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Generalization of Coefficient Inequalities and Convolution Properties of a Class of Analytic Functions with Respect to aDifferential Operator

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Abstract

Researchers have written extensively on some properties of a new class of analytic and univalent functions in the unit disk. In this work, we used a differential operator to redefine these classes of functions. The coefficient and convolution results were different from the previous results obtained except when we allow n=0. This provides an obvious extension to A. T. Oladipo's, coefficient inequalities & convolution properties for certain New classes of analytic and univalent functions in the unit disk.

Key words: Starlike function, convex function, Ruscheweyh derivative,convolution

1.0 Introduction

Let A denote the class functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.0}$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$.
We let $A(\omega)$ be the class of functions of the form

$$f(z) = z - \omega + \sum_{k=p}^{\infty} a_k (z-\omega)^k, (k \in \mathbb{N} = 2, 3, 4, ...) , \tag{1.1}$$

which are analytic in the unit disk and normalized with $f(\omega) = 0$ and $f'(\omega) - 1 = 0$
where ω is a fixed point in $E = \{z : |z| < 1\}$.
The function introduced in (1.1) was introduced in 1999 by Kanas and Ronning. They applied this concept to define and study the following classes of $\omega-$starlike and $\omega -$ convex functions respectively

$$ST(\omega) = S^{\star}(\omega) = \{f \in S(\omega) : Re\left(\frac{(z-\omega)f'(z)}{f(z)}\right) > 0, \, z \in E\} \tag{1.2}$$

$$CV(\omega) = S^{c}(\omega) = \{f \in A : Re\left(\frac{1+(z-\omega)f''(z)}{f'(z)}\right) > 0, \, z \in E\} \tag{1.3}$$

For some of the properties of of the class introduced by Kanas and Ronning, see [1-5]
We say that the class $S(\omega)$is defined by geometric property that the image of any circular arc centered at ω is starlike with respect to $f(\omega)$ and the corresponding class $S^c(\omega)$ is defined by the property that the image of any circular arc centered at ω is convex . We define the Ruscheweyh derivative of these classes of analytic functions

$$D^n f_j(z) = z - \omega + \sum_{k=p}^{\infty} \frac{n!(m-1)!}{(m+n-1)!} a_{k,j} (z-\omega)^k \tag{1.4}$$

If $\omega = 0$ we obtain the Ruscheweyh derivative of order n, see [2]. A function $D^n f(z)$ is in $S^*(\omega, \beta, n)$ is said to be $\omega -$ *starlike* of order β in $z \in E$ with respect to Ruscheweyh derivative of order n if and only if

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$$ST(\omega) = S'(\omega) = \left\{f \in S(\omega): Re\left(\frac{(z-\omega)f'(z)}{f(z)}\right) > 0, z \in E\right\} \quad (1.5)$$

We denote by $S^{*}(\omega,\beta,n)$ the class of all functions in S(ω). A function $D^n f(z)$ in $S^c(\omega,\beta,n)$ is said to be convex of order β in E with respect to Ruscheweyh derivative of order $n + 1$ if and only if

$$Re\left(1 + \frac{(z-\omega)(D^n f_j(z))^*}{(D^n f_j(z))'}\right) > \beta \quad (1.6)$$

We define the convolution of the functions $f_1(z)$, $f_2(z)$ $\in A(\omega)$ by

$$(f_1 * f_2) = z - \omega + \sum_{k=n}^{\infty} a_{k,1}a_{k,2}(z-\omega)^k$$

(1.7)

In general , (

$$f_1 * f_2 * \cdots * f_l)(z) = (z-\omega) + \sum_{k=p}^{\infty}\left(\prod_{j=1}^{l}\right)(z-\omega)^k$$

We shall study the coefficient inequalities and obtain the extremal functions and convolution properties of the redefined classes of Kanas and Ronning expressed by (1.5)

2.0 Coefficient Inequalities

LEMMA 2.1 A function $D^n f(z) \in A(\omega)$ is in the class $S^*(\omega,\beta,n)$ if and only

$$\sum_{k=p}^{\infty} \Psi_{n,m}(kn + k + \beta_j - 2)(r+d)^{k-1}a_k \le 1 - \beta_j \quad (1.8)$$

Where $\Psi^{n,m} = \dfrac{n!(m-1)!}{(m+n-1)!}$, $|z| = r <$ 1 and $|\omega| = d$.

