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Finite Element Method for the Numerical Solution of Second-Order Differential Equation for the Vibration of Automotive System

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Abstract

This paper considered Finite Element Method (FEM) as an alternative numerical method to approximate a Second-Order Differential Equation with boundary conditions which was derived from an existing model equation for the vibration of an Automotive system with three (3) arbitrary coefficients (M, C and K). This method discretizes the differential equation into N-elements with (N+1) nodes and then obtained its weak formulation and the basis function, leading to a system of tridiagonal matrix equations. The approximate solution of the differential equation obtained using FEM is in a good agreement with the exact solution of the equation. Thus, the method is incredibly precise and efficient enough to be used for the numerical approximation of Second-Order Differential Equations with arbitrary coefficients.

Keywords: Finite Element Method, Discretization, Weak Formulation, Basis function, Thomas algorithm

1.0 Introduction

Several papers on differential equations often give the impression that most differential equations can be solved in closed form, but experience does not bear this out. It remains true that solutions of the vast majority of Second-Order Differential Equations with boundary value problems cannot be found by analytical means because it is only limited to finding the general solutions and very difficult to obtain its approximate solution. Therefore, it is very important to be able to approach the problem in other ways.

In this paper, we introduce an incredibly precise and accurate method called the "Finite Element Method" (FEM), which was developed in 1943 by R. Courant. This method doesn't actually approximate the original equation but rather the weak formulation of the original differential equation. The purpose of the weak formulation is to satisfy the equation in "average sense", so that we can approximate the solutions that are discontinuous or otherwise poorly behaved. If a function is a solution to the original form of the ODE then it also satisfies the weak form of the ODE and such function is called the "Smooth function" or Test function" and can be any function that is sufficiently well-behaved for the integral to exist.

Therefore, it is important to know that our focus in this paper is to consider Finite Element Method as an alternative and efficient numerical method for approximating the Second-Order Differential Equation derived for the vibration of an Automotive system and not to analyzed any consistent theoretical background.

To acquaint more closely with Finite Element Method and its applications in problem solving, there are many excellent and exhaustive texts on these subjects that may be consulted. [1-6].

2.0 Description of the Existing Second–Order Model Equation.

We consider an automobile of mass m supported by a spring and shock absorber. According to Hooke's law,

$$\textit{Spring force} = -kX \qquad (1)$$

Where k is the force constant, X is the length and negative sign indicates the restoring force acts to return the car towards the towards the position of equilibrium.

The damping force of the shock absorber is given by,

$$\textit{Damping force} = -c\dot{X} \qquad (2)$$

Where c is the damping coefficient, $\dot{X}$ is the vertical velocity and the negative sign the indicate the damping force acting in opposite direction.

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The equation of motion for the system is given by the newton's second law $(f = ma)$ which is expressed as;
Mass × acceleration = damping force + spring force,

$$M\ddot{X} = -C\dot{X} + (-KX) \qquad (3)$$

$$M\ddot{X} + C\dot{X} + KX = 0 \qquad (4)$$

Thus, equations (3) and (4) are the model equations for the vibration of an automotive system.

2.1 Analytical Solution of The Second–Order Model Equation

We assumed that the solution of equation (4) takes the form of $X(t) = e^{\omega t}$, we write the characteristics equation as,

$$M\omega^2 + C\omega + K = 0 \qquad (5)$$

Thus, the solution of equation (5) for ω is given by,

$$\omega_{1,2} = -C \pm \frac{\sqrt{C^2-4MK}}{2M} \qquad (6)$$

We then consider the three cases for equation (6) as follows,
CASE 1: If $C > 2\sqrt{KM}$, ω_1 and ω_2 are negative real numbers and the general solution is of the form,

$$X(t) = Ae^{\omega_1 t} + Be^{\omega_2 t} \qquad (7)$$

Where A and B are constants and such systems are called over damped.
CASE 2: If $C < 2\sqrt{KM}$, the roots are complex,
$\omega_{1,2} = \pm\mu i,$
Where $\mu = \frac{\sqrt{C^2-4MK}}{2M}$ And the general solution is of the form

$$X(t) = (A\cos\mu t + B\sin\mu t)e^{-t} \qquad (8)$$

And such systems are called under damped.
CASE3: If $C = 2\sqrt{KM}$, the characteristics equation has a double root and the general solution is of the form,

$$X(t) = (A + Bt)e^{-t} \qquad (9)$$

Where $\omega = \frac{C}{2M}$ and such systems are called critically damped.
Therefore, the analytical solutions presented above is only limited to finding the general solutions of the existing model equations and is no longer possible in the case of obtaining the approximate solutions especially with boundary conditions. Thus, we employ the Finite Element Method.

