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Extended Block Adams-Moulton Method for Stiff Initial Value Problems

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Abstract

Addition of future points to classical linear multistep methods has been used to circumvent the popular second Dahlquist order barrier and develop new extended Linear Multistep Methods with superior linear stability over Linear Multistep Methods they are developed from. However, adding future points to Adams-Moulton methods failed to yield extended methods with superior stability regions over popular Adams-Moulton methods. This paper seeks to improve stability region of extended Adams-Moulton methods via block formulation. The new block methods developed in this paper are $A(\alpha)-$stable for order $p \le 7$..

Keywords: $A-$stable, $A(\alpha)-$stable, Block Methods, Stiff IVPs, Zero stable.

1.0 Introduction

The second Dahlquist order barrier places a severe restriction on Linear Multistep methods (LMMs) in that no explicit LMM can be A-stable and Implicit A-stable methods cannot exceed order $p = 2$. A-stability is a requirement for methods suitable for integrating stiff initial value problems (IVPs) in ordinary differential equations (odes) [1-4]. Interestingly, the best classical order $p = 2$ LMMs for integrating stiff IVPs is the Trapezoidal rule method which is a member of the family of Adams-Moulton method [4].

The development of high order LMMs which circumvents the Dahlquist order barrier has been achieved through two broad search directions, these are: (a) by incorporating higher derivative of the exact solution to the classical LMMs or (b) by incorporating supplementary stages, extra division points or future points [5]. These two directions can be applied simultaneously in developing new methods; examples of such are the second derivative block methods developed in Muka and Ikhile [6, 7, 8], andMusa et al [9]. Examples of methods which incorporate higher derivative of exact solution to classical LMMs include second derivative multistep methods by Enright [10] and Second derivative Backward Differentiation formulas[3]. Methods that fall into the second search direction include classes of methods called (i) block methods [11,12, 13] (ii) hybrid methods [5] and (iii) extended methods ([5], pg.45). Block methods are methods which are used to generate r approximation solutions of IVPs at every computational cycle [11]. That is, at every computational cycle an r point block method, yields approximate solutions at r nodes simultaneously. Numerous block methods have been proposed: Shampine and Watts [12] with A-stable implicit one-block method, Chu and Hamilton [13] with multi-block methods, Chartier [14] and Sommeijer et al [15] with parallel one-block methods. More recently, Muka [16] developed Second derivative parallel block methods. Hybrid methods generate approximate solutions at the node points and at sub-intervals examples are hybrid methods developed in Patricio [17] and Carroll [18]. The extended methods are methods in which future points are added to classical LMMs; first developed by Cash [19] when in an effort to improve the stability region of the Backward Differentiation formulae (BDF) added a future point $x_{n+k+1.}$

The BDF has the structure

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_k f_{n+k} \qquad (1)$$

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And is stable for order $p \le 6.$ Cash [19] improved the stability of (1) by adding a future point $x_{n+k+1.}$ to develop the extended multistep methods with the following structure:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_k f_{n+k} + h\beta_{k+1} f_{n+k+1} \tag{2}$$

The extended multistep methodsare stable for order $p \le 9,$ and A-stable for order $p \le 4,$ [3]. BDF are widely used for integrating stiff initial value problems though they are not A-stable for order $p > 2,$ they are however $A(\alpha)-$stable for $p \le 6,$ [4]. $A(\alpha)-$stability is an alternative requirement for LMMs for integrating stiff IVPs in odes.

A sub-class of LMM is the Adams type whose formulae is given as

$$y_{n+k} = y_{n+k-1} + h\sum_{j=0}^{k} \beta_j f_{n+j} \tag{3}$$

they are characterized by been zero-stable ([4], pg. 46). As remarked earlier, the only member of the Adams type family for integrating stiff IVPs is the Trapezoidal rule (Adams-Moulton method for k=1). Attempts to improve the stability region of (3) by addition of future points as in (4),

$$y_{n+k} = y_{n+k-1} + h\sum_{j=0}^{k+1} \beta_j f_{n+j}. \tag{4}$$

yielded extended multistep methods with inferior stability regions compared with Adams-Moulton method. Sommeijer et al [15]developed block methods which generalizes Adams-Moulton method, of which their block method is A-stable for order $p = 4.$ In this paper, we present block formulation of the extended Adams-Moulton methods (4). The advantage of block formulation of extended Adams-Moulton method is not only in the development of methods with superior stability regions but on the possible implementation on parallel computers like other block methods. This paper is arranged as follows: section 2 is on derivation of method, stability analysis of methods derived is in section 3, while in section 4 is on numerical test and conclusion is in section 5.

