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Rhotrix-Decomposition

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Abstract

Matrix is decomposable if it can be expressed as a product of any two non singular lower and upper triangular matrices. A number of results on matrix decompositions are known in the literature. In this paper we introduce rhotrix decomposition which is a factorization of rhotrix into a product of rhotrices and also present some of its result.

Keywords: Rhotrix, Rhotrix multiplication, Rhotrix decompositions, Left and Right triangular rhotrices.
AMS Subject Classifications [2010]: 15A15, 15A18

1.0 Introduction

The concept of rhotrices was first introduced by Ajibade [1] as an extension of the initiative on matrix-tertions and matrix-noitrets suggested by Atanassov and Shannon [2] and several works with modifications have presented on rhotrices. Ajibade [1] discussed the initial algebra and analysis on rhotrices and also set up some relationships between rhotrices and their hearts. The multiplication of rhotrices defined by Ajibade [1] was modified to similar multiplication of matrices proposed by Sani [3] defined as row column multiplication as follows: let $\boldsymbol{A}$and $\boldsymbol{B}$ be any two rhotrices defined as

$\boldsymbol{A} = \left\langle \begin{matrix} & a_{11} & \\ a_{21} & h(A) & a_{12} \\ & a_{22} & \end{matrix} \right\rangle$ and $\boldsymbol{B} = \left\langle \begin{matrix} & b_{11} & \\ b_{21} & h(B) & b_{12} \\ & b_{22} & \end{matrix} \right\rangle$ then $\boldsymbol{A} \cdot \boldsymbol{B}$**is** defined as

$$\boldsymbol{A} \cdot \boldsymbol{B} = \left\langle \begin{matrix} & a_{11}b_{11} + a_{12}b_{21} & \\ a_{21}b_{11} + a_{22}b_{21} & h(A) \cdot h(B) & a_{11}b_{12} + a_{12}b_{22} \\ & a_{21}b_{12} + a_{22}b_{22} & \end{matrix} \right\rangle$$

This alternative multiplication method was then generalized for n-dimensional rhotrices by Sani [4]. The idea of the conversion of rhotrices to 'coupled matrices' was suggested by Sani in [5]. This idea was used to solve systems of $n \times n$and $(n-1) \times (n-1)$ matrix problems simultaneously. This method was proved to be a linear transformation by Aminu [6]. The concept of vectors, one-sided system of equations and eigenvector-engenvalue problem in rhotrices were introduced by Aminu [7]. A necessary and sufficient condition for the solvability of one sided system of rhotrix was also presented in [8]. It was shown in the paper how a solution can be found provided the system is solvable. Rhotrix vector spaces and their properties were presented by Aminu [9]. Cayley–Hamilton theorem in rhotrix, determinant method for solving system of equation in rhotrices and minimal polynomial of a rhotrix were discussed in [10,11,12] respectively.

In this article, we will discuss and present the necessary and sufficient conditions for the solvability of such systems. If this system is solvable, we show how a solution can be found.

Decomposition of some special rhotrix 'Vandamonde Rhotrix' was discussed in Kumar and Sharma [13].

Our aim in this chapter is to present rhotrix decomposition, a concept which to the best of our knowledge has not been studied before.

Rhotrix has applications several applications in coding theory, cryptography, combinatorial design graph theory, see Kumar and Sharma [14, 15].

Definition 1.1

An n-dimensional rhotrix is represented and given as follows:

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$$\mathcal{R}_n = \langle a_{ij}, c_{lk} \rangle = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & \\ \ldots & \cdots & \cdots & \cdots & \cdots & \cdots & \ldots \\ a_{t1} & \vdots & \vdots & \vdots & \vdots & \vdots & a_{1t} \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\ & a_{tt-2} & a_{(t-1)\,(t-2)} & a_{(t-1)\,(t-1)} & a_{(t-2)\,(t-1)} & a_{(t-2)t} & \\ & & a_{tt-1} & c_{t-1\,t-1} & a_{t-1\,t} & & \\ & & & a_{tt} & & & \end{array} \right\rangle$$

Where $t = (n+1)/2$. The multiplication method of rhotrices is a similar as that of matrices.