PROOF. Suppose $f(z)$ belongs to S^* (ω,β,n) with $|z| = r < 1$ and $|\omega| = d$. then we have

$$\frac{(z-\omega)(D^n(f_j(z))'}{D^n f_j(z)} = \frac{(z-\omega) + \sum_{k=p}^{\infty} \frac{k(n+1)n!(m-1)!a_{k,j}(z-\omega)^k}{(n+m)!}}{(z-\omega) + \sum_{k=p}^{\infty}\frac{n!(m-1)!}{(n+m-1)!}a_{k,j}(z-\omega)^k}$$

$$\mathrm{Re}\left(\frac{(z-\omega)(D^n(f_j(z))'}{D^n f_j(z)}\right) = \left|\frac{(z-\omega)(D^n(f_j(z))'}{D^n f_j(z)} - 1\right|$$

and

$$Re\left(\frac{(z-\omega)(D^n(f_j(z))'}{D^n f_j(z)}\right) = \left|\frac{(z-\omega)(D^n(f_j(z))'}{D^n f_j(z)} - 1\right|$$

$$\le \left|\frac{\sum_{k=p}^{\infty}\frac{k(n+1)n!(m-1)!a_{k},j(z-\omega)^k}{(n+m)!} - \sum_{k=p}^{\infty}\frac{n!(m-1)!}{(n+m-1)!}a_k,j(z-\omega)^k}{(z-\omega) + \sum_{k=p}^{\infty}\frac{n!(m-1)!}{(n=m-1)!}a_k,j(z-\omega)^k}\right| \le 1 - \beta_j (1.9)$$

Since $(n + m)! > (n + m - 1)!$ then

$$\frac{1}{(n+m)!} < \frac{1}{(n+m-1)!} \quad (2.0)$$

Using (2.0) in (1.9) we obtain

$$\frac{\sum_{k=p}^{\infty}\frac{n!(m-1)!}{(n+m-1)!}a_{k,j}(r+d)^{k-1}[kn+k-1]}{1 + \sum_{k=p}^{\infty}\frac{n!(m-1)!}{(n+m-1)!}a_{k,j}(r+d)^{k-1}} \le 1 - \beta_j$$

(2.1)

Let us take $\Psi^{n,m} = \dfrac{n!(m-1)!}{(m+n-1)!}$, so (2.1) yields

$$\sum_{k=p}^{\infty}\Psi_{n,m}(kn+k+\beta_j-2)(r+d)^{k-1}a_{k,j} \le 1 - \beta_j \quad (2.2)$$

Which is the required result . If we let n=0, we obtain the same result of Lemma (2.1) of [1].
Our claim in (1.8) of Lemma 2.1 is sharp with the extremal function defined with respect to Ruscheweyh derivative of order n

$$D^n f_j(z)=z-\omega+\frac{1-\beta_j}{\sum\limits_{k=p}^{\infty}\Psi_{n,m}(kn+k+\beta_j-2)}(z-\omega)^k\ , \quad k\geq 2 \tag{2.3}$$

If n=0 in (2.2) then the result of equation (10) of [1] is established

CORROLLARY 2.1 Let $D^n f_j(z)$ $\in A(\omega)$ be in the class $S^{*}(\omega,\beta,n)$. then we have

$$a_{k,j}\leq\frac{1-\beta_j}{\psi_{n.m}(kn+k+\beta_j-2)(r+d)^{k-1}}\ ,\ k\geq 2 \tag{2.4}$$

where $d=|\omega|$ and equality in (2.4) holds true for functions $D^n f_j(z)$ given by (2.3)

