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On the Rank Equation for a Class of BOL Algebras

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Abstract

In this paper, we investigate the right alternative algebras of quaternion type. It is proved that Bol algebras of quaternion type satisfy a rank equation of the form $x^2 - t(x)x + n(x)1 = 0$ ***and by extension, the form*** $(xy)^2 - 2k_0 xy + n(xy) = 0$ ***is satisfied where*** x, y ***are elements of the algebra and the scalar*** $k_0 = 1/2t(xy)$ ***. In these algebras, the norm,*** $n(x)$, ***is non-constant but reduces to a constant (a positive real number) if the underlying loop elements are anti-commutative.***

Keywords: Right alternative, norm, rank equation, Bol quaternion algebras.

1.0 Introduction

The study of non-associative algebras have gained the attention of various researchers and authors over the years, particularly those loop algebras whose underlying loops satisfy the so called Bol-Moufang identities over some field F . For instance, it is well known [1,2] that Cayley algebras satisfyan equation of the form

$$x^2 - t(x)x + n(x)e = 0$$

where the trace $t(x)$ and the norm $n(x)$, for any non-zero element x , are real numbers.

An algebra satisfying this minimal (or rank) equation is said to be quadratic.

Our contribution in this paper is to complement such efforts by investigating the rank equation for Bol algebras of quaternion type.

2000hematics Subject Classification .20N05.

2.0 Preliminaries

We first recall the following important definitions, readers interested in more detailed information should consult Bruck [3], Plugfelder [4] and Schafer [2].

Definition 0.1*A loop L satisfying the property that for all x, y, z in L, $(xy.z)y = x(yz.y)$ is called a right Bol loop. Equivalently, if for all x, y, z in L, $y(z.yx) = (y.zy)x$ then L is called a left Bol loop.*

A right Bol loop (or a left Bol loop) is called a Bol loop since whatever properties hold for a right Bol loop also hold for its dual (left Bol loop). A loop satisfying both left and right Bol loop properties is called Moufang. A loop identity $f(x, y, z)$ say, is of Bol-Moufang type if 2 of its 3 variables occur once on either side of the equality, the third variable occurs twice on each side and the order in which they appear on both sides is the same.

Definition 0.2 [2] *Let L be a loop written multiplicatively and F an arbitrary field. If we define multiplication in the vector space A of all formal sums of a finite number of elements in L with coefficients in F by the use of both distributive laws and the definition of multiplication in L then the resulting loop algebra $A(L)$ over F is a linear non-associative algebra (associative if and only if L is a group).*

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Definition 0.3 *Let $A(L)$ be a loop algebra over a real field F whose basis elements generate a non-abelian Bol loop L of order n, where*

$$L = \{< e_i >: e_0 = 1, e_i^2 = -1, e_i e_j = \pm e_j e_i, 1 \le i \le n-1\} \quad (1)$$

then A is said to be a Bol algebra of quaternion type. L is called a Bol loop of quaternion type (simply Bol quaternion loop). An element x of a loop L such that $x^2 = -1, x^4 = 1$ and $xy = \pm yx$ for all y in L is called a Bol quaternion element [5].

Remark 0.1 The definition above show that Bol algebras of quaternion type are unital algebras of dimension $2^r, r \ge 2$ whose basis elements generate right alternative non-commutative Bol loops of order 2^{r+1}. Consequently, this includes the Hurwitz algebras R, C, H, and, O (viz. Real numbers, Complex numbers, Quaternions and Octonions) as well as their generalized extensions by Cayley-Dickson process; viz: algebras of the form $(F,-1,-1)$ and $(F,-1,-1,-1)$ etc., where F is any field [6].

Definition 0.4 *An algebra A over F is said to be alternative if for all x, y in A the identities*

$$x^2.y = x(xy) \text{ and } y.x^2 = (yx)x$$

known respectively as the Left and Right Alternative laws, hold in A. An algebraA is called Left (or, Right) Alternative Algebra ifonly the Left (or the Right) alternative law holds in A. Analternative algebra A with unity is a division algebra if every non-zero element of A is invertible [2].

Definition 0.5 *An algebra A over F is said to be commutative if $xy = yx$ for all pairs of elements x, y in A. Otherwise, A is non-commutative. An algebra A over F is said to be anti-commutative if and only if $x^2 = 0$ for all x in A.*

Definition 0.6 *An involution of an algebra A is a linear map $q \to \bar{q}$ of A satisfying*

$$\overline{q_1 q_2} = \bar{q}_2 . \bar{q}_1 \text{ and } \bar{\bar{q}} = q \quad (2)$$

for all q, q_1,$q_2$in A.