3.0 Finite Element Method

In this section, Finite Element Method (FEM) is numerically used to approximate the model equation (4) as follows;
The model equation (4) can be written as

$$\ddot{X} + \frac{C}{M}\dot{X} = -\frac{K}{M}X \qquad (10)$$

Let $P = \frac{C}{M}$, $Q = \frac{K}{M}$,
Equation (10) becomes

$$X'' + PX' + QX = 0 \qquad (11)$$

Subject to the boundary conditions

$$\begin{cases} X'(0) = PX_0 \\ \\ X'(L) = 0 \end{cases} \qquad (12)$$

3.1 Discretization

Now, we consider the discretization of equation (11) in one-dimensional space with N elements of length h, where $Nh = L$. Thus, the systems consists of N elements and $N + 1$ nodes, that is, $[x_o, \ \ x_1, \ x_2, \ldots x_{N-2}, \ x_{N-1}, \ x_N]$.

3.2 The Weak Formulation

We multiply equation (11) by a smooth function $V(x)$ and integrate over the interval $[0, L]$ to give

$$\int_0^L (X'' + PX' + QX)V\,dx = 0 \qquad (13)$$

Integrating equation (8) by parts, we have

$$X'(x)V(x)\,\Big|_0^L - \int_0^L X'(x)V'(x)dx + \int_0^L PX'(x)V(x)dx + \int_0^L QX(x)V(x)dx$$

$$= X'(L)V(L) - X'(0)V(0) - \int_0^L X'(x)V'(x)dx + P\int_0^L X'(x)V(x)dx + Q\int_0^L XVdx = 0 \qquad (14)$$

Equation (14) is the weak formulation of equation (11).

3.3 The Basis Function

Let

$$X(x) = \sum_{j=o}^{N} \alpha_j\, \varphi_j \;\; and \;\; V(x) = \varphi_i, for\ 0,1,\dots,N$$

And

$$X_i \approx X(x_i), \qquad (15)$$

Where

$$\varphi_i(x_j) = \begin{cases} 0, i \neq j \\ \\ 1, i = j \end{cases}$$

Therefore, equation (14) can be written as a basis function,

$$-\sum_{j=o}^{N} \alpha_j\, \varphi'_j(0)\varphi_i(0) - \sum_{j=o}^{N} \alpha_j \int_0^L \varphi'_i\varphi'_j dx + P\sum_{j=o}^{N} \alpha_j \int_0^L \varphi_i\varphi'_j dx + Q\sum_{j=o}^{N} \alpha_j \int_0^L \varphi_i\varphi_j dx = 0 = \qquad (16)$$

Which can be solve for

$$\mathrm{P}\ \alpha_0\varphi_i(0) - \sum_{j=o}^{N} \alpha_j \int_0^L \varphi'_i\varphi'_j dx + P\sum_{j=o}^{N} \alpha_j \int_0^L \varphi_i\varphi'_j dx + Q\sum_{j=o}^{N} \alpha_j \int_0^L \varphi_i\varphi_j dx = 0 \qquad (17)$$

Rearranging the terms of equation (16), we have

$$\mathrm{P}\alpha_0\varphi_i(0) + \sum_{j=o}^{N} \alpha_j\,(-\int_0^L \varphi'_i\varphi'_j dx + P\int_0^L \varphi_i\varphi'_j dx + Q\int_0^L \varphi_i\varphi_j dx) = 0 \qquad (18)$$

Therefore, let

$$\varphi_k = \begin{cases} (x - x_{k-1})\big/h\,, x \in (x_{k-1}, x_k) \\ \\ (x_{k+1} - x)\big/h\,, x \in (x_k, x_{k+1}) \end{cases}, \varphi'_k = \begin{cases} 1\big/h\,, x \in (x_{k-1}, x_k \\ \\ -1\big/h\,, x \in (x_k, x_{k+1} \end{cases}, \quad for \quad k = 1,2,\dots,N-1.$$

Then, equation (16) forms a system of equations which leads to

$$AC = B \qquad (19)$$

Where

$$a_{0,0} = a_{N,N} = \frac{-1}{h} + \frac{P}{2} + \frac{Qh}{3}, \qquad (20)$$

$$a_{i,i} = \frac{-2}{h} + \frac{2Qh}{3}, \qquad i = 1,2,\dots,N-1, \qquad (21)$$

$$a_{i,i-1} = \frac{1}{h} - \frac{P}{2} + \frac{Qh}{6}, \;\; i = 1,2,\dots,N-1 \qquad (22)$$

$$a_{i,i-1} = \frac{1}{h} - \frac{P}{2} + \frac{Qh}{6}, \;\; i = 1,2,\dots,N-1 \qquad (23)$$

$$B_i = 0, \quad i = 1,2,\dots,N \qquad (24)$$

Thus, equations (20), (21), (22), (23) and (24) forms a system of tridiagonal matrix equations when substituted into equation (19).

