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Economic Order QuantityModel for DelayedDeteriorating Items with Time Dependent Exponential Declining Demand

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Abstract

In this study, an Economic Order Quantity(EOQ) Inventory model is constructed fordelayed (Non-instantaneous) deteriorating items with the assumption that the demand rate before deterioration begins is exponentially declining function of time while the demand rate when deterioration sets in becomes constant up to when the inventory is completely depleted. The deterioration rate is constant throughout the cycle, shortages are not allowed. The model is solved analytically by minimizing the total inventory cost which leads to the determination of the best cycle length. Numerical examples were provided to illustrate the application of the model developed.

Keywords: EOQ,Inventory, Delayed deterioration, Exponential declining demand

1.0 Introduction

Generally, deterioration refers to decay, damage, spoilage, expiration, evaporation, devaluation, invalidity, degradation, loss of utility or loss of marginal values and so on of a product through time. The process of deterioration occurs in some categories of items: some items like radioactive substances and highly volatile chemicals start deterioration process as soon as they are held in stock; these items are referred to as **instantaneous deteriorating items**. However, items like food grains, food stuffs, fruits, flowers, vegetables, medicines, films, blood in blood banks, fish, meat, milk etc. do not start deteriorating immediately they are stocked until later, these items are referred to as **delayed** or **non-instantaneous deteriorating items**. Deterioration process also occurred in items like fashion and seasonal goods, electronic equipments, computer chips, mobile phones and so on. These items loss part or their total values through time; which is mostlydue to introduction of new technology or the alternatives. As a result of demand and deterioration, inventory system faces depletion until its value becomes zero.

Harris [1]in 1915 was the first researcher to develop aninventory, Economic Order Quantity model, which was generalized by Wilson [2], who gave a formula to obtain economic order quantity. In 1963, Ghare and Schrader [3] developed classical EOQmodel for an exponentially decaying inventory with constant deterioration rate and without shortages. The work of Ghare and Schrader was extended by Covert and Philip [4]in 1973, which developed the modelwith a three parameter Weibull distribution rate and no shortages. Shah [5] extended the model developed by Covert and Philip by introducing shortages. An inventory model for stock-dependent consumption rate was first considered by Gupta and Vrat [6]. Many researchers such as Hollier and Mark [7] and Wee [8] studied the constant partial backlogging rates during the shortage period in their inventory models. Mishra and Singh [9] studied deteriorating inventory model with time-dependent demand and partial backlogging. This work was extended by Mishra *etal.* [10] considering time-varying holding cost under partial backlogging. Sharma and Rani [11] developed an inventory model for deteriorating items with Weibull deterioration with time dependent demand and assume shortages. Mandal [12] provided an inventory model for random deteriorating items with a linear trended demand and partial backlogging.Dash *etal.* [13]developed deteriorating inventory model with exponential declining demand and time-varying holding cost (taken as a linear function of time), deteriorating rate as constant and shortages not allowed.

Inventory model for Non-instantaneous (delayed) deteriorating items with stock-dependent demand was developed by Singh and Malik [14]. Later, Musa and Adam [15] developed an ordering policy of delayed deteriorating items with unconstrainedretailer’s capital, linear trended demand and shortages.

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It is worth noting that in developing an inventory model for delayed deteriorating items, two faces of inventory status must be considered: inventory status before deterioration sets in and the status of the inventory when the deterioration begins. In thisstudy, an effort has been made to develop an inventory model fordelayed (Non-instantaneous) deteriorating items with the assumption that the demand rate before the deterioration sets in is exponentially declining function of time while demand rate when deterioration begins becomes constant up to when the inventory is completely depleted. Deterioration rate is constant and shortage is not allowed. The model is an extension of Dash *etal.*modelwhich developed an inventory model for deteriorating items with exponential declining demand and time-varying holding cost. Numerical examples were provided to illustrate the application of the model developed.The model can be applied to optimize the total inventory cost in business environment where the demand rate before deterioration sets in is observed to be exponentially declining.

1.1 Assumptions and Notations

1.1.1 Assumptions

(i) The demand rate is deterministic and is an exponential declining function of time before deterioration sets in, i.e.in the interval,$0 \leq t \leq T_1$.

(ii) The demand rate is constant when deterioration sets in the interval, $T_1 \leq t \leq T$.

(iii) Deteriorating rate is considered as constant, $\theta(t) = \theta,\ 0 < \theta < 1$.

