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Fekete-Szego Inequality for subclasses of Analytic Functions

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Abstract

In this paper, we obtained sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the subclasses of the classes of starlike and convex functions in the unit disc.

1.0 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, n = 2,3,... \tag{1}$$

which are analytic in the unit disc $E = z :|z|< 1$. Let S be class of functions of the form (1) which are analytic univalent in E. With the known estimate $|a_2| \le 2$ and $|a_3| \le 3$ proved by Bieberbach [1] in 1916 and Lowner [2] in 1923.It is natural to seek some relation between a_3 and a_2^2 for the class S. Fekete and Szego [3] used lowner's method to prove the following well known result for the class S.

Let $f \in S$. Then

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} 3-4\mu; \text{if } \mu \le 0 \\ 1+2exp(\frac{-2\mu}{1-\mu})\text{if } 0 \le \mu \le 1 \\ 4\mu - 3 \text{ if} \mu \ge 1 \end{cases} \tag{2}$$

We denote by S^*, the class of univalent starlike functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A$ and satisfying the condition $\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in E$ and K, the class of univalent convex functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A$ and satisfying the condition $\Re \frac{(zf'(z))'}{f'(z)} > 0, z \in E$

In this paper , we study subclasses of $S^*(A,B)$ defined as

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \{-1 \le B < A \le 1\}, z \in E \tag{3}$$

and K(A,B) defined as

$$\frac{(zf'(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz}, \{-1 \le B < A \le 1\}, z \in E \tag{4}$$

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.These subclasses were studied by Goel and Mehrok [4,5]. In particular $S^*(1,-1) \equiv S^*$ the class of starlike functions and $K(1,-1) \equiv K$, the class of convex functions.
symbol $\prec$ stands for subordination.

Principle of Subordination [6]

Let $f(z)$ and $g(z)$ be two functions analytic in E. Then $f(z)$ is subordinate to $g(z)$ in E, if there exist a function $w(z)$ analytic in E satisfying the condition $W(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)) : z \in E$

We denote the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n , w(0) = 0, | w(z) | < 1 \tag{5}$$

It is known that

$$| c_1 | \le 1, | c_2 | \le 1 - | c_1 |^2 \tag{6}$$

.

2.0 Preliminary Lemma

For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz} \right)$ so that

$$\frac{1+w(z)}{1-w(z)} = 1 + 2cz + 2z^2 + ...$$

3.0 Main Result

Theorem 3.1 let $f \in S^*(A,B)$, then

$$| a_3 - \mu a_2^2 | \le \begin{cases} \frac{(A-B)(A-2B)}{2} - (A-B)^2 \mu ; \text{if } \mu \le \frac{(A-2B-1)}{2(A-B)} \\ \frac{(A-B)}{2} ; \text{if } \frac{(A-2B-1)}{2(A-B)} \le \mu \le \frac{(A-2B+1)}{2(A-B)} \\ (A-B)^2 \mu - \frac{(A-B)(A-2B)}{2} ; \text{if} \mu \ge \frac{(A-2B+1)}{2(A-B)} \end{cases}$$

and the result is sharp.

Proof

By definition of $S^*(A,B)$, we have

$$\frac{zf^{'}(z)}{f(z)} = \frac{1+Aw(z)}{1+Bw(z)} ; w(z) \in U \tag{7}$$

Expanding (7), we get

$$1 + 2a_2 z + 3a_3 z^2 + ... = 1 + (a_2 + (A-B)c_1)z + ((a_3 + a_2(A-B)c_1) + (A-B)(c_2 - Bc_1^2)z^2 + ... \tag{8}$$

by comparing with respect to the power of z, we have

$$2a_2 = a_2 + (A-B)c_1 \tag{9}$$

$$3a_3 = a_3 + a_2(A-B)c_1 + (A-B)(c_2 - Bc_1^2) \tag{10}$$

from (9) and (10) ,

$$a_2 = (A-B)c_1 \tag{11}$$

$$a_3 = \frac{(A-B)c_2}{2} + \frac{(A-B)(A-2B)c_1^2}{2} \tag{12}$$

from (11) and (12)

$$a_3-\mu a_2^2=\frac{(A-B)c_2}{2}+\left(\frac{(A-B)(A-2B)}{2}-\mu(A-B)^2\right)c_1^2 \tag{13}$$

taking the absolute value of (13)

$$|a_3-\mu a_2^2|\leq\frac{A-B}{2}|c_2|+|\frac{(A-B)(A-2B)}{2}-(A-B)^2\mu||c_1^2| \tag{14}$$

using (6), we obtain

$$|a_3-\mu a_2^2|\leq\frac{A-B}{2}(1-|c_1|^2)+|\frac{(A-B)(A-2B)}{2}-(A-B)^2\mu||c_1|^2 \tag{15}$$

$$\frac{A-B}{2}+\left(|\frac{(A-B)(A-2B)}{2}-(A-B)^2\mu|-\frac{A-B}{2}\right)|c_1|^2 \tag{16}$$

case I: $\mu\leq\frac{(A-B)}{2(A-B)}$, (16) can be written as

$$|a_3-\mu a_2^2|\leq\frac{A-B}{2}+\left(\frac{(A-B)(A-2B)}{2}-(A-B)^2\mu-\frac{A-B}{2}\right)|c_1|^2 \tag{17}$$

$$\frac{A-B}{2}+\left(\frac{(A-B)(A-2B-1)}{2}-(A-B)^2\mu\right)|c_1|^2 \tag{18}$$

