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Balls in Partial b-Metric Spaces

Aribike Emmanuella Ehui

Department of Mathematics and Statistics, Lagos State Polytechnic, Ikorodu.

Abstract

This paper discusses balls in partialb-metric spaces. Let (X, p_b)be a partial b-metric space in the sense of Mustafa et-al. for the familyΔ of all ρ_b-open balls in (X,ρ_b).This paper proves that there are x, y $\in$ B $\in$ Δ such that B' $\not\subseteq$ B for all B' $\in$ Δ, where B and B' are with centres x and y, respectively. This result shows that Δ is not a base of any topology on X, which shows that a proposition and claim on partial b-metric spaces are not true.

Keywords: Ball, Partial b-metric space, Topology, T-Space, Non-emptyset, sub-base

1.0 Introduction

Partial b-metric spaces and cone metric spaces are important generalizations of metric spaces, which were introduced and investigated by Shukla in [1] and Huang-Zhang in [2], respectively.

Recently, Mustafa *et al.* introduced a new concept of partial b-metric by modifying partial b-metric in the sense of [1] in order to guarantee that each partial b-metric p_b can induce a b-metric [3]. Furthermore, they proved the following proposition.

Proposition1.1[3] *Let (X,p_b) be a partial b-metric space in the sense of [3]. For each $x \in X$ and $\varepsilon > 0$, the p_b –open ball with center x and radius ε is*

$$Bp_b(x, \varepsilon) = \{y \in X : p_b(x,y) < p_b(x,x) + \varepsilon\}.$$

Then for each $B_{pb}(x,\varepsilon)$ and each $y \in B_{pb}(x, \varepsilon)$, there is $\delta > 0$ such that $B_{pb}(y, \delta) \subseteq B_{pb}(x, \varepsilon)$.

Thus, from Proposition 1.1, the following claim arose naturally [3].

Claim 1.2:[3]*Let (X,p_b) be a partial b-metric space, in the sense of [3].*

Put $\Delta = \{B_{pb}(x,\varepsilon): x \in X$ and $\varepsilon > 0\}$, i.e.,$\Delta$is the family of all p_b-open Balls. Then $\Delta$is a base of some topology on X.

It is also worthy nothing that proposition 1.1 and Claim 1.2 were cited in [4].

Equality 1.3 $\{y \in X: d(x,y) << \varepsilon\} = \{y \in X: d(x,y) \leq \varepsilon\}$.

In this paper, we discuss Proposition 1.1, Claim 1.2, and Equality 1.3. For Proposition 1.1 and Claim 1.2, we construct a partial b-metric space (X,p_b) in the sense of [3], and show that there are ap_b-open ball $B_{pb}(x,\varepsilon)$ *and* $y \in B_{pb}(x, \varepsilon)$ *such that* $B_{pb}(y, \delta) \not\subseteq B_{pb}(x, \varepsilon)$ for all $\delta>0$, and hence Δ is not a base of any topology on *X,* which shows that Proposition 1.1 (including its proof) and Claim 1.2 are not true. For Equality 1.3, we establish some relations between balls and their closures in cone metric spaces by <, and ≤, and we give an example to show that Equality 1.3 is not true. However, it must be emphasized that these corrections do not affect the rest of the results in [3,5].

Throughout this paper $\mathbb{N}, \mathbb{R}, and \mathbb{R}^+$ denote the set of all natural numbers, the set of all real numbers and the set of all nonnegative real numbers, respectively. For an subset A of a space X,$\bar{F}$ denotes the closure of F in X. For undefined notations and terminology, one can refer to [3,5].

2.0 P_b-Open Balls in Partial B-Metric Spaces

The following partial b-metric spaces introduced by Shukla in[1].

Definition 2.1 [1] Let X be a non-empty set. A mapping $p_b : X \times X \rightarrow \mathbb{R}^+$is called a partial b-metric with coefficients ≥ 1 and (x, p_b) is called a partial b-metric space with coefficients ≥ 1 if the following are satisfied for all x, y, z, $\in$ X:

Corresponding author: Aribike Emmanuella Ehui, E-mail:aribike.ella@yahoo.com, Tel.: +2348034012186

(1) $x = y \Leftrightarrow p_b(x,x) = p_b(y,y) = p_b(x,y)$
(2) $p_b(x,y) = p_b(y,x)$
(3) $p_b(x,x) \le p_b(x,y)$
(4) $p_b(x,y) \le s(p_b(x,z) + p_b(z,y)) - p_b(z,z)$.