(iv) Shortages are not allowed.

(v) Replenishment is instantaneous.

(vi) Lead time is zero.

1.1.2 Notations

$\boldsymbol{D_1}$**(t)**The exponential demand rate (units per unit time) before deterioration set in,$\boldsymbol{D_1}$**(t)= K**$e^{-\alpha t}$,
K> 0,$\alpha > 0$,$K \gg \alpha$ are constants, K denotes initial demand and α denote the decreasing rate of the demand .

$\boldsymbol{D_2}$The demand rate (units per unit time) when the deterioration sets in

$\boldsymbol{T_1}$The time when the deterioration begins

$\boldsymbol{T_2}$The period of deterioration

$\boldsymbol{T}$The inventory cycle length that is time interval between two successive orders$\boldsymbol{\theta}$Constantdeterioration rate of an item (units/ unit time), ($0< \theta < 1$)and ($0< \alpha < \theta$)

Where α is the decreasing rate of demand.

$\boldsymbol{C}$The unit cost of the item (Naira per unit).

$\boldsymbol{i}$Inventory carrying charge per unit time (Naira per unit per unit time).

A The fixed ordering or replenishment cost (Naira per order)

$\boldsymbol{I_0}$The initial inventory.

$\boldsymbol{I(t)}$The inventory level at any time t before deterioration begins.

$\boldsymbol{I_d}$The inventory level at the time deterioration begins.

$\boldsymbol{I_d}$**(t)**The inventory level at any time t after deterioration sets in.

$\boldsymbol{T_d}$Total demand over the period of deterioration [$\boldsymbol{T_1}, \boldsymbol{T}$]

$\boldsymbol{TC(T)}$Total inventory cost per unit time (Naira per unit time).

2.0 Mathematical Formulation and Solution

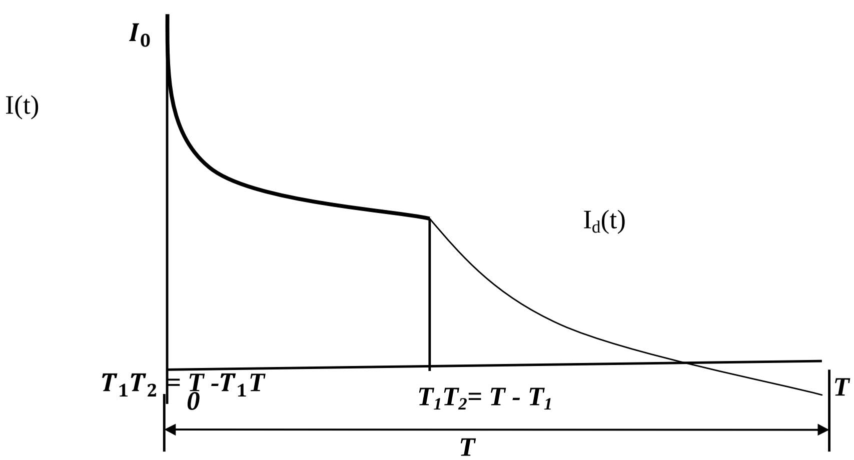

Figure 1: Inventory depletion in a delayed deterioration situation with noshortages.

Theinventory status before the deterioration sets in is governed by the differential equation:

$$\frac{dI(t)}{dt} = -D_1(\text{t}), \qquad 0 \le t \le T_1 \tag{1}$$

The differential equation representing the status of the inventory when the deterioration sets in is given by:

$$\frac{\mathrm{dI_d(t)}}{\mathrm{dt}} + \theta \mathrm{I_d(t)} = -D_2, \qquad T_1 \le t \le T \tag{2}$$

Equation (1) is first order variable separable, thus,

$\frac{dI(t)}{dt} = -Ke^{-\alpha t}$, where $D_1(\text{t}) = Ke^{-\alpha t}$, $\alpha$is a constant.

$$\Rightarrow I(t) = -K \int e^{-\alpha t} dt$$

$$\Rightarrow -K\left(-\frac{1}{\alpha}e^{-\alpha t}\right) + c$$

$$\therefore I(t) = \frac{K}{\alpha}e^{-\alpha t} + c \tag{3}$$

Substituting the boundary condition, at $t = 0, I(t) = I_0$ in (3), we get:$I_0 = \frac{K}{\alpha} + c$