Subcase I(a): $\mu\leq\frac{(A-2B-1)}{2(A-B)}$. Using (6),(18) becomes

$$|a_3-\mu a_2^2|\leq\frac{A-B}{2}+\left(\frac{(A-B)(A-2B-1)}{2}-(A-B)^2\mu\right)$$

$$=\frac{A-B}{2}+\left(\frac{(A-B)(A-2B-1)}{2}-(A-B)^2\mu\right) \tag{19}$$

Subcase I(b): $\mu\geq\frac{(A-2B-1)}{2(A-B)}$

$$|a_3-\mu a_2^2|\leq\frac{A-B}{2}-\left((A-B)^2\mu-\frac{(A-B)(A-2B-1)}{2}\right)|c_1^2|\leq\frac{A-B}{2} \tag{20}$$

Case II: $\mu\geq\frac{(A-2B)}{2(A-B)}$, preceding as in case I, we get

$$|a_3-\mu a_2^2|\leq\frac{A-B}{2}-\left((A-B)^2\mu-\frac{(A-B)(A-2B)}{2}-\frac{A-B}{2}\right)|c_1^2|$$

$$\leq\frac{A-B}{2}-\left((A-B)^2\mu-\frac{(A-B)(A-2B+1)}{2}\right)|c_1^2| \tag{21}$$

Subcase II(a): $\mu\leq\frac{(A-2B+1)}{2(A-B)}$,(21) take the form

$$|a_3-\mu a_2^2|\leq\frac{A-B}{2} \tag{22}$$

combining subcase I(a) and subcase II(a), we obtain

$$|a_3-\mu a_2^2|\le \frac{A-B}{2}; if\ \frac{(A-2B-1)}{2(A-B)}\le\mu\le\frac{(A-2B+1)}{2(A-B)} \tag{23}$$

Subcase II(b): $\mu\ge\frac{(A-2B+1)}{2(A-B)}$. Preceding as in subcase I(a), we get

$$|a_3-\mu a_2^2|\le (A-B)^2\mu-\frac{(A-B)(A-2B)}{2} \tag{24}$$

combining (19) , (23) and (24) , the theorem is proved

$$|a_3-\mu a_2^2|\le\begin{cases}\frac{(A-B)(A-2B)}{2}-(A-B)^2\mu;\text{if } \mu\le\frac{(A-2B-1)}{2(A-B)}\\ \frac{(A-B)}{2};\text{if } \frac{(A-2B-1)}{2(A-B)}\le\mu\le\frac{(A-2B+1)}{2(A-B)}\\ (A-B)^2-\frac{(A-B)(A-2B)}{2};\text{ if}\mu\ge\frac{(A-2B+1)}{2(A-B)}\end{cases}$$

Corollary 3.2 : put $A=1, B=-1,$ we get

$$|a_3-\mu a_2^2|\le\begin{cases}3-4\mu; if\mu\le\frac{1}{2}\\ 1;\text{if } \frac{1}{2}\le\mu\le 1\\ 4\mu-3;\text{ if}\mu\ge 1\end{cases}$$

Theorem 3.3 let $f\in k(A,B)$, then

$$|a_3-\mu a_2^2|\le\begin{cases}\frac{(A-B)(A-2B)}{6}-\frac{(A-B)^2}{4}\mu;\text{if } \quad \mu\le\frac{2(A-2B-1)}{3(A-B)}\\ \frac{(A-B)}{6};\text{if } \quad \frac{2(A-2B-1)}{3(A-B)}\le\mu\le\frac{2(A-2B+1)}{3(A-B)}\\ \frac{(A-B)^2}{4}\mu-\frac{(A-B)(A-2B)}{6};\text{ if}\mu\ge\frac{2(A-2B+1)}{3(A-B)}\end{cases}$$

and the result is sharp

Proof

By definition of $K(A,B)$, we have

$$\frac{(zf'(z))'}{f'(z)}=\frac{1+Aw(z)}{1+Bw(z)}; w(z)\in U \tag{25}$$

Expanding the (25), we get

$$1+4a_2z+9a_3z^2+...=1+(2a_2+(A-B)c_1)z+((3a_3+2a_2(A-B)c_1)+$$

$$(A-B)(c_2-Bc_1^2))z^2+... \tag{26}$$

by comparing with respect to the power of z, we have

$$2a_2=a_2+(A-B)c_1 \tag{27}$$

$$6a_3=a_3+a_2(A-B)c_1+(A-B)(c_2-Bc_1^2) \tag{28}$$

from (27) and (28) , we obtain

$$a_2 = \frac{(A-B)c_1}{2} \tag{29}$$

$$a_3 = \frac{(A-B)(A-2B)c_1^2}{6} + \frac{(A-B)c_2}{6} \tag{30}$$

from (29) and (30)