Definition 1.2 The heart or centre of a rhotrix is the element that lies at the middle of a given rhotrix thereby dividing it into two equal parts. It lies at the position $\frac{1}{2}\left[\frac{1}{2}(n^2+1)+1\right]$, Where n is the dimension of the rhotrix.

Definition 1.3A system of linear equations in rhotrix form is given by:

$\mathcal{R}_n \cdot \langle x^{nj} \rangle = \langle b^{nj} \rangle$

Where $\mathcal{R}_n = \langle a_{ij}, c_{lk} \rangle$for:$i, j = 1,2, \ldots t$, and $k, l = 1, 2, \ldots, t-1$.

Definition 1.4 The rhotrix defined in Definition 1.1 above is said to be **left triangular rhotrix** or simply**left rhotrix** if $a_{ij} = 0$ for $2 < i \leq t$ and $1 \leq j \leq t$ for $t = 1, 2, \ldots, n$. We denote $\boldsymbol{R}^l{}_n$ to represent n dimensional left rhotrix. Examples of 3 and 4 dimensional left rhotrix is given as$\boldsymbol{R}^l{}_3 = \left\langle \begin{array}{ccc} & a_{11} & \\ a_{21} & a & 0 \\ & a_{22} & \end{array} \right\rangle$or$\boldsymbol{R}^l{}_5 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & & a_{22} \quad 0 & \\ a_{31} & & a_{12} \quad 0 & & 0 \\ & a_{11} & & a_{33} \quad 0 & \\ & & 0 & & \end{array} \right\rangle$ respectively.

And n dimensional left rhotrix is

$$\mathcal{R}_n{}^l = \left\langle \begin{array}{ccccccc} & & & l_{11} & & & \\ & & l_{21} & c^l{}_{11} & 0 & & \\ & l_{31} & c^l{}_{21} & l_{22} & 0 & 0 & \\ \ldots & \cdots & \cdots & \cdots & \cdots & \cdots & \ldots \\ l_{t1} & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\ & l_{tt-2} & l_{(t-1)\,(t-2)} & l_{(t-1)\,(t-1)} & 0 & 0 & \\ & & l_{tt-1} & c^l{}_{t-1\,t-1} & 0 & & \\ & & & l_{tt} & & & \end{array} \right\rangle$$

Where c^l, is an element in the **left** rhotrix.

Definition 1.5 The rhotrix defined in Definition 1.1 above is said to be **right triangular rhotrix** or simply **right rhotrix** if $a_{ij} = 0$ for $2 < j \leq t$ and $1 \leq i \leq t$ for $t = 1, 2, \ldots, n$. Here we denote $\boldsymbol{R}^r{}_n$ to represent n dimensional right rhotrix. An example of 3 dimensional right rhotrix is given as: $\boldsymbol{R}_3 = \left\langle \begin{array}{ccc} & a_{11} & \\ 0 & r & a_{12} \\ & a_{22} & \end{array} \right\rangle$

An n dimensional right rhotrix is

$$\mathcal{R}_n{}^r = \left\langle \begin{array}{ccccccc} & & & r_{11} & & & \\ & & 0 & c^r{}_{11} & r_{12} & & \\ & 0 & 0 & r_{22} & c^r{}_{12} & r_{13} & \\ \ldots & \cdots & 0 & \cdots & \cdots & \cdots & \ldots \\ 0 & \vdots & \vdots & \vdots & \vdots & \vdots & r_{1t} \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\ & 0 & 0 & r_{(t-1)\,(t-1)} & r_{(t-2)\,(t-1)} & r_{(t-2)t} & \\ & & 0 & c^r{}_{t-1\,t-1} & r_{t-1\,t} & & \\ & & & r_{tt} & & & \end{array} \right\rangle$$

Where c^r, is an element in the **right** rhotrix.