Remark 2.2: If s =1 in Definition 2.1, then (X, p_b) is a partial metric space which was introduced by Matthews (for example, see [3]). Further, put $d_{pb} : X \times X \to \mathbb{R}^+$:by $d_{pd}(x,y)= 2p_b(x,y) - p_b(x,x) - p_b(y,y)$ for all $x, y \in X$, then d_{pb} is a metric on X and (X,d_p) is a metric space.
However, if s >1, then we cannot guarantee that each partial b-metric can induce a b-metric by the method in Remark 2.2. So Mustafa *et al.* gave the following partial b-metric p_b by modifying Definition 2.1 (4) and proved that the p_b induces a b-metric by the method in Remark 2.2.
Definition 2.3 [3] Let X be a non-empty set. A mapping p_b: $X \times X \to \mathbb{R}^+$ is called a partial b-metric with coefficients ≥ 1 and (x, p_b) is called a partial b-metric space with coefficients ≥ 1 if the following are satisfied for all $x, y, z, \in X$:
(1) $x = y \Leftrightarrow p_b(x,x) = p_b(y,y) = p_b(x,y)$
(2) $p_b(x,y) = p_b(y,x)$
(3) $p_b(x,x) \le p_b(x,y)$
(4) $p_b(x,y) \le s(p_b(x,z) + p_b(z,y)) - p_b(z,z) + \frac{1-s}{2}(p_b(x,x) + p_b(y,y))$

Remark 2.4 If x,y,z satisfy Definition 2.3(1), (2) (3) and are different from each other, then it is easy to check that x, y, z Definition 2.3(4) holds.
As a known fact, Proposition 1.1 and Claim 1.2 are not true if (X, p_b) is a partial b-metric space in the sense of Definition 2.1 ([6]). So it is it is important to check whether Proposition 1.1 and Claim 1.2 are true if (X, p_b) is a partial b-metric space in the sense of Definition 2.3 The following example shows that the result of the check is negative, which comes from [6]. In the following, all partial b-metric spaces are in the sense of Definition 2.3.

3.0 Results and Discussion

Example 3.1 Let X = [u, v, w] and put $p_b : X \times X \to \mathbb{R}^+$ as follows:
(i) $p_b(u,u) = p_b(w,w) = 1$ *and* $p_b(v,v) = 0.5$
(ii) $p_b(u,w) = p_b(w,u) = 1.5$.
(iii) $p_b(v,w) = p_b(w,v) = 1$.
(iv) $p_b(u,v) = p_b(v,u) = 3$
Let $B_{pb}(u,\varepsilon)$ be described in Proposition. Then the following hold:
(1) p_b is a partial b-metric with coefficient s = 3.
(2) $W \in B_{pb}(u,1)$ and for any ε> 0, $B_{pb}(w,\varepsilon) \not\subseteq B_{pb}(u,1)$
Proof (1) it is not difficult to check that p_b satisfies Definition 2.3(1), (2), (3). In order to check that p_b satisfies Definition 2.3(4), we only need to consider the following three cases by Remark 2.4
(1) $x = u, y = v, z = w$:
$P_b(u,v) = 3$.
$3(p_b(u,w) + p_b(w,v) - p_b(w,w)) + \frac{1-3}{2}(p_b(u,u) + p_b(v,v)) = 3$.
So $p_b(u,v) \le 3(p_b(u,w) + p_b(w,v) - p_b(w,w) + \frac{1-3}{2}(p_b(u,u) + p_b(v,v))$.
(2) $x = u, y = w, z = v$:
$P_b(u,w) = 1.5$.
$3(p_b(u,v) + p_b(v,w) - p_b(v,v)) + \frac{1-3}{2}(p_b(u,u) + p_b(w,w)) = 8.5$.
So $p_b(u,w) \le 3(p_b(u,v) + p_b(v,w) - p_b(v,v) + \frac{1-3}{2}(p_b(u,u) + p_b(w,w))$.
(3) $x = v, y = w, z = u$:
$P_b(v,w) = 1$,
$3(p_b(v,u) + p_b(u,w) - p_b(u,u)) + \frac{1-3}{2}(p_b(v,v) + p_b(w,w)) = 9$.
So $p_b(v,w) \le 3(p_b(v,u) + p_b(u,w) - p_b(u,u) + \frac{1-3}{2}(p_b(v,v) + p_b(w,w))$.
Thus p_b is a partial b-metric with coefficient s = 3