2.0 Derivation of Method

Block methods are direct generalization of LMMs [15]. To generalize (4), we set k=1 to obtain

$$y_{n+1} = y_n + h(\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2}) \tag{5}$$

the one block r-point generalization of (5) is now

$$A_1 Y_m = A_0 Y_{m-1} + h(B_0 F_{m-1} + B_1 F_m + B_2 F_{m+1}) \tag{6}$$

where h is the step-size, r the block-size; and $n = mr, \quad m = 0,1,\cdots$

Y_m and F_m are r-dimensional vectors given as $Y_m = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ \vdots \\ y_{n+r} \end{pmatrix}, F_m = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ \vdots \\ f_{n+r} \end{pmatrix}$ respectively; $A_i, i = 0,1$ and $B_i, i = 0,1,2$ are $r \times r$ matrices given as

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & 0 & & 1 \end{pmatrix}, A_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, B_0 = \begin{pmatrix} b_{11}^0 & b_{12}^0 & \cdots & b_{1r-1}^0 & b_{1r}^0 \\ b_{21}^0 & b_{22}^0 & \cdots & b_{2r-1}^0 & b_{2r}^0 \\ b_{31}^0 & b_{32}^0 & \cdots & b_{3r-1}^0 & b_{3r}^0 \\ \vdots & \vdots & & \vdots & \vdots \\ b_{r1}^0 & b_{r1}^0 & \cdots & b_{r-1r-1}^0 & b_{rr}^0 \end{pmatrix}, B_1 = \begin{pmatrix} b_{11}^1 & b_{12}^1 & 0 & \cdots & 0 \\ 0 & b_{22}^1 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & b_{r-2r-1}^1 & \vdots \\ \vdots & \vdots & & b_{r-1r-1}^1 & b_{r-1r}^1 \\ 0 & 0 & \cdots & 0 & b_{rr}^1 \end{pmatrix} \text{ and}$$

$$B_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & 0 \\ b_{r1}^2 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

$b^{i}_{v\tau},\quad i=0,1,2;\ v=1,2,\cdots,r;\tau=v=1,2,\cdots,r$ are non-zero elements of matrices $B_i, i=0,1,2$ to be determined. Note that for r=1 in (6) is the one step extended Adams-Moulton method in (5).

To determine the element $b^{i}_{v\tau},\quad i=0,1,2;\ v=1,2,\cdots,r;\tau=v=1,2,\cdots,r$ in (6), we use the Taylor's series expansion and method of undetermined coefficient. First, we define the Linear difference operator associated with (6) as

$$L(Y(x);h)=\begin{pmatrix} L_1(y(x);h) \\ L_2(y(x);h) \\ \vdots \\ L_r(y(x);h)\end{pmatrix}=Y_m-A_0Y_{m-1}-h(B_0F_{m-1}+B_1F_m+B_2F_{m+1}) \tag{7}$$

whose components are given by

$$L_1(y(x);h)=y(x_n+h)-y(x_n)-h(b^0_{11}y'(x_n+(1-r)h)+b^0_{12}y'(x_n+(2-r)h)+\cdots+b^0_{1r}y'(x_n)+b^1_{11}y'(x_n+h)+b^1_{12}y'(x_n+2h))$$

$$L_2(y(x);h)=y(x_n+2h)-y(x_n)-h(b^0_{21}y'(x_n+(1-r)h)+b^0_{22}y'(x_n+(2-r)h)+\cdots+b^0_{2r}y'(x_n)+b^1_{22}y'(x_n+2h)+b^1_{23}y'(x_n+3h))$$

$$\vdots$$

$$L_r(y(x);h)=y(x_n+rh)-y(x_n)-h(b^0_{r1}y'(x_n+(1-r)h)+b^0_{r2}y'(x_n+(2-r)h)+\cdots+b^0_{rr}y'(x_n)+b^1_{rr}y'(x_n+rh)+b^2_{r1}y'(x_n+(r+1)h))$$