Definition 1.6 An n dimensional rhotrix $\boldsymbol{\mathcal{R}_n}$, is said to have an $\boldsymbol{R}^r{}_n \cdot \boldsymbol{R}^l{}_n$ or simply $\boldsymbol{RL}$ rhotrix decomposition if there exists $\boldsymbol{R}^r{}_n$ and $\boldsymbol{R}^l{}_n$rhotriceswhere $\boldsymbol{L}$ and $\boldsymbol{R}$ are left and Right rhotrices respectively. This is defined as $\boldsymbol{\mathcal{R}_n} = \boldsymbol{R}^r{}_n \cdot \boldsymbol{R}^l{}_n$.where

$$\boldsymbol{R}^l{}_n \boldsymbol{R}^r{}_n = \left\langle \begin{array}{ccccccc} & & & l_{11} & & & \\ & & l_{21} & c^l{}_{11} & 0 & & \\ & l_{31} & c^l{}_{21} & l_{22} & 0 & 0 & \\ \ldots & \cdots & \cdots & \cdots & \cdots & \cdots & \ldots \\ l_{t1} & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\ & l_{tt-2} & l_{(t-1)\,(t-2)} & l_{(t-1)\,(t-1)} & 0 & 0 & \\ & & l_{tt-1} & c^l{}_{t-1\,t-1} & 0 & & \\ & & & l_{tt} & & & \end{array} \right\rangle \cdot \left\langle \begin{array}{ccccccc} & & & r_{11} & & & \\ & & 0 & c^r{}_{11} & r_{12} & & \\ & 0 & 0 & r_{22} & c^r{}_{12} & r_{13} & \\ \ldots & \cdots & 0 & \cdots & \cdots & \cdots & \ldots \\ 0 & \vdots & \vdots & \vdots & \vdots & \vdots & r_{1t} \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \\ & 0 & 0 & r_{(t-1)\,(t-1)} & r_{(t-2)\,(t-1)} & r_{(t-2)t} & \\ & & 0 & c^r{}_{t-1\,t-1} & r_{t-1\,t} & & \\ & & & r_{tt} & & & \end{array} \right\rangle$$

$\boldsymbol{R}^r{}_n \cdot \boldsymbol{R}^l{}_n$

$$= \left\langle \begin{array}{cccccccc} & & & l_{11}r_{11} & & & & \\ & & l_{21}r_{11} & c^l{}_{11}c^r{}_{11} & l_{11}r_{12} & & & \\ & l_{31}r_{11} & c^l{}_{21}c^r{}_{21} & l_{21}r_{12+}l_{22}r_{22} & 0 & l_{11}r_{13} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ l_{t1}r_{11} & \vdots & \vdots & \vdots & \vdots & \vdots & & l_{11}r_{1t} \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \\ & l_{t1}r_{11}+\cdots+l_{tt-2}r_{t-2t} & l_{31}r_{11}+\cdots+l_{3t}r_{tt} & \vdots & 0 & l_{t1}r_{1t+}l_{t2}r_{2t}+\cdots+l_{tt}r_{tt} & & \\ & & l_{t1}r_{11}+\cdots+l_{tt-1}r_{t-1t} & c^l{}_{t-1\,t-1}c^r{}_{t-1\,t-1} & 0 & & & \\ & & & l_{t1}r_{1t}+l_{t2}r_{2t}+\cdots+l_{tt}r_{tt} & & & & \end{array} \right\rangle$$

2.0 $\boldsymbol{LR}$–Rhotrix Decomposition

Given a System of linear equations:$\boldsymbol{\mathcal{R}_n X = b}$, let $\boldsymbol{\mathcal{R}_n = LR}$ where$\boldsymbol{R}^l{}_n$ and $\boldsymbol{R}^r{}_n$are Left and Right triangular rhotrices. Let denotes $\boldsymbol{L}$ and$\boldsymbol{R}$ as follows: here we give more consideration for $n = 3$, which is the order of the rhotrices.