3.0 Main Result

In our investigation of this minimal equation for Bol quaternion algebras we recognize the underlying loops as those of Bol quaternion loops of order 8 and above since these loops generate the respective Bol quaternion algebras. In Burn [7], it was shown that there are six non-associative Bol loops of order 8. Of these six, the first one named π_1 in [7] is the only one that is of quaternion type. We shall name it BQ_8 in accordance with the nomenclature of its associative counterpart (the quaternion group Q_8) so that [5],

$$BQ_8 = \{< i, j, k >: ij = ji, jk = kj, ik = -ki, i^2 = j^2 = k^2 = -1\}$$

For loops of higher orders in this class, one may consult Sharma and Solarin [8], Moorhous [9] or Naggy and Vojtechovsky [10].

Theorem 0.1*Let A(L) be a Bol algebra of quaternion type defined over the real field F, then for any q∈ $A(L)$, (i) N(q) is not a constant (ii) N(q) reduces to a positive real number if and only if A is alternative or Char(F)=2*

Proof: (i) Let A be finite dimensional of order n, and let

$$L = \{< e_i >: e_0 = 1, e_1^2 = -1, e_i e_j = \pm e_j e_i, 1 \le i \le n-1\}.$$

Then, for any q in A, the norm

$$N(q) = q.\bar{q} = (a_0 + \sum_{i=1}^{n-1} a_i e_i)(a_0 - \sum_{i=1}^{n-1} a_i e_i) = \sum_{i=0}^{n-1} a_i{}^2 - 2\sum_{1 \le i < 1}^{n-1} a_i a_j e_i e_j \quad (3)$$

where e_i, e_j are commuting elements for some i,j. Since $e_i e_j = e_k (say), k \ne 0, N(q)$ is not a constant.

(ii) Consider that from (3)

$$N(q) = \sum_{i=0}^{n-1} a_i^{\,2} - 2\sum_{1\le i<1}^{n-1} a_i a_i e_i e_j = \sum_{i=0}^{n-1} a_i \qquad (4)$$

provided $2\sum_{1\le i<j}^{n-1} a_i a_j e_i e_j = 0$ i.e. if $\sum_{1\le i<j}^{n-1} a_i a_j (2e_i e_j) = 0$ i.e. if $\sum_{1\le i<j}^{n-1} a_i a_j (e_i e_j + e_i e_j) = 0$. For non-zero $a_i a_j$, this implies

$$e_i e_j = -e_i e_j = -e_j e_i \text{ for all } i, j, \quad i \ne j \qquad (5)$$

(since e_i, e_j commute). Thus, (4) holds if $e_i e_j = -e_j e_i$ for all $i \ne j$ in (3). This is the case when L is Moufang and so holds true for alternative $A(L)$. For non-alternative $A(L)$, (4) holds only when F is of characteristic 2.

To prove the converse, one must recognize that A is alternative once the underlying loop L is alternative. That is, if

$$e_i^2 e_j = e_i(e_i e_j) \ and \ e_j e_i^2 = (e_j e_i)e_i$$

holds true in L . By definition, if $e_i e_j = e_k$ then

$$-e_j = e_i(e_k) = e_i e_k \ and \ e_j(-1) = (\pm e_k)e_i = \pm e_k e_i$$

Thus the form $e_i e_j$ is anti-symmetric in L , and so $2\sum_{1\le i<j}^{n-1} a_i a_j e_i e_j = 0$. Similarly,if $charF = 2$ for any A then (4) holds, and so $N(q) = \sum_{i=0}^{n-1} a_i^2$.

Remark 0.2 *From the theorem, one obtains, for* $n = 4$

$$N(q) = \sum_{i=0}^{3} a_i^2 - a_1 a_2(e_1 e_2 + e_2 e_1) - a_1 a_3(e_1 e_3 + e_3 e_1) - a_2 a_3(e_2 e_3 + e_3 e_2)$$

so that by consequently applying the appropriate multiplication rule for the underlying loop the right result is obtained. For instance, by considering the Bol quaternion algebra with the underlying group Q_8 one easily obtains $N(q) = \sum_{i=0}^{3} a_i^2$ *, a positive real number. This is the case for alternative Cayley algebras, and is the usual result in literature. Same is obtained, of course, for the Bol quaternion algebra with the underlying Moufang loop $M_{16}Q_8$.*

On the other hand, the Bol quarternion algebra with the underlying loop $B(Q_8)$ has $N(q) = \sum_{i=0}^{3} a_i^2 + 2a_2 a_3 e_1 - 2a_1 a_2 e_3$, *which is not a constant. Ditto for the Bol quaternion algebras of order 8 with underlying loop* L *, where* L *is of the non-associative non-Moufang type in the Moorhous collection [9]*

