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Computational Analysis of System of Volterra Integral Equations Using Homotopy Pertubation Method

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Abstract

In this paper, we have applied homotopy perturbation method to solve a system of volterra integral equations. We considered steady state laminar boundary layer flow on a flat plate. The boundary layer on a flat plate which is often called a zero pressure gradient boundary layer, the momentum equation in the x direction for boundary layer is transformed by means of a similarity variable η from partial differential equation into ordinary differential equation. The obtained equation is called Blasius equation. The nonlinear third order ordinary differential equation is further transformed into a system of first order ordinary differential equations which were further written as a system of volterra integral equations. The solution obtained has been compared with variational iterative method. It is shown that this method is easy to implement. It is more efficient and converges faster than variational iterative method.

Keywords: Homotopy Perturbation Method, Volterra Integral Equation, Blasius Equation, Laminar Flow.

1.0 Introduction

In developing a mathematical theory of boundary layer, a solution of a limiting form of equation of motion as Reynolds number becomes large is expected to describe approximately the flow in laminar boundary layers. We consider a flat plate at $y = 0$ with a stream with constant velocity u_∞ parallel to the plate. At the surface there is no flow across it which implies that the rate of flow, $v = 0$ at $y = 0$. Due to the viscosity we have the no slip condition at the plate. In other words $u = 0$ at $y = 0$. At infinity outside the boundary layer we have $u = u_\infty$ as y becomes large. Because of its numerous applications in science and engineering, the laminar boundary layer flow has attracted the interest of physicists, engineers, mathematicians and numerical analysts alike since discovered by Prandtl in 1904 and solved by Blasius in 1908[1]. Since one can elegantly reduce these equations to ODE by similarity transform, mathematicians have found their fulfilment in uncovering the underlying symmetries and proving the existence and uniqueness of its solutions. The first numerical solution was obtained by Howarth [2]. More recently, the Blasius equation was solved using the shooting method [3, 4]. The Adomian decomposition method was employed to solve the Blasius equation as outline in [5 - 8]. The same equation was solved using variational iterative method [9]. Also, research work on application of homotopy perturbation method for non-linear Blasius equation to boundary layer flow over a flat plate has been investigated [10]. We are to solve the equation using Homotopy perturbation method.

Our aim is to compute the solution to Blasius equation by Homotopy perturbation method using MAPLE, and to compare solutions obtained with variational iterative method and Blasius' solution.

Perturbation method called homotopy perturbation method (HPM) is a series expansion method used in the solution on nonlinear ODE and PDE. It was introduced by Ji-Huan He in 1998 and systematic description in 2000 which is, in fact, coupling of the traditional perturbation method and homotopy in topology [11-13]. Until recently, the application of the HPM in

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non-linear problems has been developed by scientists and engineers, because this method is the most effective and convenient for both weakly and strongly non-linear equations [14 - 19]. In this paper, we have applied HPM to compute the solution of Blasius equation.

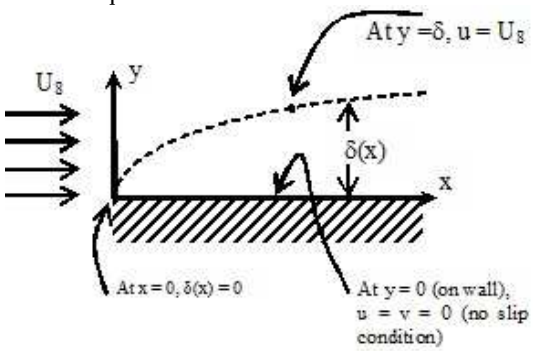

Figure 1: Boundary layer flow

2.0 Perturbation Method

A brief description of the standard HPM on volterra integral equations will be presented. Without loss of generality, consider the following system of the integral equations:

$$M(t) = N(t) + \lambda \int_0^t K(t,s)M(s)ds, \tag{1}$$

where

$$M(t) = \left(m_1(t), m_2(t), ..., m_n(t)\right)^T \tag{2}$$

$$N(t) = \left(n_1(t), n_2(t), ..., n_n(t)\right)^T \tag{3}$$

$$K(t,s) = \left[k_{ij}(t,s)\right], i = 1,2,3...n : j = 1,2,3...n \tag{4}$$

To convey an idea of the homotopy perturbation method, we consider a general equation of the type