Note that, we will be using $\boldsymbol{\mathcal{R}_n}^{\boldsymbol{l}} = \boldsymbol{L}$ and $\boldsymbol{\mathcal{R}_n}^{r} = \boldsymbol{R}$**,** throughout the work, represent, left and right rhotrices respectively.

$$\boldsymbol{L} = \left\langle \begin{array}{ccc} & l_{11} & \\ l_{21} & l & 0 \\ & l_{22} & \end{array} \right\rangle \text{ and } \boldsymbol{R} = \left\langle \begin{array}{ccc} & r_{11} & \\ 0 & r & r_{12} \\ & r_{22} & \end{array} \right\rangle \qquad (2.1)$$

Then,

$$\boldsymbol{LR} = \left\langle \begin{array}{ccc} & l_{11} & \\ l_{21} & l & 0 \\ & l_{22} & \end{array} \right\rangle \cdot \left\langle \begin{array}{ccc} & r_{11} & \\ 0 & r & r_{12} \\ & r_{22} & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & l_{11}r_{11} & \\ l_{21}r_{11} & lr & l_{11}r_{12} \\ & l_{21}r_{12}+l_{22}r_{22} & \end{array} \right\rangle = \boldsymbol{\mathcal{R}_3} \qquad (2.2)$$

Crout's and Doolittle's methods were first introduced on matrices and have received a number of attentions; the reader is referred to Bunch [16] for more information.Here we need to define Crout's and Doolittle's Methods using rhotrices.

2.1 Crout's Algorithm for Rhotrices:

Using (2.2) above we have: $r_{ii} = r = 1$, $i = 1, 2$. Then

$$LR = \left\langle \begin{array}{ccc} & l_{11} & \\ l_{21} & l & l_{11}r_{12} \\ & l_{21}r_{12}+l_{22} & \end{array} \right\rangle = \mathcal{R}_3 \qquad (2.3)$$

In **Crout's Methodfor rhotrix**, we use: $r_{ii} = r = 1$, where r is the centre of the right rhotrix

2.2 Doolittle's Algorithm for Rhotrices:

using (2.2) as above where$l_{ii} = l = 1$, then for $i = 1, 2$.

$$\boldsymbol{LR} = \left\langle \begin{array}{ccc} & r_{11} & \\ l_{21}r_{11} & r & r_{12} \\ & l_{21}r_{12}+r_{22} & \end{array} \right\rangle = \boldsymbol{\mathcal{R}_3} \qquad (2.4)$$

Similarly, as in Crout Method for rhotrices,**Doolittle's Method for rhotrix** we use: $l_{ii} = l = 1$, where l is the centre of the left rhotrix.

For example, let's solve a system below using any of the above algorithms.

A system $2x_1 - 3x_2 = 1$

$x_1 + x_2 = 3$

Can be solve using any of these algorithms, but here in particular, we are going to deploy the first algorithm, which is Crout's Method.

The system in rhotrix form is

$$\left\langle \begin{array}{ccc} & 2 & \\ 1 & 4 & -3 \\ & 1 & \end{array} \right\rangle \cdot \left\langle \begin{array}{ccc} & x_1 & \\ x_2 & 0 & 0 \\ & 0 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ 3 & 0 & 0 \\ & 0 & \end{array} \right\rangle \qquad (2.5)$$

By choosing an arbitrary Heart, here choose a heart (4). This can be represented in an equivalent form as:

$\mathcal{R}_n\langle x^{nj}\rangle = \langle b^{nj}\rangle$

For $j = 1, 2$. $n = 3$We have:

$\mathcal{R}_3\langle x^{31}\rangle = \langle b^{31}\rangle$

Where $\langle x^{31}\rangle$ and $\langle b^{31}\rangle$ are column vector rhotrices. Re-writing the rhotrix system in the form

$\mathcal{R}_3 = LR$, where L and R are Left and Right triangular rhotrices.

Now, using Crout's algorithms to rhotrices: $r_{ii} = r = 1$. Using (2.3) we have:

$$\boldsymbol{LR} = \left\langle \begin{array}{ccc} & l_{11} & \\ l_{21} & l & l_{11}r_{12} \\ & l_{21}r_{12} + l_{22} & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 2 & \\ 1 & 4 & -3 \\ & 1 & \end{array} \right\rangle = \boldsymbol{\mathcal{R}_3}$$

So we get: $l_{11} = 2$, $l_{21} = 1$, $l = 4$, $l_{11}r_{12} = -3$ which implies $r_{12} = -\frac{3}{2}$,
$l_{21}r_{12} + l_{22} = 1$ or $(1)\left(-\frac{3}{2}\right) + l_{22} = 1 \Rightarrow l_{22} = \frac{5}{2}$.With $r_{11} = r_{22} = 1$.