Here we assume that $y(x_n+\sigma h)$ and $y'(x_n+\sigma h),\quad \sigma=0,1,2,\cdots,r$ are differentiable as often as we need. Taylor expanding $L_\sigma(y(x);h), \sigma=1,2,\cdots,r$ about x_n and solving the arising nonlinear equation we obtain block methods of order $p=r+2,\ r=2,3,4,5,6.$

For r=2;

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{pmatrix}y_{n+1}\\y_{n+2}\end{pmatrix}=\begin{bmatrix}0&1\\0&1\end{bmatrix}\begin{pmatrix}y_{n-1}\\y_n\end{pmatrix}+h\begin{bmatrix}-\frac{1}{24}&\frac{13}{24}\\-\frac{2}{9}&\frac{11}{9}\end{bmatrix}\begin{pmatrix}f_{n-1}\\f_n\end{pmatrix}+h\begin{bmatrix}\frac{13}{24}&-\frac{1}{24}\\0&\frac{11}{9}\end{bmatrix}\begin{pmatrix}f_{n+1}\\f_{n+2}\end{pmatrix}+\begin{bmatrix}0&0\\-\frac{2}{9}&0\end{bmatrix}\begin{pmatrix}f_{n+3}\\f_{n+4}\end{pmatrix}$$

For r=3;

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{pmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{pmatrix}$$
$$=\begin{bmatrix}0&0&1\\0&0&1\\0&0&1\end{bmatrix}\begin{pmatrix}y_{n-2}\\y_{n-1}\\y_n\end{pmatrix}+h\begin{bmatrix}\frac{11}{720}&-\frac{37}{360}&\frac{19}{30}\\\frac{19}{150}&-\frac{29}{45}&\frac{74}{45}\\\frac{87}{200}&-\frac{747}{400}&\frac{63}{20}\end{bmatrix}\begin{pmatrix}f_{n-2}\\f_{n-1}\\f_n\end{pmatrix}+h\begin{bmatrix}\frac{173}{360}&-\frac{19}{720}&0\\0&\frac{91}{90}&-\frac{31}{225}\\0&0&\frac{651}{400}\end{bmatrix}\begin{pmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\end{pmatrix}$$
$$+h\begin{bmatrix}0&0&0\\0&0&0\\-\frac{69}{200}&0&0\end{bmatrix}\begin{pmatrix}f_{n+4}\\f_{n+5}\\f_{n+6}\end{pmatrix}$$

For r=4;

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{pmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\end{pmatrix}=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix}\begin{pmatrix}y_{n-3}\\y_{n-2}\\y_{n-1}\\y_n\end{pmatrix}+h\begin{bmatrix}-\frac{11}{1440}&\frac{77}{1440}&-\frac{43}{240}&\frac{511}{720}\\-\frac{19}{225}&\frac{38}{75}&-\frac{23}{18}&\frac{31}{15}\\-\frac{29}{80}&\frac{783}{400}&-\frac{3321}{800}&\frac{707}{160}\\-\frac{328}{315}&\frac{328}{63}&-\frac{736}{75}&\frac{1838}{225}\end{bmatrix}\begin{pmatrix}f_{n-3}\\f_{n-2}\\f_{n-1}\\f_n\end{pmatrix}$$

$$+h\begin{bmatrix}\frac{637}{1440} & -\frac{3}{160} & 0 & 0\\ 0 & \frac{199}{225} & -\frac{43}{450} & 0\\ 0 & 0 & \frac{1099}{800} & -\frac{189}{800}\\ 0 & 0 & 0 & \frac{3026}{1575}\end{bmatrix}\begin{pmatrix}f_{n+1}\\ f_{n+2}\\ f_{n+3}\\ f_{n+4}\end{pmatrix}+h\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -\frac{232}{525} & 0 & 0 & 0\end{bmatrix}\begin{pmatrix}f_{n+5}\\ f_{n+6}\\ f_{n+7}\\ f_{n+8}\end{pmatrix}$$