$$a_3 - \mu a_2^2 = \left(\frac{(A-B)(A-2B)c_1^2}{6} + \frac{(A-B)c_2}{6} \right) - \mu \left(\frac{(A-B)c_1}{2} \right)^2 \tag{31}$$

$$= \frac{(A-B)(A-2B)c_1^2}{6} + \frac{(A-B)c_2}{6} - \mu \frac{(A-B)^2 c_1^2}{4} \tag{32}$$

$$\frac{(A-B)c_2}{6} + \left(\frac{(A-B)(A-2B)}{6} - \mu \frac{(A-B)^2}{4} \right) c_1^2 \tag{33}$$

Taking the absolute value, (33) can be written as

$$| a_3 - \mu a_2^2 | = \frac{(A-B)}{6} | c_2 | + | \frac{(A-B)(A-2B)}{6} - \mu \frac{(A-B)^2}{4} c_1^2 | \tag{34}$$

using (6), we obtain

$$| a_3 - \mu a_2^2 | \le \frac{(A-B)}{6} (1 - | c_1 |^2) + | \frac{(A-B)(A-2B)}{6} - \mu \frac{(A-B)^2}{4} \| c_1 |^2 \tag{35}$$

$$= \frac{(A-B)}{6} + \left(| \frac{(A-B)(A-2B)}{6} - \mu \frac{(A-B)^2}{4} | - \frac{(A-B)}{6} \right) | c_1 |^2 \tag{36}$$

Case I : $\mu \le \frac{2(A-2B)}{3(A-B)}, (36)$ can be written as

$$| a_3 - \mu a_2^2 | \le \frac{(A-B)}{6} + \left(\left(\frac{(A-B)(A-2B)}{6} - \mu \frac{(A-B)^2}{4} \right) - \frac{(A-B)}{6} \right) | c_1 |^2$$

$$= \frac{(A-B)}{6} + \left(\frac{(A-B)(A-2B-1)}{6} - \mu \frac{(A-B)^2}{4} \right) | c_1 |^2 \tag{37}$$

Subcase I(a) : $\mu \le \frac{2(A-2B-1)}{3(A-B)}$, using (6),(37) becomes

$$| a_3 - \mu a_2^2 | \le \frac{(A-B)}{6} + \left(\frac{(A-B)(A-2B-1)}{6} - \mu \frac{(A-B)^2}{4} \right) = \frac{(A-B)(A-2B)}{6} - \mu \frac{(A-B)^2}{4} \tag{38}$$

Subcase I(b) : $\mu \ge \frac{2(A-2B-1)}{3(A-B)}$, We obtain from (38)

$$| a_3 - \mu a_2^2 | \le \frac{(A-B)}{6} - \left(\mu \frac{(A-B)^2}{4} - \frac{(A-B)(A-2B-1)}{6} \right) | c_1 |^2 \le \frac{(A-B)}{6} \tag{39}$$

Case II : $\mu \ge \frac{2(A-2B)}{3(A-B)}$ preceeding as in case I, we get

$$|a_3-\mu a_2^2|\le\frac{(A-B)}{6}+\left(\left(\frac{(A-B)^2}{4}\mu-\frac{(A-B)(A-2B)}{6}\right)-\frac{(A-B)}{6}\right)|c_1|^2$$

$$\le\frac{(A-B)}{6}+\left(\frac{(A-B)^2}{4}\mu-\frac{(A-B)(A-2B+1)}{6}\right)|c_1|^2 \qquad (40)$$

Subcase II(a) : $\mu\le\frac{2(A-2B+1)}{3(A-B)}$.(40) takes the form

$$|a_3-\mu a_2^2|\le\frac{(A-B)}{6} \qquad (41)$$

combining I(b) and subcase II(a), we obtain

$$|a_3-\mu a_2^2|\le\frac{(A-B)}{6};if\ \frac{2(A-2B-1)}{3(A-B)}\le\mu\le\frac{2(A-2B+1)}{3(A-B)} \qquad (42)$$

Subcase II(b) : $\mu\ge\frac{2(A-2B+1)}{3(A-B)}$, preceeding as in case subcase I(a), we get

$$|a_3-\mu a_2^2|\le\frac{(A-B)^2}{4}\mu-\frac{(A-B)(A-2B)}{6} \qquad (43)$$

combining (38), (42) and (43), the theorem is proved.

Corollary 3.4 : put $\{A=1,B=-1\}$, we get

$$|a_3-\mu a_2^2|\le\begin{cases}1-\mu;if\mu\le\frac{2}{3}\\ \frac{1}{3};\text{if }\frac{2}{3}\le\mu\le\frac{4}{3}\\ \mu-1;\text{if}\mu\ge\frac{4}{3}\end{cases}$$

The Functional considered in this work is closely associated with the bounds on $|a_3-\mu a_2^2|$ known as fekete-Szego in geometry functions theory satisfying equations (3) and (4).

4.0 References

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