CORROLLARY 2.2 Let $D^n f_j(z)$ $\in A(\omega)$ be in the class $S^{*}(\omega,\beta,0)$. Then we have

$$a_{k,j}\leq\frac{1-\beta_j}{\psi_{0,m}(k+\beta_j-2)(r+d)^{k-1}}\ , \quad k\geq 2\ , \quad \psi_{om}=1 \tag{2.5}$$

LEMMA2.2 A function $D^n f_j(z)$ $\in A(\omega)$ is in the class $S^c(\omega,\beta,n)$ if and only if

$$\sum_{k=p}^{\infty}\psi_{n,m}[k(k+kn-n+\beta_j-2)(r+d)^{k-1}a_{k,j}\leq(1-\beta_j) \tag{2.6}$$

PROOF : We use the same method as in Lemma 2.1, we write

$$1+\frac{(z-\omega)\ \ D^{n+1}f_j(z))''}{(D^n f_j(z))'}=\frac{\sum_{k=p}^{\infty}\left(\frac{k(k-1)(n+1)!(m-1)!}{(n+m)!}a_{k,j}\right)(z-\omega)^{k-1}}{1+\sum\limits_{k=p}^{\infty}\left(\frac{kn!(m-1)!}{(n+m-1)!}a_{k,j}\right)(z-\omega)^{k-1}}+1$$

hence,

$$Re\left(1+\frac{(z-\omega)\ \ D^{n+1}f_j(z))''}{(D^n f_j(z))'}\right)$$

$$=\frac{\sum\limits_{k=p}^{\infty}\left(\frac{k(k-1)(n+1)!(m-1)!}{(n+m)!}a_{k,j}\right)(z-\omega)^{k-1}+\sum\limits_{k=p}^{\infty}\left(\frac{kn!(m-1)!}{(n+m-1)!}a_{k,j}\right)(z-\omega)^{k-1}}{1+\sum\limits_{k=p}^{\infty}\left(\frac{kn!(m-1)!}{(n+m-1)!}a_{k,j}\right)(z-\omega)^{k-1}}$$

Using relation (2.0) in equation (2.7) and allow $\Psi_{n,m}=\dfrac{n!(m-1)!}{(n+m-1)!}$, we have

$$\mathrm{Re}\left(1+\frac{(z-\omega)(D^{n+1}f_j(z))''}{(D^n f_j(z))'}\right)\leq\frac{\sum\limits_{k=p}^{\infty}\psi_{n,m}[k(k+kn-n+\beta_j-2)(r+d)^{k-1}a_{k,j}}{1+\sum\limits_{k=p}^{\infty}\psi_{n,m}k(r+d)^{k-1}a_{k,j}}\leq 1-\beta_j$$

further simplification will give

$$\sum_{k=p}^{\infty}\psi_{m,n}k[(n+1)(k-1)](r+d)^{k-1}a_{k,j}\leq(1-\beta_j)\left(1+\sum_{k=p}^{\infty}\psi_{n,m}k(r+d)^{k-1}a_{k,j}\right)$$

And finally one obtains

$$\sum_{k=p}^{\infty} \psi_{m,n} k(k+kn-n+\beta_j)(r+d)^{k-1} a_{k,j} \le (1-\beta_j) \qquad (2.8)$$

We observe that when $n = 0$,equation (2.8) yields result of Lemma 2.2 of [1]. Our extremal function is

$$D^n f_j(z) = (z-\omega) + \frac{1-\beta_j}{\sum_{k=p}^{\infty} \Psi_{n,m}[k(kn+k+\beta_j-2)]}(z-\omega)^k \quad k \ge 2 \qquad (2.9)$$

CORROLLARY 2.3 Let $D^n f_j(z) \in A(\omega)$ is in the class $S^c(\omega,\beta,0)$. Then

$$a_{k,j} \le \frac{1-\beta_j}{\sum_{k=p}^{\infty} \Psi_{n,m}[k(kn+k+\beta_j-2)](r+d)^{k-1}}, \qquad (3.0)$$

Where $d = |\omega|$. Equality holds true for functions given by (2.9)

CORROLLARY 2.4 $D^n f_j(z) \in A(\omega)$ be in the class $S^c(\omega,\beta,0)$.Then

$$a_{k,j} \le \frac{1-\beta_j}{\sum_{k=p}^{\infty} \Psi_{0,m}[k(k+\beta_j-2)](r+d)^{k-1}}, \quad \Psi_{0,m}=1 \qquad (3.1)$$

Which agrees with corollary (2.2) of [1].