For r=5;

$$\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{pmatrix}y_{n+1}\\ y_{n+2}\\ y_{n+3}\\ y_{n+4}\\ y_{n+5}\end{pmatrix}=\begin{bmatrix}0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{pmatrix}y_{n-4}\\ y_{n-3}\\ y_{n-2}\\ y_{n-1}\\ y_{n}\end{pmatrix}$$

$$+h\begin{bmatrix}\frac{271}{60480} & -\frac{29}{840} & \frac{811}{6720} & -\frac{254}{945} & \frac{5221}{6720}\\ \frac{1621}{26460} & -\frac{404}{945} & \frac{179}{140} & -\frac{2018}{945} & \frac{9433}{3780}\\ \frac{19683}{62720} & -\frac{3991}{1960} & \frac{6129}{1120} & -\frac{4293}{560} & \frac{26357}{4480}\\ \frac{2972}{2835} & -\frac{1576}{245} & \frac{11752}{735} & -\frac{18784}{945} & \frac{418}{35}\\ \frac{99475}{36288} & -\frac{9125}{567} & \frac{236375}{6272} & -\frac{227375}{5292} & \frac{266215}{12096}\end{bmatrix}\begin{pmatrix}f_{n-4}\\ f_{n-3}\\ f_{n-2}\\ f_{n-1}\\ f_{n}\end{pmatrix}$$

$$+h\begin{bmatrix}\frac{349}{840} & -\frac{863}{60480} & 0 & 0 & 0\\ 0 & \frac{3019}{3780} & -\frac{94}{1323} & 0 & 0\\ 0 & 0 & \frac{4729}{3920} & -\frac{10881}{62720} & 0\\ 0 & 0 & 0 & \frac{1214}{735} & -\frac{6392}{19845}\\ 0 & 0 & 0 & 0 & \frac{33965}{15876}\end{bmatrix}\begin{pmatrix}f_{n+1}\\ f_{n+2}\\ f_{n+3}\\ f_{n+4}\\ f_{n+5}\end{pmatrix}+h\begin{bmatrix}0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ -\frac{262775}{508032} & 0 & 0 & 0 & 0\end{bmatrix}\begin{pmatrix}f_{n+6}\\ f_{n+7}\\ f_{n+8}\\ f_{n+9}\\ f_{n+10}\end{pmatrix}$$

For r=6

$$\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{pmatrix}y_{n+1}\\ y_{n+2}\\ y_{n+3}\\ y_{n+4}\\ y_{n+5}\\ y_{n+6}\end{pmatrix}=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{pmatrix}y_{n-5}\\ y_{n-4}\\ y_{n-3}\\ y_{n-2}\\ y_{n-1}\\ y_{n}\end{pmatrix}$$

$$+h\begin{bmatrix} -\frac{13}{4480} & \frac{2999}{120960} & -\frac{1283}{13440} & \frac{2987}{13440} & -\frac{44797}{120960} & \frac{11261}{13440} \\ -\frac{23}{490} & \frac{9901}{26460} & -\frac{176}{135} & \frac{363}{140} & -\frac{3053}{945} & \frac{11089}{3780} \\ -\frac{4989}{17920} & \frac{263871}{125440} & -\frac{106763}{15680} & \frac{5445}{448} & -\frac{16227}{1280} & \frac{67681}{8960} \\ -\frac{200}{189} & \frac{21722}{2835} & -\frac{17228}{735} & \frac{85256}{2205} & -\frac{33784}{945} & \frac{1754}{105} \\ -\frac{8345}{2688} & \frac{225125}{10368} & -\frac{9220625}{145152} & \frac{3713125}{37632} & -\frac{14160625}{169344} & \frac{807815}{24192} \\ -\frac{537}{70} & \frac{7299}{140} & -\frac{15446}{105} & \frac{8717}{40} & -\frac{169461}{980} & \frac{60667}{980} \end{bmatrix}\begin{pmatrix} f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}$$

$$+h\begin{bmatrix} \frac{5311}{13440} & -\frac{275}{24192} & 0 & 0 & 0 & 0 \\ 0 & \frac{19477}{26460} & -\frac{733}{13230} & 0 & 0 & 0 \\ 0 & 0 & \frac{136361}{125440} & -\frac{16773}{125440} & 0 & 0 \\ 0 & 0 & 0 & \frac{9676}{6615} & -\frac{4892}{19845} & 0 \\ 0 & 0 & 0 & 0 & \frac{1898375}{1016064} & -\frac{400375}{1016064} \\ 0 & 0 & 0 & 0 & 0 & \frac{13561}{5880} \end{bmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \end{pmatrix}$$

$$+h\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{113}{196} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+6} \\ f_{n+7} \end{pmatrix}$$