So we have: $L = \left\langle \begin{array}{ccc} & l_{11} & \\ l_{21} & l & 0 \\ & l_{22} & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 2 & \\ 1 & 4 & 0 \\ & \frac{5}{2} & \end{array} \right\rangle$ and

$$R = \left\langle \begin{array}{ccc} & r_{11} & \\ 0 & r & r_{12} \\ & r_{22} & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & -\frac{3}{2} \\ & 1 & \end{array} \right\rangle$$

We solve the system by **back substitution** by first solving the system below:
So that: $LR \cdot \mathcal{R}_3 \langle x^{31} \rangle = \langle b^{31} \rangle$
Let $R \cdot \langle x^{31} \rangle = \langle y^{31} \rangle$ then $L \cdot \langle y^{31} \rangle = \langle b^{31} \rangle$ now we have

$$\left\langle \begin{array}{ccc} & 2 & \\ 1 & 4 & 0 \\ & \frac{5}{2} & \end{array} \right\rangle \cdot \left\langle \begin{array}{ccc} & y_1 & \\ y_2 & 0 & 0 \\ & 0 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 1 & \\ 3 & 0 & 0 \\ & 0 & \end{array} \right\rangle$$

The matrix equivalent form is $\begin{pmatrix} 2 & 0 \\ 1 & \frac{5}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$,

$\Rightarrow 2y_1 = 1$, or $y_1 = \frac{1}{2}$ and $y_1 + \frac{5}{2}y_2 = 3$, $\frac{5}{2}y_2 = 3 - \frac{1}{2} = \frac{5}{2}$ therefore $y_1 = \frac{1}{2}$ and $y_1 = 1$.
Now, **forward substitution:**solving for $\langle x^{31} \rangle$ in the system :
$R \cdot \langle x^{31} \rangle = \langle y^{31} \rangle$

$$\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & -\frac{3}{2} \\ & 1 & \end{array} \right\rangle \cdot \left\langle \begin{array}{ccc} & x_1 & \\ x_2 & 0 & 0 \\ & 0 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & y_1 & \\ y_2 & 0 & 0 \\ & 0 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & \frac{1}{2} & \\ 1 & 0 & 0 \\ & 0 & \end{array} \right\rangle$$

This implies that $\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & -\frac{3}{2} \\ & 1 & \end{array} \right\rangle \cdot \left\langle \begin{array}{ccc} & x_1 & \\ x_2 & 0 & 0 \\ & 0 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & \frac{1}{2} & \\ 1 & 0 & 0 \\ & 0 & \end{array} \right\rangle$

Converting to its matrix equivalent form: we have $x_2 = 1$ and $x_1 - \frac{3}{2}x_2 = \frac{1}{2}$ and
$x_1 = \frac{1}{2} + \frac{3}{2} = \frac{4}{2} = 2$. So $x_1 = 2$ and $x_2 = 1$.
Hence we clearly check the system for $x_1 = 2$ and $x_2 = 1$ the truth is clear.

Theorem 2.1

A necessary and sufficient condition for an $n$dimensional rhotrix, $\boldsymbol{\mathcal{R}_n}$**,** to be $\boldsymbol{R}^r{}_n \cdot \boldsymbol{R}^l{}_n$ or simply $\boldsymbol{LR} -$ rhotrix decomposition is that $\boldsymbol{\mathcal{R}_n}$ is invertible.

Proof:

Let $\boldsymbol{\mathcal{R}_n} = \langle a_{ij}, c_{lk} \rangle$ for $1 \le i \le n$, $1 \le j \le n$ and $1 \le k \le n$, suppose $\boldsymbol{\mathcal{R}_n}$ admits $\boldsymbol{LR}$ decomposition, we need to show that $\boldsymbol{det}\ (\boldsymbol{\mathcal{R}_n})$exists.

Since $\mathcal{R}_n$ can be express as follows $\mathcal{R}_n = LR$, where L and R are Left and Right triangular rhotrices. Now, the $det\ (L) \neq 0$ and $det\ (R) \neq 0$ hence the determinant $det\ (LR) \neq 0$which implies that $det\ (\mathcal{R}_n) \neq 0$. Thus $\mathcal{R}_n$ is invertible.