$\Rightarrow c = I_0 - \frac{K}{\alpha}$so that, $I(t) = \frac{K}{\alpha}e^{-\alpha t} + I_0 - \frac{K}{\alpha}$

$$= I_0 + \frac{K}{\alpha}(e^{-\alpha t} - 1) \tag{4}$$

Putting the condition $\text{t} = T_1$, $I(t) = \mathrm{I_d}$ in equation (4) we get,

$$I_d = I_0 + \frac{K}{\alpha}(e^{-\alpha T_1} - 1)$$

$$\Rightarrow I_0 = I_d - \frac{K}{\alpha}(e^{-\alpha T_1} - 1) \tag{5}$$

Substituting (5) in (4) we get,

$$I(t) = I_d - \frac{K}{\alpha}(e^{-\alpha T_1} - 1) + \frac{K}{\alpha}(e^{-\alpha t} - 1)$$

$$= I_d + \frac{K}{\alpha}[(e^{-\alpha t} - 1) - (e^{-\alpha T_1} - 1)]$$

$$= I_d + \frac{K}{\alpha}(e^{-\alpha t} - 1 - e^{-\alpha T_1} + 1)$$

$$= I_d + \frac{K}{\alpha}(e^{-\alpha t} - e^{-\alpha T_1}) \tag{6}$$

Equation (2) is first order linear differential equation with integrating factor $\rho = e^{\int \theta dt} = e^{\theta t}$.The solution of (2) is thus given by:

$$\rho I_d(t) = -\int \rho D_2 \, dt$$

$$\Rightarrow e^{\theta t} I_d(t) = -D_2 \int e^{\theta t} dt$$

$$= -\frac{d_2}{\theta}e^{\theta t} + c_1$$

$$\Rightarrow I_d(t) = -\frac{D_2}{\theta}e^{\theta t}.e^{-\theta t} + c_1 e^{-\theta t}$$

$$= -\frac{D_2}{\theta} + c_1 e^{-\theta t} \tag{7}$$

Putting the condition, at $\text{t} = T_1$, $I_d(t) = I_d$ in equation (7), we get,

$$I_d = -\frac{D_2}{\theta} + c_1 e^{-\theta T_1}$$

$$\Rightarrow c_1 = \left(I_d + \frac{D_2}{\theta}\right) e^{\theta T_1} \tag{8}$$

Putting equation (8) in (7) we get,

$$I_d(t) = -\frac{D_2}{\theta} + (I_d + \frac{D_2}{\theta})e^{\theta T_1} \, . \, e^{-\theta t}$$

$$= -\frac{D_2}{\theta} + (I_d + \frac{D_2}{\theta})e^{(T_1 - t)\theta}$$

$$= -\frac{D_2}{\theta} + I_d e^{(T_1 - t)\theta} + \frac{D_2}{\theta}e^{(T_1 - t)\theta}$$

$$\therefore I(t) = I_d e^{(T_1 - t)\theta} + \frac{D_2}{\theta}(e^{(T_1 - t)\theta} - 1) \tag{9}$$

Putting the condition at $\text{t} = \text{T}$, $I_d(t) = 0$ in equation (9) we get,

$$0 = I_d e^{(T_1 - T)\theta} + \frac{D_2}{\theta}(e^{(T_1 - T)\theta} - 1)$$

$$\Rightarrow I_d e^{(T_1 - T)\theta} = -\frac{D_2}{\theta}(e^{(T_1 - T)\theta} - 1)$$

$$\Rightarrow I_d = -\frac{D_2}{\theta}(e^{(T_1 - T)\theta} - 1)\, e^{-(T_1 - T)\theta}$$

$= -\frac{D_2}{\theta}(1 - e^{-(T_1 - T)\theta})$

$= -\frac{D_2}{\theta}(1 - e^{(T - T_1)\theta})$ (10)

Substituting equation (10) in to equation (6) we get,

$I(t) = -\frac{D_2}{\theta}(1 - e^{(T-T_1)\theta}) + \frac{K}{\alpha}(e^{-\alpha t} - e^{-\alpha T_1})$ (11)

Again substituting (10) into equation (9) we get,

$I_d(t) = -\frac{D_2}{\theta}(1 - e^{(T-T_1)\theta})e^{(T_1 - t)\theta} + \frac{D_2}{\theta}(e^{(T_1 - t)\theta} - 1)$

$= \frac{D_2}{\theta}(-e^{(T_1 - t)\theta} + e^{(T-T_1)\theta + (T_1 - t)\theta} + e^{(T_1 - t)\theta} - 1)$

$= \frac{D_2}{\theta}\left(e^{(T-t)\theta} - 1\right)$ (12)

The total demand over the period of deterioration$[T_1, T]$is$T_d = D_2 \times (T - T_1) = D_2 T_2$.

