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On the Parameterised Representation of Quasi Linear Processes

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Abstract

The study concerns the analysis of quasi linear differential equation for the evolution of kinematic phenomena. The traditional method of solution involving characteristic curves is generalized. This is, thus, applicable to any form of initial datae and thus, appears to suggest the related significant advantage of this approach.

1.0 Introduction

Mathematical problem associated with kinetic phenomena in nonlinear physics had been widely studied for years. The investigations as related to kinematic waves may be regarded as pioneering and quite touching [1,2]. Follow-up on additional investigationshas been done [3,4,5].

Interesting applications were reported in [2]. These concerned those phenomena that are slowly moving. Such evolutional processes model motor traffic and flood flows. Their investigation proved to be no less significant in the understanding of these evolutional processes .

This study tends more towards application. Method of parametersation of initial datae isthe intended goal. The datae are taken to be those associated with integral surfaces on which the problems are based for the process.

2.0 Specifications for the Quasi-Linear Processes and Formulations

Take x- as the coordinate axis normal to the direction of the process, y- the horizontal coordinate, z=u, perpendicular to the x-y plane, t represents time.

Consider a curve in (x,y,u) space with parametric equation provided by x=x(t), y=y(t) and u=u(t) . The component of the vector tangential to the surface is $(\frac{dx}{dt}, \frac{dy}{dt}, \frac{du}{dt})$. Take (a,b,c) as the tangential components of the vector field called, characteristic directions, then, in this consideration,

$\frac{dx}{dt}$ = a(x,y,u) , $\frac{dy}{dt}$ =b(x,y,u) and $\frac{du}{dt}$ =c(x,y,u). These are the system of the equation of the characteristic curve and depends on the system; from which are obtained:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} \quad \ldots \qquad (1)$$

Further, the equation of a smooth surface is given by f(x,y,u)=u(x,y)-u=C_0 (C_0 being a constant) The spacial gradient of the this surface is

$$\Delta f = (f_x, f_y, f_u) = (u_x, u_y, -1) \quad \ldots \qquad (2)$$

(2) is a vector perpendicular to the surface $f(x, y, u) = C_0$. The tangent to this surface is

$$\underline{dr} = (a, b, c). \text{------} \qquad (3)$$

Thus

$$\underline{dr}.\nabla f = au_x + bu_y - c = 0$$

i.e

$$au_x + bu_y = c \quad \ldots \qquad (4)$$

(4) is the quasi-linear differential equation and its solution is determined through (1)

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3.0 Generalization of Solution

We consider an arbitrary function F(M,N) where M(x,y,u) and N(x,y,u) are two independent functions of the space (x,y,u).M(x,y,u) = c_1, N(x,y,u) = c_2, c_1, and c_2, are constants.
We now differentiate F(M,N) =0

$$\frac{\partial F}{\partial M}\left(\frac{\partial M}{\partial x}+P\frac{\partial M}{\partial u}\right)+\frac{\partial F}{\partial N}\left(\frac{\partial N}{\partial x}+P\frac{\partial N}{\partial u}\right)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$\frac{\partial F}{\partial M}\left(\frac{\partial M}{\partial y}+q\frac{\partial M}{\partial u}\right)+\frac{\partial F}{\partial N}\left(\frac{\partial N}{\partial y}+q\frac{\partial N}{\partial u}\right)=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

(5) and (6) constitute two linear equations for $\frac{\partial F}{\partial M}$ and $\frac{\partial F}{\partial N}$ and the solution exist if the determinant

$$\begin{vmatrix} M_x+PM_uN_x+PN_u \\ \\ M_y+qM_uN_y+qN_u \end{vmatrix}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Where $M_x=\frac{\partial M}{\partial x} andsoon, P=\frac{\partial u}{\partial x}, \quad q=\frac{\partial u}{\partial y}$

From (6), we obtain

$$(M_x+PM_u)(N_y+qN_u)-\ (M_y+qM_u)(N_x+PN_u)=0$$

$$i.e\big((M_xN_y-M_yN_x)+P(M_uN_y-N_uM_y)+q(N_uM_x-M_uN_x)=0$$

Define theJacobiandeterminantas$\frac{\partial(M,N)}{\partial(x,y)}=M_xN_y-M_yN_x$, hence

$$P\frac{\partial(M,N)}{\partial(y,u)}+q\frac{\partial(M,N)}{\partial(u,x)}+\frac{\partial(M,N)}{\partial(x,y)}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

From the relationship $f(M,N)=0$ involved in the general solution of (3), $M(x,y,u)=c_1$
$N(x,y,u)=c_2$
$(gradM).\underline{dr}=0,$ provides

$$M_xdx+M_ydy+M_ud_u=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

$(gradN).dr=0$provides

$$N_xdx+N_ydy+N_ud_u=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

$$\underline{dr}=\hat{\imath}dx+\hat{\jmath}dy+\hat{k}du$$

By the method of algebraic elimination, involving (8) and (9)

$$\frac{dx}{\frac{\partial(M,N)}{\partial(y,u)}}=\frac{dy}{\frac{\partial(M,N)}{\partial(u,x)}}=\frac{du}{\frac{\partial(M,N)}{\partial(x,y)}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

From (10) we elliminatethe Jecobian in (7) to obtain

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

Thus $f(M,N)=\ 0$ is the solution of the quasi linear differential eqution.

4.0 Illustration with Group Velocity $\mathcal{G}_F$(x, t)

It has been proved that the evolution of group velocity $\mathcal{G}_F\ (x,t)$ is governed by quasi-linear partial differential equation [6]. This idea has been appliedin various forms [7]. Consequently, method of parameterization will be applied in this presentation to provide additional insight into this area of interest.

