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Analysis of a Sixth Stage Sixth-Order Explicit Runge Kutta Method For The Solution of Initial Value Problems In Ordinary Differential Equations

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Abstract

In This paper, a sixth stage sixth-order explicit Runge-Kutta formula is derived through a tactful application of Taylor series expansion techniques to generate its parameters. The derived method is constructed from the traditional Runge-Kutta's formula which is capable of solving initial value problems (IVPs) in ordinary differential equations (ODEs).

The performance of the derived formula is tested by numerical computation of some selected IVPs and the results compared favorably well with other existing Runge-Kutta Methods. The numerical results show that the method is accurate and effective and has very fast computing time.

Keywords and Phrases: Analysis, sixth stage sixth-order explicit Runge-Kutta Methods, initial value problems, ordinary differential equations

1.0 Introduction

Many methods have been proposed and used by different authors with the aim of providing accurate solutions to the various types of differential equations, although many of such methods already exist. Runge-Kutta method is one of such method. This method has come of age [1]. It was discovered that a sixth-order Runge-Kutta method possess the ability of improving results if the parameters are varied [2, 3, 4]. Error analysis is considerably easier with linear multi-step method (LLM) than the one-step method [5]. RKM is a class of one-step formulae derived for solving IVPs in ODEs [6]. This prompted us to work on the sixth stage sixth-order Runge-Kutta methods that is capable of solving initial value problems of the form [7, 8]

$$y' = f(x, y),\ y(x_0,) = y_0,\ a \le x \le b \tag{1.1}$$

According to Agbeboh et al [9], the general R-stage Runge-Kutta method is

$$y_{n+1} - y_n = h\phi(x_n, y_n; h) \tag{1.2}$$

$$\phi(x_n, y_n; h) = \sum_{r=1}^{R} c_r k_r \tag{1.3}$$

$$k_i = f(x, y) \tag{1.4}$$

$$k_r = f\left(x + ha_r, y + h\sum_{s=1}^{r-1} b_{rs} k_s\right), \qquad r = 2,3,\ldots, R \tag{1.5}$$

$$a_r = \sum_{s=1}^{r-1} b_{rs}, \qquad r = 2,3,\ldots, R \tag{1.6}$$

From the above, we arrive at a sixth stage sixth-order method written as:

$$y_{n+1} - y_n = h(c_1k_1 + c_2k_2 + c_3k_3 + c_4k_4 + c_5k_5 + c_6k_6)$$

Where $k_1 = f(x_n, y_n) = f$ (1.7a)

$$k_2 = f(x_n + a_1h, y_n + a_2hk_1) \tag{1.7b}$$

$$k_3 = f(x_n + a_3h, y_n + h(a_4k_1 + a_5k_2)) \tag{1.7c}$$

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$$k_4 = f(x_n + a_6 h, y_n + h(a_7 k_1 + a_8 k_2 + a_9 k_3)) \quad (1.7d)$$

$$k_5 = f(x_n + a_{10} h, y_n + h(a_{11} k_1 + a_{12} k_2 + a_{13} k_3 + a_{14} k_4)) \quad (1.7e)$$

$$k_6 = f(x_n + a_{15} h, y_n + h(a_{16} k_1 + a_{17} k_2 + a_{18} k_3 + a_{19} k_4 + a_{20} k_1)) \quad (1.7f)$$

Where $\phi(x_n, y_n; h) = \sum_{r=1}^{R} C_r K_r = c_1 k_1 + c_2 k_2 + c_3 k_3 + c_4 k_4 + c_5 k_5 + c_6 k_6$

2.0 Steps For The Derivation of the Method

The following steps is taken for the derivation of our method, the parameters a_j, c_j are to be determined from the system of non-linear equations generated by the following steps:

i. Set $a_1 = a_3 = a_6 = a_{10} = a_{15} = 0$

ii. Obtain the Taylor series expansion of $k_{j's}$ about the point (x_n, y_n) $for\ j = 1,2,...6$ and discard everything that has to do with the partial derivative of x.

iii. Insert the series expansion into (1.3)

iv. Compare the final expansion with the Taylor series expansion of $\phi(x_n, y_n; h)$.