3.0 Stability Analysis of Proposed Block Method

Definition 1 [13]

Block method of the form (6) is said to be zero stable if the roots $\zeta_i, i = 1,2,\cdots,r$, of the first characteristics polynomial satisfies the condition $|\zeta_i| \le 1$ with one of its root $|\zeta_i| = 1$.

The first characteristics polynomial of (6) is

$$\rho(\zeta) = Det(A_1\zeta - A_0), \tag{8}$$

for matrices $A_i, i = 0,1$ specified above; roots of the first characteristics polynomial (8) are all situated at the origin except one on boundary of the unit circle. This clearly shows that (6) is zero-stable and exhibits the characteristic of Adams type method [2,4].

Applying (6) to the test equation $y' = \lambda y, \quad y(x_0) = y_0$, yields the characteristic polynomial

$$\Pi(\zeta, z) = \det(A_1\zeta - A_0 - z(B_0 - B_1\zeta - B_2\zeta^2)) \tag{9}$$

Definition 2 [4]

The block method (6) is said to be absolutely stable for given z, if for that z all the roots of the characteristic polynomial (9) satisfies $|\zeta_t| < 1, t = 1,2,\ldots,r,$ and to be absolutely unstable for that z otherwise.

Definition 3 [4]

The block method (6) is said to have region of absolute stability R_A, where R_A is a region of the complex z-plane, if it is absolutely stable for all $z \in R_A$.

The boundary locus technique is the most convenient method for finding regions of absolute stability R_A, [4]. The absolute stability region R_A associated with the method (6) for r =2, 3, 4 ,5, 6 are shown in Figures 1 –5.

Definition 4[4]

The block method (6) is said to be $A(\alpha)-$stable, $\alpha \in (0, \frac{\pi}{2})$ if $R_A \supseteq \{z \mid -\alpha < \pi - \arg z < \alpha\}$.

Definition 5[4]

The block method (6) is said to be $A-$stable, if $R_A \supseteq C^-$.

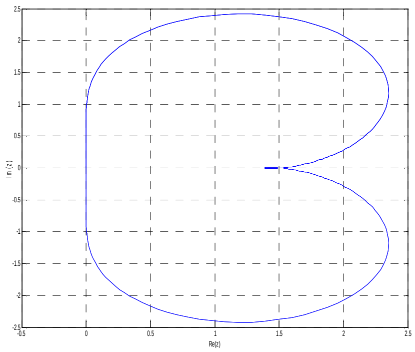

Fig. 1: Stability Plot of Method for r=2.

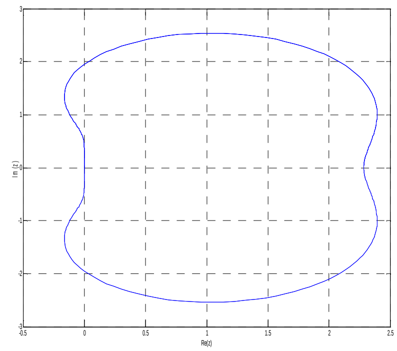

Fig. 2: Stability Plot of Method for r=3.

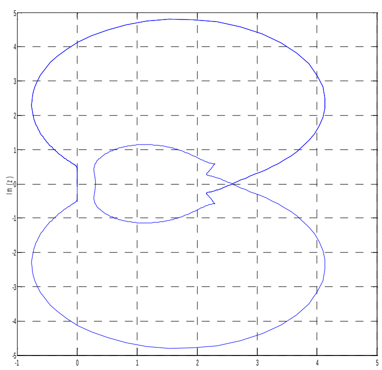

Fig. 3: Stability Plot of Method for r=4.