Theorem 0.2 Let A be a loop algebra generated by a Bol quaternion loop L over the real field F, then, every element q in A satisfies:

$$q^2 - 2a_0 q + N(q) = 0 \qquad (6)$$

where $2a_0 = q + \bar{q}$ and $N(q) = q.\bar{q}$

Proof: Let A be finite dimensional, of order n, and let

$$L = \{<e_i>: e_0 = 1, e_i^2 = -1, e_i e_j = \pm e_j e_i, 1 \le i \le n-1\}$$

Then

$$q^2 = (a_0 + \sum_{i=1}^{n-1} a_i e_i)^2 = a_0^2 + (\sum_{i=1}^{n-1} a_i e_i)^2 + 2a_0 \sum_{i=1}^{n-1} a_i e_i$$

$$= a_0^2 - \sum_{i=1}^{n-1} a_i^{\,2} + 2a_0 \sum_{i=1}^{n-1} a_i e_i + 2\sum_{i,j=1}^{n-1} a_i a_j e_i e_j \qquad (7)$$

where i < j and $e_i e_j$ commute. Now,

$$2a_0 q = 2a_0(a_0 + \sum_{i=1}^{n-1} a_i e_i) = 2a_0^{\,2} + 2a_0 \sum_{i=1}^{n-1} a_i e_i$$

N (q) = q. $\bar{q} = \sum_{i=1}^{n-1} a_i^{\,2} - 2\sum_{i,j=1}^{n-1} a_i a_j e_i e_j$ (8)

Substituting these into (6) proves the theorem.

Lemma 0.1 Let A be a Bol quaternion algebra over a field F, then the trace t(x) is linear and the Norm N(x) is quadractic.

Proof: Follows from (8) and the definition of t(x).

Theorem 0.3 Let A be an algebra generated by a Bol loop L over F where

$L = \{ < e_i >: e_0 = 1, e_i^2 = -1, e_i e_j = \pm e_j e_i, 1 \le i \le n-1 \}$

Then, for all q_1, q_2 in A, the following equation is satisfied:

$$(q_1 q_2)^2 - 2K_0 q_1 q_2 + N(q_1 q_2) = 0 \quad (9)$$

Proof: We follow similar steps as above. Let $q_1 = a_0 + \sum_{i=1}^{n-1} a_i e_i$, and $q_2 = b_0 + \sum_{i=1}^{n-1} b_j e_j$. Then,

$$q_1 q_2 = (a_0 + \sum_{i=1}^{n-1} a_i e_i)(b_0 + \sum_{i=1}^{n-1} b_i e_i) = a_0 b_0 + bo\sum_{i=1}^{n-1} a_i e_i + ao\sum_{j=1}^{n-1} b_j e_j + \sum_{i,j=1}^{n-1} a_i b_j e_i e_j$$

$$= a_0 b_0 - \sum_{i=1}^{n-1} a_i b_i + \sum_{k=1}^{n-1}\sum_{i,j=0}^{n-1} a_i b_j e_k, \qquad i \ne j \quad (10)$$

where $e_i e_j = e_k$. Consequently,

$$(q_1 q_2)^2 = (a_0 b_0 - \sum_{i=1}^{n-1} a_i b_i + \sum_{k=1}^{n-1}\sum_{i,j=0}^{n-1} a_i b_j e_k)^2$$

$$= (a_0 b_0 - \sum_{i=1}^{n-1} a_i b_i)^2 + (\sum_{k=1}^{n-1}\sum_{i,j=0}^{n-1} a_i b_j e_k)^2 + 2(a_0 b_0 - \sum_{i=1}^{n-1} a_i b_i)(\sum_{k=1}^{n-1}\sum_{i,j=0}^{n-1} a_i b_j e_k)$$

$$= (a_0 b_0)^2 + (\sum_{i=1}^{n-1} a_i b_i)^2 - 2a_0 b_0 \sum_{i=1}^{n-1} a_i b_i + (\sum_{k=1}^{n-1}\sum_{i,j=0}^{n-1} a_i b_j e_k)^2$$

$$+ 2a_0 b_0 \sum_{k=1}^{n-1}\sum_{i,j=0}^{n-1} a_i b_j e_k - 2\sum_{i=1}^{n-1} a_i b_i \sum_{k=1}^{n-1}\sum_{i,j=0}^{n-1} a_i b_j e_k$$

$$= (a_0 b_0)^2 + (\sum_{i=1}^{n-1} a_i b_i)^2 - 2a_0 b_0 \sum_{i=1}^{n-1} a_i b_i + \sum_{k=1}^{n-1}(\sum_{i,j=0}^{n-1} a_i b_j e_k)^2 +$$

$$2(\sum_{k=1}^{n-1}\sum_{i,j=0}^{n-1} a_i b_j e_k)(\sum_{k=1}^{n-1}\sum_{i,j=0}^{n-1} a_i b_j e_k) + 2a_0 b_0 \sum_{k=1}^{n-1}\sum_{i,j=0}^{n-1} a_i b_j e_k -$$

$$2(\sum_{i=1}^{n-1} a_i b_i)(\sum_{k=1}^{n-1}\sum_{i,j=0}^{n-1} a_i b_j e_k) \quad (11)$$