3.0 Derivation of the Method

The function evaluation using Taylor series expansion about the point (x_n, y_n) becomes

$$k_1 = f(x_n, y_n) \quad (3.1a)$$

$$k_2 = \sum_{r=0}^{\infty} \frac{1}{r!} (a_2 h \frac{\partial}{\partial x} + a_2 h k_1 \frac{\partial}{\partial y})^r f(x, y) \quad (3.1b)$$

$$k_3 = \sum_{r=0}^{\infty} \frac{1}{r!} (a_3 h \frac{\partial}{\partial x} + h(a_4 k_1 + a_5 k_2) \frac{\partial}{\partial y})^r f(x, y) \quad (3.1c)$$

$$k_4 = \sum_{r=0}^{\infty} \frac{1}{r!} (a_6 h \frac{\partial}{\partial x} + h(a_7 k_1 + a_8 k_2 + a_9 k_3) \frac{\partial}{\partial y})^r f(x, y) \quad (3.1d)$$

$$k_5 = \sum_{r=0}^{\infty} \frac{1}{r!} (a_{10} h \frac{\partial}{\partial x} + h(a_{11} k_1 + a_{12} k_2 + a_{13} k_3 + a_{14} k_4) \frac{\partial}{\partial y})^r f(x, y) \quad (3.1e)$$

$$k_6 = \sum_{r=0}^{\infty} \frac{1}{r!} (a_{15} h \frac{\partial}{\partial x} + h(a_{16} k_1 + a_{17} k_2 + a_{18} k_3 + a_{19} k_4 + a_{20} k_5) \frac{\partial}{\partial y})^r f(x, y) \quad (3.1f)$$

For the purpose of reducing complicated derivation, we will concentrate on the y function only,

Then Setting

$a_1 = a_3 = a_6 = a_{10} = a_{15} = 0$ and discarding everything that has to do with the derivatives of x, we have the following results:

$$k_1 = f(y_n) = K_1$$

$$k_2 = K_1 + h a_2 K_1 f_y + \frac{h^2}{2} a_2{}^2 K_1^2 f_{yy} + \frac{h^3}{3!} a_2{}^3 K_1^3 f_{yyy} + \frac{h^4}{4!} a_2{}^4 K_1^4 f_{yyyy} + \frac{h^5}{5!} a_2{}^5 K_1^5 f_{yyyyy}$$

$$k_3 = K_1 + h(a_4k_1 + a_5k_2)f_y + \frac{h^2}{2}(a_4k_1 + a_5k_2)^2 f_{yy} + \frac{h^3}{3!}(a_4k_1 + a_5k_2)^3 f_{yyy} + \frac{h^4}{4!}(a_4k_1 + a_5k_2)^4 f_{yyyy}$$
$$+ \frac{h^5}{5!}(a_4k_1 + a_5k_2)^5 f_{yyyyy}$$

$$k_4 = k_1 + h(a_7k_1 + a_8k_2 + a_9k_3)f_y + \frac{h^2}{2}(a_7k_1 + a_8k_2 + a_9k_3)^2 f_{yy} + \frac{h^3}{3!}(a_7k_1 + a_8k_2 + a_9k_3)^3 f_{yyy}$$
$$+ \frac{h^4}{4!}(a_7k_1 + a_8k_2 + a_9k_3)^4 f_{yyyy} + \frac{h^5}{5!}(a_7k_1 + a_8k_2 + a_9k_3)^5 f_{yyyyy}$$

$$k_5 = k_1 + h(a_{11}k_1 + a_{12}k_2 + a_{13}k_3 + a_{14}k_4)f_y + \frac{h^2}{2!}(a_{11}k_1 + a_{12}k_2 + a_{13}k_3 + a_{14}k_4)^2 f_{yy} + \frac{h^3}{3!}(a_{11}k_1$$
$$+ a_{12}k_2 + a_{13}k_3 + a_{14}k_4)^3 f_{yyy} + \frac{h^4}{4!}(a_{11}k_1 + a_{12}k_2 + a_{13}k_3 + a_{14}k_4)^4 f_{yyyy} + \frac{h^5}{5!}(a_{11}k_1 + a_{12}k_2$$
$$+ a_{13}k_3 + a_{14}k_4)^5 f_{yyyyy}$$

$$k_6 = k_1 + h(a_{16}k_1 + a_{17}k_2 + a_{18}k_3 + a_{19}k_4 + a_{20}k_5)f_y + \frac{h^2}{2!}(a_{16}k_1 + a_{17}k_2 + a_{18}k_3 + a_{19}k_4 + a_{20}k_5)^2 f_{yy}$$
$$+ \frac{h^3}{3!}(a_{16}k_1 + a_{17}k_2 + a_{18}k_3 + a_{19}k_4 + a_{20}k_5)^3 f_{yyy} + \frac{h^4}{4!}(a_{16}k_1 + a_{17}k_2 + a_{18}k_3 + a_{19}k_4 + a_{20}k_5)^4 f_{yyyy}$$
$$+ \frac{h^5}{5!}(a_{16}k_1 + a_{17}k_2 + a_{18}k_3 + a_{19}k_4 + a_{20}k_5)^5 f_{yyyyy}$$