(3) Since $p_b(u,w) = 1.5 < 1 + 1 = p_b(u,u) + 1$ $w \in B_{pb}(u,1)$. In addition, for any $\varepsilon > 0$, $p_b(w, v) = 1 < 1 + \varepsilon = p_b(w,w) + \varepsilon$, so $v \in B_{pb}(w, \varepsilon)$. On the other hand, $p_b(u,v) = 3 \not< 2 = 1+1 = p_b(u,u) + 1$, so $v \notin B_{Pb}(u,1)$. This shows that $B_{Pb}(w,\varepsilon) \not\subseteq B_{Pb}(u, 1)$.

Remark 3.2 Example 3.1 shows that Proposition 1.1 and Claim 1.2 are not true if (X, p_b) is a partial b-metric space. However, we have the following.

Proposition 3.3 [7] *Let (X, p_b) be a partial b-metric space and Δbe described in Claim 1.2. Then Δ is a subbase for some topology on X. We denote the topology by $\mathcal{T}_{Pb}$.*
It is well known that the space $(X,\mathcal{T}_{Pb})$ is T_0 but does not need to be T_1 *[7]*. The following proposition give a sufficient and necessary such that $(X, \mathcal{T}_{Pb})$ is a T_1*-space.*

Proposition 3.4*Let (X,p_b) be a partial b-metric space in the sense of Definition 2.3. Thenthe following are equivalent:*
(1) $(X,\mathcal{T}_{Pb})$*is a* T_1*-space.*
(2) $p_b(x,y) > \max \{p_b(x,x), p_b(y,y)\}$ *for each pair of distinct points* $x, y \in X$
Proof (1) $\Rightarrow$ (2): Let $(X,\mathcal{T}_{Pb})$ be a T_1*- space. If* $x, y, \in X$ and$x \neq y$, then there is a neighborhood U of x such that $y \notin U$. SinceΔ is a subbase of $(X,\mathcal{T}_{Pb})$ from Proposition 2.7, there are $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k > 0$ such that $y \notin \cap \{B_{Pb}(x,\varepsilon_i) : I = 1, 2, \ldots, k\}$, *and hence there is* $i_0 \in \{1,2,\ldots,k\}$ such that $y \notin B_{Pb}(x,\varepsilon_{i0})$.*So* $p_b(x,y) \geq p_b(x,x) + \varepsilon_{i0} > p_b(x,x)$. In the same way, $p_b(x,y) > p_b(y,y)$. *So* $p_b(x,y) > \max \{p_b(x,x), p_b(y,y)\}$
(2)$\Rightarrow$ (1): Let $x,y \in X$ *and* $x \neq y$. *if* $p_b(x,y) > \max \{p_b(x,x), p_b(y,y)\}$. *Then* $p_b(x,y) > p_b(x,x)$. *Put* $\varepsilon = p_b(x,y) - p_b(x,x) > 0$,*then* $p_b(x,y) - p_b(x,x) + \varepsilon$, and so $y \notin B_{Pb}(X, \varepsilon)$. In the same way, there is $\varepsilon' > 0$ such that $x \notin B_{Pb}(y,\varepsilon')$. *Consequently,* $(X,\mathcal{T}_{Pb})$ *is a* T_1*-space.*

4.0 Conclusion

Let (X,p_b) be a partial b-metric space in the sense of [3]. For the familyΔ of all p_b- open balls in (X,p_b), this paper proves that there are $x,y, \in B \in \Delta$ such that $B' \not\subseteq B$ for all $B' \in \Delta$, where B and B' are with centres x and y, respectively. This result shows that Δ is not a base of any topology on X, which shows that [3], Proposition 4 and the claim following [3] and Proposition 4 are not true.

5.0 References

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