Conversely, if $\mathcal{R}_n$ is invertible its determinant exists. That is
$det\ (\mathcal{R}_n) \neq 0 \Rightarrow det\ (LR) \neq 0$

Now, since $det\ (LR) \neq 0$ then $det\ (L) \neq 0$ and $det\ (R) \neq 0$, thus $\mathcal{R}_n$ can be express in the form $\mathcal{R}_n = LR$. Hence LR decompositions exist when $\mathcal{R}_n$ is invertible.

Definition 2.1 An $\boldsymbol{LDR}$**-rhotrix decomposition** is of the form: $\mathcal{R}_n = LDR$, where D is a diagonal rhotrix and L and U are *unit* triangular rhotrices, meaning that all the entries on the diagonals of L and U are one. It is equivalent to $\boldsymbol{LR}$ rhotrix decomposition.

Theorem 2.2

If aninvertible rhotrixhas an $\boldsymbol{LDR}$rhotrix factorization, then its unique, in that case, the LRfactorization is also unique if we require the diagonal of L or R consists of ones.

Proof:

By using any of the algorithms, Crout or Doolittle Methods we can arrive at the proof.

Now, using Crout's method where $r_{ii} = r = 1$ and D a diagonal rhotrix

Suppose $\mathcal{R}_n$has $\boldsymbol{LDR}$rhotrix factorization, then it can be expressed in the form:

$\mathcal{R}_n = RDL = \langle x^{ii} \rangle \cdot \langle x^{nn} \rangle \cdot \langle x^{jj} \rangle$ where D is a diagonal rhotrix. Since $\mathcal{R}_n$ is invertible it follows from theorem 1.1 that $\boldsymbol{\mathcal{R}_n = RDL}$ exists.

And since it exists then $\boldsymbol{RDL} = \boldsymbol{LDR}$ hence the factorization is unique.

It then follows from $\boldsymbol{RDL} = \boldsymbol{LDR}$

Pre-multiplying through by$\boldsymbol{D^{-1}}(\boldsymbol{RDL}) = \boldsymbol{D^{-1}}(\boldsymbol{LDR})$ where

$$\boldsymbol{RD^{-1}DL} = \boldsymbol{LD^{-1}DR}$$

Or $$\boldsymbol{RL} = \boldsymbol{LR}$$

Hence $\boldsymbol{RL}$ factorization is unique provided that $\boldsymbol{LDR}$ factorization exists.

Theorem 2.3

If $\mathcal{R}_n$ is a non-singular rhotrix, then there exists a permutation rhotrix P so that $P\mathcal{R}_n$ has an LR- decomposition. i.e. $\boldsymbol{P\mathcal{R}_n} = \boldsymbol{LR}$.

Proof:

Since $\mathcal{R}_n$ is invertible being non singular, it follows from theorem 1.2. note we use P only when $\mathcal{R}_n$ requires row interchanges to row echelon form, and $P\mathcal{R}_n$ requires no row interchanges. So we have $P\mathcal{R}_n$ can be express as $P\mathcal{R}_n = LR$. Hence the proof.

Definition 2.2 An n dimensional rhotrix which arises by a finite number of row interchange is called a permutation rhotrix. It is usually denotes as $\boldsymbol{P}$.

Theorem 2.4 The system $\mathcal{R}_n X = b$ is equivalent to the system $P\mathcal{R}_n X = Pb$, where $P\mathcal{R}_n$ has LR decomposition and P is a permutation rhotrix.

Proof:

We need to show that the system $\mathcal{R}_n X = b$ is equivalent to $P\mathcal{R}_n X = Pb$. It suffices to verify that if $P\mathcal{R}_n X = Pb$, P interchanges the rows of the rhotrix $\mathcal{R}_n$,

Now, if $P\mathcal{R}_n X = Pb$ is given: pre-multiply both sides with P^{-1} we get:

$$P^{-1}P\mathcal{R}_n X = P^{-1}Pb$$

$\Rightarrow IP\mathcal{R}_n X = Ib$ or $\mathcal{R}_n X = b$

Hence $\mathcal{R}_n X = b$ is equivalent to:$P\mathcal{R}_n X = Pb$.