The number of items in the inventory that deteriorated

$N_d = I_d - D_2 T_2$ (13)

Substituting for I_d from equation (10) into equation (13) we get,

$N_d = -\frac{D_2}{\theta}(1 - e^{(T-T_1)\theta}) - D_2 T_2$

$= -\frac{D_2}{\theta} + \frac{D_2}{\theta}e^{(T-T_1)\theta} - D_2 T_2$

$= -\frac{D_2}{\theta}[1 - e^{(T-T_1)\theta} + \theta T_2]$

$= -\frac{D_2}{\theta}[1 - e^{(T-T_1)\theta} + (T - T_1)\theta]$ (14)

Computation of variable cost

The total inventory (variable) cost is made up of the sum of the following inventory related costs:

(i) Ordering Cost, O_c = A

(ii) Inventory holding(carrying) cost, H_c

(iii) Cost of deteriorated items, CN_d

These costs are calculated separately and then sum up to get the total variable cost.

The inventory holding (carrying) cost, H_c

$$H_c = iC\int_0^{T_1} I(t)dt + ic\int_{T_1}^{T} I_d(t)dt,$$

$=iC\left\{\int_0^{T_1}\left[-\frac{D_2}{\theta}\left(1 - e^{(T-T_1)\theta}\right) + \frac{K}{\alpha}(e^{-\alpha t} - e^{-\alpha T_1})\right]dt + \int_{T_1}^{T}\frac{D_2}{\theta}(e^{(T-t)\theta} - 1)dt\right\}$

$=iC\left\{-\frac{D_2}{\theta}T_1 + \frac{D_2}{\theta}T_1 e^{(T-T_1)\theta} - \frac{D_2}{\theta^2} - \frac{D_2}{\theta}T + \frac{D_2}{\theta^2}e^{(T-T_1)\theta} + \frac{D_2}{\theta}T_1 - \frac{K}{\alpha^2}e^{-\alpha T_1} - \frac{K}{\alpha}e^{-\alpha T_1}T_1 + \frac{K}{\alpha^2}\right\}.$

$=iC\left\{\frac{D_2}{\theta}T_1 e^{(T-T_1)\theta} + \frac{D_2}{\theta^2}e^{(T-T_1)\theta} - \frac{D_2}{\theta^2} - \frac{D_2}{\theta}T - \frac{K}{\alpha^2}e^{-\alpha T_1} - \frac{K}{\alpha}e^{-\alpha T_1}T_1 + \frac{K}{\alpha^2}\right\}.$

$= \frac{iCT_1\alpha}{\theta}\left\{\frac{D_2}{\theta\alpha T_1}[(T_1\theta + 1)e^{(T-T_1)\theta} - (1 + \theta T)] + \frac{K\theta}{\alpha^3 T_1}[1 - (1 + \alpha T_1)e^{-\alpha T_1}]\right\}$ (15)

Cost of deteriorated items

Cost of deteriorated items, CN_d= C$\left[-\frac{D_2}{\theta}\left(1 - e^{(T-T_1)\theta}\right) + (T - T_1)\theta\right]$ (16)

where C is a unit cost of item.

The total variable cost per unit time is given by:

TC(T) = $\frac{1}{T}${inventory ordering cost, O_c + inventory holding, H_c+ cost of deteriorated items, CN_d}.

$= \frac{1}{T}\{O_c + H_c + CN_d\}$

$$= \frac{A}{T} + \frac{iCT_1\alpha}{\theta T}\left\{\frac{D_2}{\theta\alpha T_1}[(T_1\theta + 1)e^{(T-T_1)\theta} - (1 + \theta T)] + \frac{K\theta}{\alpha^3 T_1}[1 - (1 + \alpha T_1)e^{-\alpha T_1}]\right\}$$

$+ \frac{C}{T}\left[-\frac{D_2}{\theta}\left(1 - e^{(T-T_1)\theta}\right) + (T - T_1)\theta\right]$ (17)

Our aim is to optimize the variable cost per unit time. The necessary and sufficient conditions that minimize our TC(T) for a given value of T are respectively:$\frac{dTC(T)}{dT} = 0$ and $\frac{d^2TC(T)}{dT^2} > 0$.