$$L(u) = 0 \tag{5}$$

where L is an integral or differential operator, we define a convex homotopy $H(u,p)$ by

$$H(u,p) = (1-p)F(u) + pL(u) \tag{6}$$

where $F(u)$ is a functional operator with known solutions v_0, which can be obtained easily. It is clear that

$$H(u,p) = 0 \tag{7}$$

From, which we have $H(u,0) = F(u)$ and $H(u,1) = L(u)$. This shows the $H(u,p)$ continuously traces an implicitly defined curve from a starting point $H(v_0,0)$ to a solution $H(f,1)$. The embedding parameter increases monotonically from zero to unit as the problem $F(u) = 0$ is continuously deforms the original problem $L(u) = 0$. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter to obtain

$$u = \sum_{i-0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + \cdots \tag{8}$$

If $p \to 1$, then (7) corresponds to (5) and becomes the approximate solution of the form

$$f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i \tag{9}$$

It is an established fact that the series (9) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$, we assume that problem (1) has a unique solution.

Considering the i th equation of (1), which take the form

$$f_1(t) = \sum_{i=0}^{\infty} p^i u_i$$

$$f_2(t) = \sum_{i=0}^{\infty} p^i v_i \tag{10}$$

$$f_3(t) = \sum_{i-0}^{\infty} p^i w_i$$

$$\vdots$$

The comparison of like powers of p gives solution of various orders.

3.0 Materials and Method

For the flow along the flat plate with constant velocity u_∞, we assume no pressure gradient, so the continuity and momentum equation in the x direction for steady boundary layer is govern by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{11}$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} \tag{12}$$

The appropriate boundary conditions are:

$$y = 0,\ x > 0;\ u = v = 0 \tag{13}$$

$$y \to \infty,\ x \geq 0;\ u = U_\infty \tag{14}$$

We reduced the boundary layer equation (12) to partial differential equation with a single dependent variable by considering the stream function ψ related to the velocity u and v according to the equations

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x} \tag{15}$$

We substitute equation (15) into the equation (12) to obtain a partial differential equation given by

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \upsilon\frac{\partial^3 \psi}{\partial y^3} \tag{16}$$

with the boundary conditions

$$y = 0,\ x > 0;\ \frac{\partial \psi}{\partial y} = -\frac{\partial \psi}{\partial x} = 0 \tag{17}$$

$$y \to \infty,\ x > 0;\ \frac{\partial \psi}{\partial y} = U_\infty \tag{18}$$

Based on the observation of Blasius in 1908, we reduced the partial differential equation (16) to an ordinary differential equation using a similarity variable defined by

$$\eta = y\left(\frac{U_\infty}{vx}\right)^{\frac{1}{2}} \tag{19}$$

$$\frac{\partial \psi}{\partial y} = U_\infty f'(\eta) \tag{20}$$

We substitute equations (19) and (20) into equation (16), we obtained an ordinary differential equation given by

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0 \tag{21}$$

With the boundary conditions

$$f'(0) = 0,\ f(0) = 0,\ f'(\infty) = 1 \tag{22}$$

The function $f(\eta)$ is the solution of the boundary value problem given by the equation (21) and (22), which has no closed form solution. The ordinary differential equation (21) is nonlinear and has to be solved semi-analytically together with the boundary condition (22).

Using the transformation $\frac{df}{d\eta} = g(\eta), \frac{dg}{d\eta} = h(\eta),$ we can rewrite the boundary value problem (21) and (22) as a system of differential equations:

$$\left.\begin{aligned} \frac{df}{d\eta} &= g(\eta) \\ \frac{dg}{d\eta} &= h(\eta) \\ \frac{dh}{d\eta} &= -\frac{1}{2} f(\eta)h(\eta) \end{aligned}\right\} \tag{23}$$

with $f(0) = 0, g(0) = 0, h(0) = a$ which can be written as a system of integral equations:

$$\left.\begin{aligned} f(\eta) &= 0 + \int_0^\eta g(t)dt \\ g(\eta) &= 0 + \int_0^\eta h(t)dt \\ h(\eta) &= a - \frac{1}{2}\int_0^\eta f(t)h(t)dt \end{aligned}\right\} \tag{24}$$