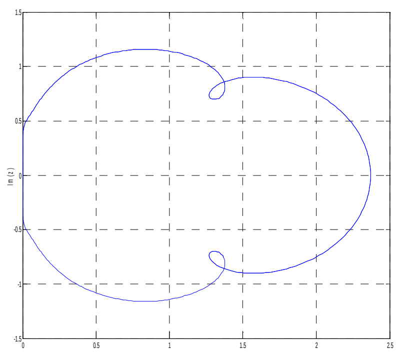

Fig. 4: Stability Plot of Method for r=5.

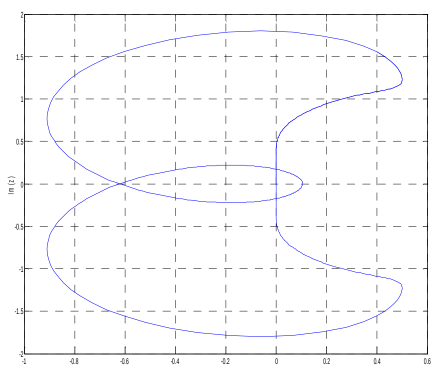

Fig. 5: Stability Plot of Method for r=6.

The enclosed regions in Figures 1-5 are the region of instability, outside the enclosure is the region of absolute stability R_A.

In Figures 1 and 5, observe that the regions of absolute stability R_A contain the entire left of the complex plane. Hence, proposed block method (6) is A-stable for r=2 and r=5 while for r=3, 4 in figures (2) and (3), the instability region invade the left of the complex plane hence both methods are $A(\alpha)-$stable. Table 1 shows the various α values for proposed methods.

Table 1: Stability measures for proposed method

r	2	3	4	5
p	4	5	6	7
α	90^0	72^0	55^0	90^0

For r=6, proposed block method is unstable.

4.0 Numerical Experiments

Proposed block method forr=2 is implemented in this section using modified Newton iteration

$$y_{n+j}^{(i+1)} = y_{n+j}^{(i)} - [F'(y_{n+j}^{(i)})]^{-1}[F'(y_{n+j}^{(i)})], \quad j = 1,2$$

set

$$F_1 = y_{n+1} - y_n - h(-\tfrac{1}{24} f_{n-1} + \tfrac{13}{24} f_n) - h(\tfrac{13}{24} f_{n+1} - \tfrac{1}{24} f_{n+2}) = 0$$

$$F_2 = y_{n+2} - y_n - h(-\tfrac{2}{9} f_{n-1} + \tfrac{11}{9} f_n) - h(\tfrac{11}{9} f_{n+2} - \tfrac{2}{9} f_{n+3}) = 0$$

$$F'(y_{n+j}^{(i)}) = \begin{pmatrix} 1 - \tfrac{13}{24} h \dfrac{\partial f_{n+1}}{\partial y_{n+1}} & \tfrac{1}{24} h \dfrac{\partial f_{n+2}}{\partial y_{n+2}} \\ 0 & 1 - \tfrac{11}{9} h \dfrac{\partial f_{n+2}}{\partial y_{n+2}} \end{pmatrix}$$

and h=0.2.

Problem[20]

$$y' = f(x,y) = -10y; \qquad y(2) = e^{-20}; \quad 2 \le x \le 3$$

$$\textit{Exact Solution} \quad y(x) = e^{-10x}$$

Maximum Error $= 0.00027692$

If the stiff problem is solved using explicit Euler method, the restriction on the step-size h is $|\lambda h| < 2.$ Therefore the problem cannot be solved with explicit Euler for h=0.2. The problem solved with proposed block method for r=2 with h=0.2 shows that the method is suitable for stiff problems because of its $A(\alpha)-$stability properties.

5.0 Conclusion

Adding future point to Adams-Moulton methods produces extended methods that are not useful because of their poor stability properties. Generalizing these extended methods through Block formulation yielded useful methods for integrating stiff IVPs. Lambert ([4], pg. 74) stated that it is not always true that as order increases, the regions of stability shrink. Lambert's conjecture holds for proposed methods as shown in Table1. Methods developed in this paper circumvents the second Dahlquist order barrier owing to A-stable methods of order $p = 4$ *and* $7.$ Numerical experiment in section 4, shows the application of proposed method on stiff IVPs in ODEs.

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