muhammad [8], and they provided a way of diagonalizing rhotrices. In this paper, we introduce another way of diagonalizing symmetric rhotrices called orthogonal diagonalization.

2.0 Rhotrix Diagonalization

The idea of finding the eigenvalue and eigenvector of a rhotrix as defined by Aminu [6] will be used here, since before diagonalizing a rhotrix, we first of all need to find the eigenvalue and eigenvector of that rhotrix. Aminu [6] defined the rhotrix eigenvalue problem as:

Given $R_n = \left\langle a_{ij}, c_{lk} \right\rangle$, we find all $\lambda \in \mathbb{R}$ (eigenvalue) and an n-dimensional rhotrix column vector $\langle x^{nj} \rangle$, $\langle x^{nj} \rangle \neq 0$ (eigenvector) such that

$$R_n \langle x^{nj} \rangle = \lambda \langle x^{nj} \rangle.$$

Two rhotrices R_n and Q_n are similar if there exist an invertible rhotrix P_n such that

$$P_n^{-1} R_n P_n = Q_n \tag{2}$$

A rhotrix R_n is diagonalizable if it is similar to a diagonal rhotrix; in other words, if there is a diagonal rhotrix D_n and an invertible rhotrix P_n such that

$$P_n^{-1} R_n P_n = D_n \tag{3}$$

If a rhotrix R_n is diagonalizable, and that $P_n^{-1} R_n P_n = D_n$, where D_n is a diagonal rhotrix

$$D_n = diag\langle \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{n-2}, \lambda_{n-1}, \lambda_n \rangle = \left\langle \begin{array}{ccccccc} & & & \lambda_1 & & & \\ & & 0 & \lambda_2 & 0 & & \\ & 0 & 0 & \lambda_3 & 0 & 0 & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & 0 & 0 & \lambda_{n-2} & 0 & 0 & \\ & & 0 & \lambda_{n-1} & 0 & & \\ & & & \lambda_n & & & \end{array} \right\rangle$$

Then we have $R_n P_n = D_n P_n$.

Theorem 2.1

If an n-dimensional rhotrix R_n has t linearly independent eigenvectors with $t = (n+1)/2$, then a rhotrix P_n can be found such that $P_n^{-1} R_n P_n$ is a diagonal rhotrix.

Proof

We prove the theorem for a 3-dimensional rhotrix i.e. R_3. The proof can be extended easily to rhotrices of higher dimension.

Let $R_3 = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$

and let $\lambda_1, \lambda_2, \lambda_3$ be its eigenvalues and $\langle x_1^{31}\rangle, \langle x_2^{31}\rangle, \langle x_3^{11}\rangle$ the corresponding eigenvectors, where

$\langle x_1^{31}\rangle = \left\langle \begin{array}{ccc} & x_1 & \\ y_1 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \langle x_2^{31}\rangle = \left\langle \begin{array}{ccc} & x_2 & \\ y_2 & 0 & 0 \\ & 0 & \end{array} \right\rangle$ and $\langle x_3^{11}\rangle = \langle x_3 \rangle$

For the eigenvalue λ_1, we get

$$\left.\begin{array}{l}(a-\lambda_1)x_1 + dy_1 = 0 \\ bx_1 + (e-\lambda_1)y_1 = 0\end{array}\right\} \tag{4}$$

Similarly for λ_2, we have

$$\left.\begin{array}{l}(a-\lambda_2)x_2 + dy_2 = 0 \\ bx_2 + (e-\lambda_2)y_2 = 0\end{array}\right\} \tag{5}$$

And for λ_3, we have

$$(c-\lambda_3)x_3 = 0 \tag{6}$$

Equations (4), (5) and (6) becomes

$$\left.\begin{array}{l}ax_1 + dy_1 = \lambda_1 x_1 \\ bx_1 + ey_1 = \lambda_1 y_1\end{array}\right\} \tag{7}$$

$$\left.\begin{array}{l}ax_2 + dy_2 = \lambda_2 x_2 \\ bx_2 + ey_2 = \lambda_2 y_2\end{array}\right\} \tag{8}$$

$$cx_3 = \lambda_3 x_3 \tag{9}$$

We now consider the rhotrix

$$P_3 = \left\langle \begin{array}{ccc} & x_1 & \\ y_1 & x_3 & x_2 \\ & y_2 & \end{array} \right\rangle$$

whose columns are the eigenvalues of R_3, then

$$R_3P_3 = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \left\langle \begin{array}{ccc} & x_1 & \\ y_1 & x_3 & x_2 \\ & y_2 & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & ax_1 + dy_1 & \\ bx_1 + ey_1 & cx_3 & ax_2 + dy_2 \\ & bx_2 + ey_2 & \end{array} \right\rangle$$

From (7), (8) and (9), we get

$$R_3P_3 = \left\langle \begin{array}{ccc} & \lambda_1 x_1 & \\ \lambda_1 y_1 & \lambda_3 x_3 & \lambda_2 x_2 \\ & \lambda_2 y_2 & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & x_1 & \\ y_1 & x_3 & x_2 \\ & y_2 & \end{array} \right\rangle \left\langle \begin{array}{ccc} & \lambda_1 & \\ 0 & \lambda_3 & 0 \\ & \lambda_2 & \end{array} \right\rangle$$

$$= P_3D_3$$