$$f_0 + pf_1 + \cdots = 0 + p\int_0^\eta (g_0 + pg_1 + \cdots)dt \tag{25}$$

$$g_0 + pg_1 + \cdots = 0 + p\int_0^\eta (h_0 + ph_1 + \cdots)dt \tag{26}$$

$$\left.\begin{aligned} &h_0 + ph_1 + p^2h_2 + \cdots = a - \frac{p}{2}\int_0^\eta \big(g_0h_0 + P\big(g_0h_1 \\ &+ g_0h_1\big) + p^2(g_0h_2 + g_1h_1 + g_2h_0) + \cdots\big) \end{aligned}\right\} \tag{27}$$

Comparing the coefficient of like powers of p, we have

$$p^{(0)}: \begin{cases} f_0 = 0 \\ g_0 = 0 \\ h_0 = a \end{cases} \quad p^{(1)}: \begin{cases} f_1 = 0 \\ g_1 = a\eta \\ h_1 = 0 \end{cases} \quad p^{(2)}: \begin{cases} f_2 = \frac{1}{2}a\eta^2 \\ g_2 = 0 \\ h_2 = 0 \end{cases} \quad p^{(3)}: \begin{cases} f_3 = 0 \\ g_3 = 0 \\ h_3 = -\frac{1}{12}a^2\eta^3 \end{cases} \quad p^{(4)}: \begin{cases} f_4 = 0 \\ g_4 = -\frac{1}{48}a^2\eta^4 \\ h_4 = 0 \end{cases}$$

$$p^{(5)}: \begin{cases} f_5 = -\frac{1}{240}a^2\eta^5 \\ g_5 = 0 \\ h_5 = 0 \end{cases} \quad p^{(6)}: \begin{cases} f_6 = 0 \\ g_6 = 0 \\ h_6 = -\frac{11}{2880}a^3\eta^6 \end{cases} \quad p^{(7)}: \begin{cases} f_7 = 0 \\ g_7 = \frac{11}{20160}a^3\eta^7 \\ h_7 = 0 \end{cases} \quad p^{(8)}: \begin{cases} f_8 = \frac{11}{161280}a^3\eta^8 \\ g_8 = 0 \\ h_8 = 0 \end{cases}$$

$$p^{(9)}:\begin{cases} f_9 = 0 \\ g_9 = 0 \\ h_9 = -\dfrac{25}{193536}a^4\eta^9 \end{cases} \qquad p^{(10)}:\begin{cases} f_{10} = 0 \\ g_{10} = -\dfrac{5}{387072}a^4\eta^{10} \\ h_{10} = 0 \end{cases} \qquad p^{(11)}:\begin{cases} f_{11} = -\dfrac{5}{4257792}a^4\eta^{11} \\ g_{11} = 0 \\ h_{11} = 0 \end{cases}$$

$$p^{(12)}:\begin{cases} f_{12} = 0 \\ g_{12} = 0 \\ h_{12} = \dfrac{9299}{2554675200}a^5\eta^{12} \end{cases} \qquad p^{(13)}:\begin{cases} f_{13} = 0 \\ g_{13} = \dfrac{9299}{33210777600}a^5\eta^{13} \\ h_{13} = 0 \end{cases} \tag{28}$$

$\vdots$

From equation (28) we obtained the following series solution:

$$f(\eta) = \frac{1}{2}a\eta^2 - \frac{1}{240}a^2\eta^5 + \frac{11}{161280}a^3\eta^8 - \frac{5}{4257792}a^4\eta^{11} + \frac{9299}{464950886400}a^5\eta^{14} - \frac{1272379}{3793999233024000}a^6\eta^{17} - \frac{19241647}{3460127300517888000}a^7\eta^{20} \tag{29}$$

By assuming the value $a = 0.332057$ based on the conclusions of [3], we obtained the following series solution from equation (29)

$$f(\eta) = 1.660285\text{x}10^{-1}\eta^2 - 4.5942438\text{x}10^{-4}\eta^5 + 2.497181392\text{x}10^{-6}\eta^8 - 1.427697248\text{x}10^{-8}\eta + 8.074067341\text{x}10^{-11}\eta^{14} - 4.49567692\text{x}10^{-13}\eta^{17} + 2.47536\text{x}10^{-15}\eta^{20} \tag{30}$$

4.0 Application of the Result

We have compared solution obtained with that obtained by variational iterative method in [3] and Blasius' solution as shown in **Table 1** and **2**.