4.0 Results and Discussion

4.1 Numerical Examples and Results

We assumed values for the arbitrary coefficients *M*, *C* and *K* and compute the numerical solution for the system of equation (19) as follows:

Supposed $M = 1,\ C = 1,\ h = 2.5,\ K = 0.2\ and\ L = 10,$

$$P = \frac{C}{M} = 1, \qquad Q = \frac{-K}{M} = -0.2, \qquad N = \frac{L}{h} = 4,$$

$$a_{0,0} = a_{4,4} = -\frac{1}{2.5} + \frac{1}{2} + \frac{(-0.2)}{3} = 0.03,$$
$$a_{1,1} = a_{2,2} = a_{3,3} = -\frac{-2}{2.5} + \frac{2(0.2)(2.5)}{3} = -1.13,$$
$$a_{1,0} = a_{2,1} = a_{3,2} = \frac{1}{2.5} - \frac{1}{2} + \frac{(-0.2)(2.5)}{6} = -0.18$$
$$a_{1,2} = a_{2,3} = \frac{1}{2.5} + \frac{1}{2} + \frac{(-0.2)(2.5)}{6} = 0.82$$

Thus, the system becomes,

$$\begin{bmatrix} 0.03 & 0.82 & & & \\ -0.18 & -1.13 & 0.82 & & \\ & -0.18 & -1.13 & 0.82 & \\ & & -0.18 & -1.13 & 0.82 \\ & & & -0.18 & 0.03 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (25)$$

Which can be solved by the Thomas algorithm. [7, 8].
The results are,
$X_0 = 85.41,\ X_1 = 55.72,\ X_2 = 36.36,\ X_3 = 23.84,\ X_4 = 17.73$
The Approximate Solutions X_o, X_1, X_2, X_3 and X_4 obtained above using FEM are compared with the Exact Solution of Model equation (4) in **Table I.**

Table I: Approximate Solution using FEM and the Exact Solutions for the Model Equation at $h = 0, 2.5, 5.0, 7.5\ and\ 10.0$.

H	**Exact Solution (*X*)**	**Finite Element Approximation (*X_i*)**
0	85.4102	85.400
2.5	55.7245	55.7200
5.0	36.3626	36.3600
7.5	23.8408	23.8400
10.0	17.7334	17.7300

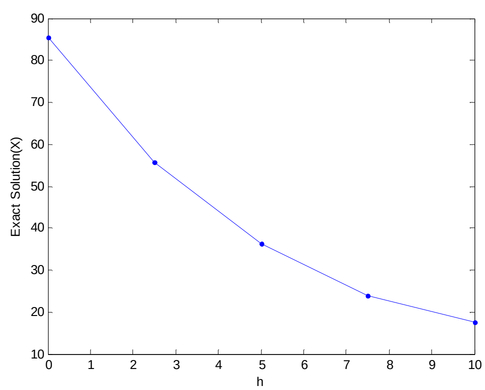

Figure 1: The graph of the Exact Solution (*X*) versus h

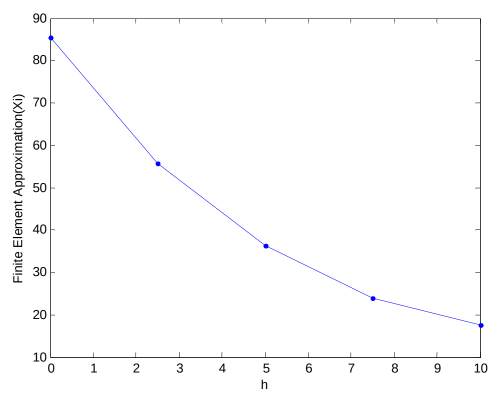

Figure 2: The graph of the Approximate Solution (X_i) using FEM versus h

5.0 Discussion of Results

The numerical results, X_0, X_1, X_2, X_3 and X_4 gotten above are the values of the X_i obtained from the Finite Element Approximation of equation (11). The values decrease for $\mathrm{h} = 0, 2.5, 5.0, 7.5$ and 10.0. Figure 2 represents the graphs of the Approximate Solution obtained using FEM.

6.0 Conclusion

FEM was used to discretized the second-order differential equation into N-elements with (N+1) nodes and then obtained its weak formulation and the basis function leading to a system of tridiagonal matrix equations. The approximate solution of the differential equation obtained using FEM is in a good agreement with the exact solution of the equation as shown in Table I. As a result of this, the graphs of the exact solutions and the approximate solutions are plotted as Figure 1 and Figure 2 respectively. Thus, the method is incredibly precise and efficient enough to be used for the numerical approximation of Second-Order Differential Equations with arbitrary coefficients.

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