3.0 Convolution Properties for Functions in the Class $S^*(\omega,\beta,\mathbf{n})$

$S^*(\omega,\beta,n)$ In this section we consider the Hadamard product of functions defined by (1.7) for the class $S^*(\omega,\beta,n)$

THEOREM3.1

If $D^n f_j(z) \in A(\omega)$ be in $S^*(\omega,\beta,n)$, ($j = 1,2,...,m$) then $D^n(f_1 * \cdots * f_m)(z) \in S^*(\omega,\beta,n)$ where

$$\alpha = 1 - \frac{(kn+k-1)\prod_{j-1}^{m}(1-\beta_j)}{\prod_{j=1}^{m}(1-\beta_j) + \prod_{j=1}^{m}(kn+k+\beta_j-2)(r+d)^{k-1}} \qquad (3.2)$$

The result is sharp for the functions $D^n f_j(z)$ ($j = 1,2...m$) given by

$$D^n f_j(z) = z-\omega+\frac{(1-\beta_j)}{(kn+k+\beta_j-2)}(z-\omega)^k \qquad (3.3)$$

Proof . Here we use the principle of mathematical induction to prove theorem (3.1)

Let $D^n f_1(z) \in S(\omega,\beta_1,n)$ and $D^n f_2(z) \in S^*(\omega,\beta_2,n)$. Then the inequality

$$\sum_{k=p}^{\infty} \psi_{n,m}(kn+k+\beta_j-2)(r+d)^{k-1} a_{k,j} \le 1-\beta_j \qquad (3.4)$$

implies that

$$\sum_{k=p}^{\infty} \sqrt{\frac{\psi_{n,m}(kn+k+\beta_j-2)(r+d)^{k-1}}{1-\beta_j} a_{k,j}} \le 1 \qquad (3.5)$$

Thus , by applying the Cauchy-Schwarz inequality ,we have

$$\le (r+d)^{k-1}\left[\sum_{k=p}^{\infty} \psi_{n,m}\left(\frac{kn+k+\beta_1-2}{1-\beta_1}\right)a_{k,1}\right]\left[\sum_{k=p}^{\infty} \psi_{n,m}\left(\frac{kn+k+\beta_2-2}{1-\beta_2}\right)a_{k,2}\right] \le 1 \qquad (3.6)$$

Therefore , if $\sum_{k=p}^{\infty} \psi_{m,n}\left(\frac{kn+k+\delta-2}{1-\delta}\right)a_{k,1}a_{k,2} \le \sum_{k=p}^{\infty} \sqrt{\frac{(\psi_{m,n})^2 (kn+k+\beta_1-2)(kn+k+\beta_2-2)a_{k,1}a_{k,2}}{(1-\beta_1)(1-\beta_2)}}$

That is if, $\sqrt{a_{k,1}a_{k,2}} \le \left(\frac{1-\delta}{kn+k+\delta-2}\right)\sqrt{\frac{(kn+k+\beta_1-2)(kn+k+\beta_2-2)(r+d)^{k-1}}{(1-\beta_1)(1-\beta_2)}}$

Then $D^n[(f_1 * f_2)(z)] \in S^{□}(\omega,\delta,n)$. We note that the inequality (3.5) yields

$\sqrt{a_{k,j}} \le \sqrt{\frac{1-\beta_j}{kn+k+\beta_j-2}(r+d)^{k-1}}$ (j=1,2 : k=p, p+1, p+2 ...)