Expanding fully and substituting the various $k_{j^s}, \ \forall \ j = 2,3,4,5,6,$ we reduce the k_j *to* k_1 for the purpose of linearity and easy of computation and collecting like terms, and also, let $B=(a_4 + a_5), D=(a_7 + a_8 + a_9), E=(a_{11} + a_{12} + a_{13} + a_{14})$ *and* $F=(a_{16} + a_{17} + a_{18} + a_{19} + a_{20})$ we have

$$k_1 = f(y_n) = f$$

$$k_2 = f + ha_2 ff_y + \frac{h^2}{2}a_2{}^2 f^2 f_{yy} + \frac{h^3}{3!}a_2{}^3 f^3 f_{yyy} + \frac{h^4}{4!}a_2{}^4 f^4 f_{yyyy} + \frac{h^5}{5!}a_2{}^5 f^5 f_{yyyyy}$$

$$k_3 = f + hBff_y + h^2 a_2 a_5 ff_y{}^2 + \frac{h^2}{2!}B^2 f^2 f_{yy} + \frac{h^3}{2!}(a_2^2 a_5 + 2a_2a_4a_5 + 2a_2a_5^2) f^2 f_y f_{yy} + \frac{h^3}{3!}B^3 f^3 f_{yyy}$$
$$+ \frac{h^4}{2!}(a_2{}^2 a_4 a_5 + a_2{}^2 a_5^2) f^3 f_{yy}{}^2 + \frac{h^4}{2!}a_2{}^2 a_5{}^2 f^2 f_y{}^2 f_{yy} + \frac{h^4}{3!}(a_4{}^3 a_5 + 3a_4^2 a_5 + 6a_2a_4a_5^2 + 3a_2 a_5{}^3) f^3 f_y f_{yyy}$$
$$+ \frac{h^4}{4!}B^4 f^4 f_{yyyy} + \frac{h^5}{4!}(a_2{}^4 a_4 + 4a_2a_4{}^3 a_5 + 8a_2a_4{}^2 a_5{}^2 + 4a_2a_5{}^4 + 12a_2a_4a_5) f^4 f_y f_{yyyy} + \frac{h^5}{2!}a_2^3 a_5^2 f^3 f_y f_{yy}^2$$
$$+ \frac{h^5}{3!}(a_2^3 a_4 a_5 + a_2^3 a_5^2 + \frac{3}{2}a_2^2 a_4^2 a_5 + \frac{3}{2}a_2^2 a_5^2) f^4 f_{yy} f_{yyy} + \frac{h^5}{2!}(a_2^2 a_4 a_5^2 + a_2^2 a_5^3) f^3 f_y^2 f_{yyy} + \frac{h^5}{5!}B^5 f^5 f_{yyyyy}$$

$$K_4 = f + hDff_y + h^2(a_2a_8 + Ba_9) ff_y^2 + \frac{h^2}{2}D^2 f^2 f_{yy} + h^3 a_2a_5a_9 ff_y^3 + h^3 a_2^2 a_7 a_8 a_9 f^2 f_y^3 + \frac{h^3}{2}(a_2^2 a_8$$
$$+ a_4^2 a_9 + a_5 a_9 + 2a_2 a_8^2 + 2Ba_9^2 + 2a_4a_5a_9 + 2a_2a_7a_8 + 2Ba_7a_9 + 2Ba_8a_9 + 2a_2a_8a_9) f^2 f_y f_{yy}$$
$$+ \frac{h^4}{2}(a_2^2 a_7 a_8 + a_4^2 a_7 a_8 + B^2 a_7 a_9 + 2a_4a_5a_7a_9 + 2a_4a_5a_8a_9 + a_5^2 a_7 a_9 + a_2^2 a_8^2 + a_4^2 a_9^2 + a_5^2 a_8 a_9$$
$$+ a_2^2 a_8 a_9 + 4a_4a_5a_9^2) f^3 f_{yy}^2 + \frac{h^3}{6}D^3 f^3 f_{yyy} + \frac{h^4}{3!}(a_2^3 a_8 + 4B^3 a_9 + 3a_2a_8^3 + 3a_4a_9^3 + 3a_5^3 a_9 + 3a_2a_7^2 a_8$$