Now, if $\mathcal{R}_n X = b$ is given, to interchange the rows of $\mathcal{R}_n$ into row echelon form we multiply through by the permutation rhotrix P, such that $P\mathcal{R}_n X = Pb$.

Hence $\mathcal{R}_n X = b \Leftrightarrow P\mathcal{R}_n X = Pb$

Theorem 2.5

Let $\mathcal{R}_n X = b$ be a rhotrix system of linear equation where$\mathcal{R}_n$has zero heart.Using$\boldsymbol{LR}$ decomposition of rhotrices, by applying either Crout or Doolittle methods the heart of either $\boldsymbol{L}$ or $\boldsymbol{R}$ must be zero.

Proof:

Let l and r be the centres (hearts) of L and Rleft and right triangular rhotrices respectively.

If $\boldsymbol{\mathcal{R}_n = LR}$, where the initial system in matrix form is transformed to rhotrix form has zero heart, so the product of the their hearts gives the heart of $\mathcal{R}_n$. If the heart of $\mathcal{R}_n$ is 0 then we have: $r \cdot l = 0$ which implies that either, $r = 0$ or $l = 0$, Hence the proof.

Definition 2.3

The total number of floating point operations $(\times, \div, +, -)$ determine the cost of computations involved in solving problems. It's usually denoted as $O(n^k)$, for natural number $k \geq 1$ and n is the dimension of the rhotrix.

Theorem 2.6

An $\boldsymbol{LR}$ **rhotrix decomposition** requires $O(n^3)$ floating point operations.

Proof:

In order to compute the total number of operations we will need the following identities:

$$\sum_{i=1}^{n} 1 = n, \quad \sum_{i=1}^{n} i = \frac{n}{2}(n+1), \quad \sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1) \text{which can be proved using induction}$$

There are $(n + 1 - i)$ and $(n - i + 2)$ multiplication$(\times)$ and division$(\div)$ operations respectively.

Therefore the total number of multiplication$(\times)$ and division$(\div)$ operations is:

$$\sum_{i=1}^{n-1}(n+1-i)(n-i+2) = (n^2+3n+2)\sum_{i=1}^{n-1}1-(2n+3)\sum_{i=1}^{n-1}i+\sum_{i=1}^{n-1}i^2$$
$$= n(n^2+3n+2)-(2n+3)\frac{n}{2}(n+1)+\frac{n}{6}(n+1)(2n+1) = \frac{1}{3}n^3+\frac{1}{2}n^2+\frac{2}{3}n \approx \frac{1}{3}n^3.$$

Similarly,
There are $(n-i)$ and $(n-i+1)$ addition$(+)$ and subtraction$(-)$ operations respectively.
Therefore the total number of addition$(+)$ and subtraction$(-)$ operations is:

$$\sum_{i=1}^{n-1}(n-i)(n-i+1) = (n^2+n)\sum_{i=1}^{n-1}1-3\sum_{i=1}^{n-1}i+\sum_{i=1}^{n-1}i^2 = \frac{1}{3}n^3+\frac{1}{2}n^2-\frac{5}{6}n \approx \frac{1}{3}n^3.$$

The approximate total solution cost is: $= \frac{1}{3}n^3+\frac{1}{2}n^2+\frac{2}{3}n+\frac{1}{3}n^3+\frac{1}{2}n^2-\frac{5}{6}n = \frac{2}{3}n^3+n^2-\frac{1}{6}n \approx \frac{2}{3}n^3$.

So, $O(n^3) \approx \frac{2}{3}n^3$.

Note:

1. If two rhotrices of order n can be multiplied in $\mathcal{R}(n)$, where $\mathcal{R}(n) \geq n^a$ for some $a > 2$, then the LR-decompositions the rhotrix decompositions can be computed in $O(\mathcal{R}(n))$.
2. Computing the LRrhotrix decomposition of rhotrices using either of these algorithms requires$O(n^3) = \frac{2n^3}{3}$ floating point operations, ignoring the lower order terms.

3.0 Conclusion

In this paper we havediscussed the concept of rhotrix decomposition and its properties.

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