Consequently, if

$$\sqrt{\frac{(1-\beta_1)(1-\beta_2)}{(kn+k+\beta_1-2)(kn+k+\beta_2-2)(r+d)^{k-1}}} \le \frac{1-\delta}{kn+k+\delta-2}\sqrt{\frac{(kn+k+\beta_1-2)(kn+k+\beta_2-2)(r+d)^{k-1}}{(1-\beta_1)(1-\beta_2)}}$$

that is , if

$$\frac{kn+k+\delta-2}{1-\delta} \le \frac{(kn+k+\beta_1-2)(kn+k+\beta_2-2)(r+d)^{k-1}}{(1-\beta_1)(1-\beta_2)} \quad k = p, p+1, p+2,. \tag{3.7}$$

then we have

$D^n((f_1 * f_2)(z)) \in S^*(\omega,\delta,n)$ It follows from (3.7) that

$$\delta \le 1-\frac{(kn+k-1)(1-\beta_1)(1-\beta_2)}{(1-\beta_1)(1-\beta_2)+(kn+k+\beta_1-2)(r+d)^{k-1}} = L(k),(k = p, p+1, p+2,...) \tag{3.8}$$

Since $L(k)$ is increasing for $k > n$, we have

$$\delta = 1-\frac{(pn+p-1)(1-\beta_1)(1-\beta_2)}{(1-\beta_1)(1-\beta_2)+(pn+p+\beta_1-2)(pn+p+\beta_2-2)(r+d)^{k-1}} \tag{3.9}$$

which shows that

$D^n(f_1 * f_2)(z) \in S^*(\omega,\delta,n)$

where
$$\delta = 1-\frac{(kn+k-1)(1-\beta_1)(1-\beta_2)}{(1-\beta_1)(1-\beta_2)+(kn+k+\beta_1-2)(kn+k+\beta_2-2)(r+d)^{k-1}} \tag{4.0}$$

Next we suppose that $D^n(f_1 *** f_m)(z) \in S^*(\omega,\gamma,n)$, where $\gamma = \frac{(kn+k-1)\prod_{j=1}^{m}(1-\beta_j)}{(r+d)^{k-1}\prod_{j=1}^{m}(kn+k+\beta_j-2)+\prod_{j=1}^{m}(1-\beta_j)}$ (4.1)

Then by means of the above technique, we can show that

$$\alpha \le 1-\frac{(kn+k-1)\prod_{j=0}^{m+1}(1-\beta_j)}{(r+d)^{k-1}\prod_{j=0}^{m+1}(kn+k+\beta_j-2)\psi_{m,n}^{m+1}+\prod_{j=1}^{m+1}(1-\beta_j)} \tag{4.2}$$

Finally , for functions given by (16) , we have $D^n(f_1 * ... * f_m)(z) = z-\omega+\left(\prod_{j}^{m}\left(\frac{1-\beta_j}{(r+d)^{k-1}(kn+k+\beta_j-2)}\right)\right)(z-\omega)^k$

4.0 Convolution Properties For Functions In The Class $S^c(\omega,\beta,n)$.

We shall derive the convolution of functions in the class $S^c(\omega,\beta,n)$.

THEOREM 4.1

If $D^n f_j(z) \in S^c(\omega,\beta,n)$ $(j = 1,2,...,m)$ then $D^n(f_1 * \cdots * f_m)(z) \in S^c(\omega,\beta,n)$ where

$$\delta = 1 - \frac{(p+pn-n-1)\prod_{j=1}^{m}(1-\beta_j)}{\Psi_{n,m}p^{m-1}(r+d)^{k-1}\prod_{j=1}^{m}p(p+pn-n+\beta_j-2)+\prod_{j=1}^{m}(1-\beta_j)} \qquad (4.3)$$

the result is sharp for the functions $f_j(z)$ given by

$$D^n f_j(z) = (z-\omega)+\left(\frac{1-\beta_j}{\Psi_{n,m}[k(k+kn-n+\beta_j-2)}\right)(z-\omega)^{k-1} \qquad (4.4)$$