Unlike Cauchy initial value problem, this method needs the specified curve to be parametarised. Thus the method provides detailed information beyond expectation from the present consideration.

Consider the quasi-linear equation.

$$\frac{\partial \mathcal{G}_F}{\partial t}+\mathcal{G}_F\frac{\partial}{\partial x}\mathcal{G}_F=\ A_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

$\mathcal{G}_F=\ \mathcal{G}_F(x,t), A_0$is the constant force powering the system. specifiedthe initial as
$\mathcal{G}_F(x,0)=x.$

As defined, the equation for characteristic direction are

$$\frac{dt}{1}=\frac{dx}{\mathcal{G}_F}=-\frac{d\mathcal{G}_F}{A_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

consequently, $2xA_0=\ \mathcal{G}_F^2,\ \ \mathcal{G}_F=\ -A_0t$

Thus,

$$t^2 = \frac{2x}{A_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

This is a form of parabola with focus a $\left(\frac{1}{2A_0}, 0\right)$ and directrix is provided by the line $x = -\frac{1}{2A_0}$. The solution of (13) is determined from the following:

$\frac{\partial \mathcal{G}_F^2}{2A_0 x} = 1$, $\frac{\mathcal{G}_F}{A_0 t} = \frac{1}{t}\sqrt{\frac{2x}{A_0}}$,

Thus,

$$F\left(\frac{\mathcal{G}_F}{A_0 t}, \frac{1}{t}\sqrt{\frac{2x}{A_0}}\right) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

It follows that

$$\mathcal{G}_F = A_0 t g\left(\frac{1}{t}\sqrt{\frac{2x}{A_0}}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

(15) is a general solution derived by using method of characteristics. However, it does not admit a realistic initial data. Resorting to the method of parameterization, (12) will be revisited.

5.0 On the Parameterization Approach

Thismethod is rather general because, the solution is usually prescribed along an arbitrary curve in (x,t) surface.
Consider two parameters s and τassume that the solution of (4) passes through the curve :

$$x = x_0(s), t = t_0(s) and \mathcal{G}_F(x,t) = \mathcal{G}_0(s) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

The equations of characteristics for (4) are thus;

$$\frac{dx}{d\tau} = \mathcal{G}_F, \quad \frac{dt}{d\tau} = 1, \frac{d\mathcal{G}_F}{d\tau} = A_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

From which

$$\mathcal{G}_F(s,\tau) = A_0\tau + \mathcal{G}_0, t(s,\tau) = \tau + t_0(s) and$$

$$x(s,\tau) = \frac{A_0\tau^2}{2} + \mathcal{G}_0\tau + x_0(s) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

Consequently, the following are realistic; representations

$$\mathcal{G}_0(s) = 0, x_0(s) = 2s^2, t_0(s) = 2s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19)$$

Equation (18) and (19) give

$$\mathcal{G}_F(s,\tau) = A_0\tau, \quad t(s,\tau) = 2s + \tau and x(s,\tau) = \frac{A_0\tau}{2} + 2s^2 \ldots\ldots\ldots (20a,b,c)$$

The fundamental curve is given by

$$s > 0, \tau = 0, \mathcal{G}_F(x,t) = 0, t^2 = 2x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21)$$

Equation (21) is identical to (13), if $A_0 = 1$. That is a parabolic curve with directrix defined by the line $x = -{}^1\!/_2$and focal point at $({}^1\!/_2, 0)$.

Using equation (20), we determine$\mathcal{G}_F(x,t)$, by eliminating $\tau and s$from the equations. Consequently,

$x = \mathcal{G}_F^2\left(\frac{1}{A_0} + \frac{1}{2A_0^2}\right) - \frac{\mathcal{G}_F}{A_0}t + {}^{t^2}\!/_2$, that is

$2A_0^2 x = \mathcal{G}_F^2 R - 2tA_0\mathcal{G}_F + t^2 A_0, \quad R = 1 + 2A_0$

$\mathcal{G}_F$, satisfies the quadratic equation

$$\mathrm{R}\mathcal{G}_F^2 - 2A_0 t\mathcal{G}_F + A_0^2(t^2 - 2x) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

If $\mathcal{G}_{F_1} and \mathcal{G}_{F_2}$ are roots of (22), then,

$$\mathcal{G}_{F_1} = \frac{1}{R}[tA_0 + B], \quad \mathcal{G}_{F_2} = \frac{1}{R}[tA_0 - B] \ldots\ldots\ldots\ldots\ldots (23a,b)$$

$$B^2 = 2xR - t^2(R - A_0^2) = R(2x - t^2) + t^2A_0^2 \ldots\ldots\ldots\ldots\ldots (23c)$$

The solution which satisfies the condition (21), is provided by

$\mathcal{G}_F = 0 \; if t^2 = 2x$which is (23c)

$\mathcal{G}_F(x,t)$exists if only $R(2x - t^2) + t^2A_0^2 > 0$. This implies that $2x \geq t_0^2$.

If $A_0 = 1, R = 3$,

$$\mathcal{G}_{F_2} = \frac{1}{3}\left[t - (3x - t^2)^{1/2}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

Equation (23b) and (24) are complete solutions and neither depends on the parameter introduced in the derivation nor does (24). They contain an arbitrary constant which needs to be determined. This is an interesting development in this analysis.

6.0 Conclusion

The parameterization approach is applied to the solution of quasi linear differential equation governing the evolutions of kinematic processes in nonlinear physics. This method has significant advantage in relation to other approaches because, it can explain events with arbitrary initial datae.
The parabolic profile so provided by this method gives an interesting but unexpected result.

7.0 References

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