We differentiate equation (17) with respect to T and equate to zero toget the necessary condition that minimize TC(T), provided that $\frac{d^2TC(T)}{dT^2} > 0$.

$$\frac{dTC(T)}{dT} = -\frac{A}{T^2} + \frac{iCT_1\alpha}{\theta T^2}\left\{\frac{D_2}{\theta\alpha T_1}\left[\left(\frac{T}{\theta} - 1\right)(T_1\theta + 1)e^{(T-T_1)\theta} + 1\right] - \frac{K\theta}{\alpha^3 T_1}[1 - (1 + \alpha T_1)e^{-\alpha T_1}]\right\}$$

$+ \frac{CD_2}{T^2\theta}\left[\left(\frac{T}{\theta} - 1\right)e^{(T-T_1)\theta} + \frac{T_1\theta^2}{D_2}(1 - T) - \frac{\theta T}{D_2} + 1\right] = 0$ (18)

Multiplying equation (18) by T^2 and equating to zero yields:

$$-A+\frac{iCT_1\alpha}{\theta}\left\{\frac{D_2}{\theta\alpha T_1}\left[\left(\frac{T}{\theta}-1\right)(T_1\theta+1)e^{(T-T_1)\theta}+1\right]-\frac{K\theta}{\alpha^3T_1}\left[1-(1+\alpha T_1)e^{-\alpha T_1}\right]\right\}$$

$$+\frac{CD_2}{\theta}\left[\left(\frac{T}{\theta}-1\right)e^{(T-T_1)\theta}+\frac{T_1\theta^2}{D_2}(1-T)-\frac{\theta T}{D_2}+1\right]=0 \qquad (19)$$

We use (19) to determine the best cycle length T which minimizes the total inventory cost. The Economic Order Quantity corresponding to the optimal cycle length T =T* can be computed as follows:

EOQ = Total demand in the period [0, T_1] before deterioration sets in + Total demand over the deterioration period [T_1 ,T] + Number of items that deteriorated.

$$= D_1(t)T_1 + D_2T_2 + N_d$$

$$= KT_1e^{-\alpha t} + D_2(T-T_1) - \frac{D_2}{\theta}\left[1-e^{(T-T_1)\theta}+(T-T_1)\theta\right].$$

$$= KT_1e^{-\alpha t} + \frac{D_2}{\theta}\left[e^{(T-T_1)\theta}-1\right] \qquad (20)$$

3.0 Numerical Examples

For the purpose of numerical examples,fivedifferent parameter values in proper units are considered (as input) and the output of the modelusing Maple (2015) Mathematical Software givesthecorresponding Optimal cycle length (T), the minimum total inventory cost(TC) and Economic Order Quantity (EOQ) as givenin Table 1:

Table1: EOQ, Total Variable Cost and Optimal Cycle length for Delayed deteriorating items with Exponential Declining Demand

S/N	A	K Units	i	α	θ	D_1 Units	D_2 Units	C	T_1 Year	T Year	T Days	TC Naira	EOQ Units
1	1000	500	0.013	0.1	0.4	500	200	9	0.0261	0.0948	35	11877	27
2	1050	550	0.023	0.3	0.5	546	273	14	0.0361	0.0743	27	16116	30
3	1100	600	0.033	0.4	0.6	592	355	20	0.0461	0.0574	21	20578	31
4	900	300	0.033	0.2	0.3	300	90	15	0.0252	0.0570	21	16551	10
5	850	150	0.043	0.3	0.5	148	74	20	0.0392	0.1530	56	6681	16

4.0 Conclusion

In this study, we developed an inventory model that is applicable to a business environment that considers the fact that deterioration of the storage item isdelayed (non-instantaneous) and in which the demand rate before deterioration begins is exponentially declining function of time and when deterioration sets in, it is constant up to the end of the cycle. The model do not allow for shortages. The model was solved analytically to determine the minimum total inventory cost, the optimal cycle length and the Economic Order Quantity (EOQ). Five different parameter values are taken to provide numerical examples on the application of the model. The results obtained from the numerical examples indicate the validity and stability of the model.

5.0 References

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