$$+6a_2a_7a_8a_9+3a_5a_7^2a_9+6a_2a_7a_8^2+12a_4a_7a_8a_9+6Ba_7a_9^2+3Ba_8^2a_9+3a_4a_7^2a_9+6a_2a_8^2a_9+6Ba_8a_9^2$$

$$+3a_2a_8a_9^2+)f^3f_yf_{yyy}+\frac{h^5}{2}(a_2^2a_4a_7a_9+\frac{h^5}{2!}(2a_2^3a_7a_8+2a_4^3a_7a_9+6a_4a_7^3a_9+2a_7^4a_9+2a_2^3a_8a_9$$

$$+6a_4^2a_7a_8a_9+6a_4a_7^2a_8a_9+3a_4^2a_8^2a_9+2a_2^3a_8^2+a_4^3a_9^2+3a_2^2a_7^2a_8+6a_2^2a_7a_8^2+9a_4^2a_7^2a_9+2a_4a_7^3a_9$$

$$+a_2^2a_9+a_2^3a_7a_9+2a_2a_4a_5^2a_9+2a_2a_7^3a_9+a_2^3a_8a_9+4a_2a_5a_7a_8a_9+3a_2a_7^2a_8a_9$$

$$+a_2a_4^2a_8a_9+a_2^2a_4a_8a_9+a_2^2a_7a_8a_9+a_2^3a_8^2+a_2^3a_9^2+2a_2a_4a_7a_9^2+2a_2a_7^2a_9^2+B^3)f^3f_yf_{yy}^2$$

$$+\frac{h^4}{2}a_2a_7^2a_8f^4f_yf_{yyy}+\frac{h^5}{2!}(a_2^2a_7^2a_9^2+2a_2a_7a_9B)f^2f_y^3f_{yy}+\frac{h^5}{2!}(2a_2^3a_7a_8+2a_4^3a_7a_9$$

$$+6a_4a_7^3a_9+2a_7^4a_9+2a_2^3a_8a_9+6a_4^2a_7a_8a_9+6a_4a_7^2a_8a_9+3a_4^2a_8^2a_9+2a_2^3a_8^2+a_4^3a_9^2$$

$$+3a_2^2a_7^2a_8+6a_2^2a_7a_8^2+9a_4^2a_7^2a_9+2a_4a_7^3a_9+a_7^4a_9^3+12a_4^2a_7a_9^2+18a_4a_7^2a_9^2+8a_7^3a_9^2$$

$$+2a_2a_4a_8a_9^2+a_2^2a_8^3+\frac{1}{3}(a_2a_7a_9^3+2a_9^3B^2))f^3f_y^2f_{yyy}+\frac{h^4}{4!}D^4f^4f_{yyyy}+\frac{h^5}{4!}(a_2^4a_8+a_4^4a_9$$

$$+4a_4^3a_7a_9+4a_4^2a_7^2a_9+4a_4a_7^3a_9+a_7^4a_9+4a_2a_7^3a_8+12a_2a_7^2a_8^2+12a_7^2a_8a_9+12a_2a_7a_8^3$$

$$+4a_7^3a_9B+12a_7^2a_9^2B+24a_7a_8^2a_9B+12a_7a_9^3B+4a_8^3a_9B+12a_8^2a_9^2B+48a_7a_8a_9^2B+4a_2a_8^4$$

$$+4a_4a_9^4+12a_8a_9^3B+4a_7a_9^4)f^4f_yf_{yyyy}+\frac{h^5}{5!}D^5f^5f_{yyyyy}$$

$$+\frac{h^4}{2!}(a_2^2a_5a_{13}+2a_2a_5^2a_{13}+a_2^2a_{12}^2+a_5^2a_{13}^2+a_2^2a_8a_{14}+B^2a_9a_{14}+a_2a_8^2a_{14}+2Ba_9^2a_{14}$$

$$+2a_2a_5a_{13}^2+a_4a_5a_{13}^2+D^2a_{14}^2+2a_2a_8a_{14}^2+2Ba_9a_{14}^2+2a_7a_8a_{14}^2+2a_7a_9a_{14}^2+2a_8a_9a_{14}^2$$

$$+2a_2a_8a_9a_{14}+2a_2a_5a_{11}a_{13}+2a_2a_8a_{11}a_{14}+2Ba_9a_{11}a_{14}+4a_2a_5a_{12}a_{13}+2a_2a_4a_{12}a_{13}$$