Proof. For $D^n f_j(z) \in S^c(\omega,\beta,n)$, (j=1,2) then we have the inequality

$$\sum_{k=p}^{\infty}\sqrt{\frac{\psi_{n,m}[k(k+kn-n+\beta_1-2]\psi_{n,m}[k(k+kn-n+\beta_2-2](r+d)^{k-1}a_{k,1}a_{k,2}}{(1-\beta_1)(1-\beta_2)}} \le 1 \qquad (4.5)$$

which implies that $D^n(f_1 * f_2)(z) \in S^c(\omega,\delta,n)$

Following the method of proof in Theorem (3.1) , we get

$$\delta \le 1 - \frac{k(k+kn-n-1)\prod_{j=1}^{2}(1-\beta_j)}{k\prod_{j=1}^{2}(1-\beta_j)+\psi_{n,m}[\prod_{j=1}^{2}k(k+kn-n+\beta_j-2](r+d)^{k-1}}, \quad k = (p,p+1,...) \qquad (4.6)$$

The right-hand side of (4.5) takes its minimum at $k = p$ because it is an increasing function of $k \ge p$ This shows that

$D^n(f_1 * f_2)(z) \in S^c(\omega,\delta,n)$ where

$$\delta = 1 - \frac{(p+pn-n-1)\prod_{j=1}^{2}(1-\beta_j)}{\prod_{j=1}^{2}(1-\beta_j)+\psi_{n,m}p^{-1}[\prod_{j=1}^{2}p(p+pn-n+\beta_j-2)](r+d)^{k-i}}$$

Observe that when $n = 0$ we get the same result as equation (27) of [1].

Suppose $D^n(f_1 * \ldots * f_m)(z) \in S^c(\omega,\gamma,n)$

Where

$$\gamma = 1 - \frac{p+pn-n-1)\prod_{j=1}^{m}(1-\beta_j)}{\prod_{j=1}^{m}(1-\beta_j)+\psi_{n,m}p^{m-1}[\prod_{j=1}^{m}p(p+pn-n+\beta_j-2)(r+d)^{k-1}}$$

Finally, we have $D^{n}(f_1 * \ldots * f_{m+1})(z) \in S^{c}(\omega, \alpha, n)$, where

$$\alpha = 1 - \frac{(p+pn-n-1)\prod_{j=1}^{m+1}(1-\beta_j)}{\prod_{j=1}^{m+1}(1-\beta_j) + \psi_{n,m} p^{m}[\prod_{j=1}^{m+1} p(p+pn-n+\beta_j-2)(r+d)^{k-1}}$$

5.0 Conclusion

It is easy to observe that in all cases considered in this work,we used Ruscheweyh derivative of order n to generalize the results of cofficient and convolution problems involvinga class of analytic functions with fixed points .Having shown that when $n = 0$ we obtained the same results as in [1],this work clearly demonstrated that Ruscheweyh derivative is a useful tool in generalization of some properties of functions.

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7.0 References

[1] A. T. Oladipo ,Coefficient Inequalities and Convolution Properties for Certain New Classes of Analytic and Univalent functions in the unit Disk. Far East Journal Of Mathematical Sciences (FJMS) Vol. 62 , Number 2 ,2012 pp 219-232.

[2] Emelike Ukeje & J. Nnadi . Convolution Properties Of a Class of Analytic Functions , International Journal Of Mathematics & Analysis (IJMA) ,Vol. 4, No 1,(2012) pp 71-78.

[3] Mugur Acu & Shigeyoshi Owa, On a Subclass of n-starlike functions . International Journal Of Mathematics & Mathematical Sciences 2005 :17(2005) 2841-2846.

[4] Sultan Aytas & H. Oziem Guney . On The Fekete-Szego like inequality For Meromorphic Functions With Fixed Residue. International Journal Of Physical Sciences, Vol. 8(17) pp750753.

[5] A. T. Oladipo .Coefficient Estimate For Some Families Of Analytic Univalent Functions Associated With $q - Analogue$ Of Dziok-Srivastava Operator. Analele Universitatii Oradea Fasc Matematica,Tom XXX (2015) , Issue No. 1, 141-153.