Table 1: Obtained solution, in comparison with VIM and Blasius' solution for $f(\eta)$.

	$f(\eta)$		
η	Blasius	VIM	HPM
0	0.0000	0.0000	0.0000
0.5	0.0415	0.0415	0.0415
1.0	0.1656	0.1656	0.1656
1.5	0.3701	0.3701	0.3701
2.0	0.6500	0.6500	0.6500
2.5	0.9963	0.9963	0.9963
3.0	1.3968	1.3966	1.3968
3.5	1.8377	1.8361	1.8377
4.0	2.3057	2.2966	2.3064

Table 2: Obtained solution, in comparison with VIM and Blasius'solutionfor $f'(\eta)$.

	$f'(\eta)$		
η	Blasius	VIM	HPM
0	0.0000	0.0000	0.0000
0.5	0.1659	0.1659	0.1659
1.0	0.3298	0.3298	0.3298
1.5	0.4868	0.4868	0.4868
2.0	0.6298	0.6298	0.6298
2.5	0.7513	0.7512	0.7513
3.0	0.8460	0.8451	0.8461
3.5	0.9130	0.9070	0.9133
4.0	0.9555	0.9262	0.9594

Table 3: Obtained solution, in comparison with Blasius'solution for $f''(\eta)$.

	$f''(\eta)$	
η	Blasius	HPM
0	0.33206	0.33206
0.5	0.33091	0.33091
1.0	0.32301	0.32301
2.0	0.26675	0.26675
3.0	0.16136	0.16142
3.5	0.10777	0.10917
4.0	0.06423	0.08498

In Table 1 the velocity distribution $f(\eta)$ is computed for $\eta \le 4$ and the solution obtained compared favourably with Blasius' solution except for $\eta = 4$ with absolute error of 0.0007. In Table 2 the first derivative of the velocity distribution $f'(\eta)$ is tabulated and compared with Blasius' solution. The solutions obtained are in good agreement with Blasius' solution except for $\eta = 3.0, 3.5 \text{ and } 4.0$ with absolute error of 0.0001, 0.0003 and 0.0039 respectively. Table 3 shows the second derivative of the velocity distribution $f''(\eta)$, obtained solution compared favourably with Blasius' solution except for $\eta = 3.0, 3.5 \text{ and } 4.0$ with absolute error of 0.00006, 0.0014 and 0.02075 respectively.

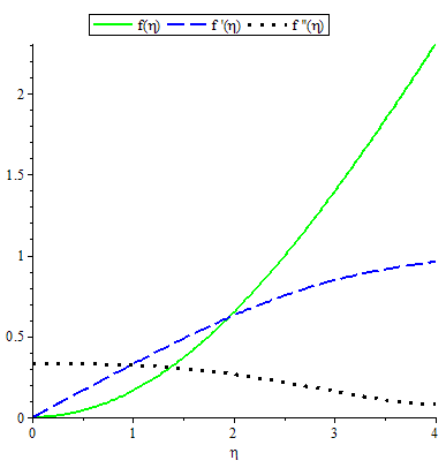

Figure 2: Function of $f(\eta)$, $f'(\eta)$ and $f''(\eta)$ against η.

Figure 2 depicts the graph of $f(\eta)$, $f'(\eta)$ and $f''(\eta)$ against η. It can be shown from the graph that $f(0) = f'(0) = 0$ is satisfied.

5.0 Conclusion

In this paper, we have applied Homotopy Perturbation Method to solve Blasius boundary layer equation. We have obtained solution with excellent accuracy for $\eta \leq 4$. In Table 1 , 2 and 3, comparison between Blasius' solution, the variational iterative method and HPM is presented. It is clear that HPM produces a velocity distribution that compare very favourably with Blasius' solution. This obtained solution can be used in situations where an analytical solution is not available as it is more accurate, converges faster than variational iterative method and confirm the exact solution of Blasius.

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