Now, since $q_1 q_2 \ne 0$ we have $2K_0 = q_1 q_2 + \overline{q_1 q_2}$ where

$$\overline{q_1 q_2} = a_0 b_0 - \sum_{i=1}^{n-1} a_i b_i + \sum_{k=1}^{n-1}\sum_{i \ne j=0}^{n-1} a_i b_j e_k, \quad (12)$$

So that $K_0 = a_0 b_0 - \sum_{i=1}^{n-1} a_i b_i$ .Therefore

$$2K_0 q_1 q_2 = 2(a_0 b_0)^2 - 4a_0 b_0 \sum_{i=1}^{n-1} a_i b_i + 2a_0 b_0 \sum_{k=1}^{n-1}\sum_{i \ne j=0}^{n-1} a_i b_j e_k +$$

$$2(\sum_{i=1}^{n-1} a_i b_i)^2 - (2\sum_{i=1}^{n-1} a_i b_i)(\sum_{k=1}^{n-1}\sum_{i\neq j=0}^{n-1} a_i b_j e_k) \qquad (13)$$

$$N(q_1 q_2) = \left[a_0 b_0 - \sum_{i=1}^{n-1} a_i b_i + \sum_{k=1}^{n-1}\sum_{i,j=0}^{n-1} a_i b_j e_k \right]\left[a_0 b_0 - \sum_{i=1}^{n-1} a_i b_i - \sum_{k=1}^{n-1}\sum_{i,j=0}^{n-1} a_i b_j e_k \right]$$

which on multiplying out gives

$$N(q_1 q_2) = (a_0 b_0 - \sum_{i=1}^{n-1} a_i b_i)^2 - (\sum_{k=1}^{n-1}\sum_{i\neq j=0}^{n-1} a_i b_j e_k)^2 \qquad (14)$$

$$= (a_0 b_0)^2 + (\sum_{i=1}^{n-1} a_i b_i)^2 - 2a_0 b_0 \sum_{i=1}^{n-1} a_i b_i - (\sum_{k=1}^{n-1}\sum_{i\neq j=0}^{n-1} a_i b_j e_k)^2$$

On substituting these values into the rank equation (9), we obtain the required result.

Corollary 0.1 Let A be a loop algebra generated by the Bol loop BQs over the real field F, and let $\{1, e_1, e_2, e_3\}$ be a basis for A, where $e_0 = 1$, then

(i) every element q in A satisfies the rank equation (6), and

(ii) for all $q_1, q_2 \in A$, the rank equation (9) is satisfied.

Proof: (i) From (8), we have, for n=4, $N(q) = \sum_{i=0}^{3} a_i^2 + 2a_2 a_3 e_1 - 2a_1 a_2 e_3$ and since

$$q^2 = a^2{}_0 - \sum_{i=1}^{3} a_i{}^2 + 2a_0(a_1 e_1 + a_2 e_2 + a_3 e_3) + 2a_2 a_3 e_3 - 2a_1 a_2 e_1$$

$$2a_0 q = 2a_0(a_0 + \sum_{i=1}^{3} a_i e_i) = 2a_0^2 + 2a_0 \sum_{i=1}^{3} a_i e_i$$

Then substituting these into (6), yields the result.

(ii) By obtaining the expressions, as above, for $(q_1 q_2)^2$, $2k_0 q_1 q_2$, $N(q_1 q_2)$ and substituting into (9) we obtain the result.

Corollary 0.2 Let A be the loop algebra generated by a Bol quaternion loop of order 16 over the real field F, and let $\{\{e_i\}, 0 \le i \le 7, e_0 = 1\}$ be a basis for A, then

(i) every element q in A satisfies the rank equation (6), and

(ii) for all elements q_1, q_2 in A, the rank equation (9) is satisfied.

Proof: The proof follows as for corollary (0.1) by taking n=8 instead of n= 4

4.0 Conclusion

In this paper, it is established that Bol Algebras of quaternion type satisfy the rank equation $x^2 - t(x)x + n(x)1 = 0$ and hence are quadratic.

5.0 References

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