$$K_5=f+hEff_y+h^2(a_2a_{12}+Ba_{13}+Da_{14})ff_y^2+\frac{h^2}{2!}E^2f^2f_{yy}+h^3(Ba_2a_8+Ba_9a_{14})ff_y^3+\frac{h^3}{3!}(a_2^2a_{12}$$

$$+B^2a_{13}+D^2a_{14}+2a_2a_{12}^2+2Ba_{13}^2+2Da_{14}^2+2a_7a_8a_{14}^2+2a_7a_9a_{11}+2a_4a_5a_{13}+4a_8a_9a_{14}$$

$$+2a_2a_{11}a_{12}+2Ba_{11}a_{13}+2Da_{11}a_{14}+2Ba_{12}a_{13}+4a_2a_{12}a_{13}+2a_7a_{12}a_{14}+2a_2a_{11}+2Ba_{11}a_{13}$$

$$+2Da_{11}a_{14}+2Ba_{12}a_{13}+2Da_{12}a_{14}+2Da_{13}a_{14}++2Ba_{13}a_{14})f^2f_yf_{yy}+\frac{h^3}{3!}E^3f^3f_{yyy}$$

$$+h^4a_2a_5a_9a_{14}ff_y^4\,4a_2a_8a_{12}a_{14}+4Ba_9a_{12}a_{14}+2Da_2a_{12}a_{14}+4Ba_9a_{13}a_{14}+2DBa_{13}a_{14})f^2f_y^2f_{yy}$$

$$+\frac{h^4}{3!}(a_2^3+B^3a_{13}+D^3a_{14}+3a_2a_{12}^3+3Ba_{13}^3+3a_7a_{14}^3+3a_8a_{14}^3+3B^3a_{13}$$

$$+3a_7^2a_8a_{14}+3a_7^2a_9a_{14}+3a_7a_8^2a_{14}+3a_7a_9^2a_{14}+3a_8^2a_9a_{14}+3a_8a_9^2a_{14}+3a_2a_{11}^2a_{12}+3Ba_{11}^2a_{13}$$

$$+3Da_{11}^2a_{14}+6a_2a_{11}a_{12}^2+6Ba_{11}a_{13}^2+6Da_{11}a_{14}^2+3Ba_{12}^2a_{13}+6a_2a_{12}^2a_{13}+3Da_{12}^2a_{14}+6a_2a_{12}^2a_{14}$$

$$+6Ba_{12}a_{13}^2+3a_2a_{12}a_{13}^2+6Da_{12}a_{14}^2+3a_2a_{12}a_{14}^2+3Da_{13}^2a_{14}+6Ba_{13}^2a_{14}+6Da_{13}a_{14}^2+3Ba_{13}a_{14}^2$$

$$+6a_8a_{13}a_{14}^2+6a_9a_{13}a_{14}^2+3Ba_{13}a_{14}^2+6a_7a_8a_9a_{14}+6a_2a_{11}a_{12}a_{13}+6Ba_{11}a_{12}a_{13}+6a_2a_{11}a_{12}a_{14}$$

$$+3Ba_{13}a_{14}^2+6a_7a_8a_9a_{14}+6a_2a_{11}a_{12}a_{13}+6Ba_{11}a_{12}a_{13}+6a_2a_{11}a_{12}a_{14}+6Da_{11}a_{12}a_{14}+6Ba_{11}a_{13}a_{14}$$

$$+6Da_{11}a_{13}a_{14}+6a_2a_{12}a_{13}a_{14}+6Ba_{12}a_{13}a_{14}++6Da_{12}a_{13}a_{14})f^3f_yf_{yyy}+\frac{h^4}{2!}(a_2^2a_{12}^2+B^2a_{13}^2+D^2a_{14}^2$$

$$+a_2^2a_{11}a_{12}+B^2a_{11}a_{13}+D^2a_{11}a_{14}+a_5^2a_{13}a_{14}+2a_7a_8a_{14}^2+2a_7a_9a_{14}^2+2a_8a_9a_{14}^2+2a_4a_5a_{11}a_{13}$$

$$+2a_7a_8a_{11}a_{14}+2a_7a_9a_{11}a_{14}+2a_8a_9a_{11}a_{14}+2a_4a_5a_{12}a_{13}+2a_7a_8a_{12}a_{14}+2a_7a_9a_{12}a_{14}+$$

$$+2a_8a_9a_{12}a_{14}+2a_7a_8a_{13}a_{14}+2a_7a_9a_{13}a_{14}+2a_8a_9a_{13}a_{14}+2a_4a_5a_{13}a_{14})f^3f_{yy}^2+\frac{h^4}{4!}E^4f^4f_{yyyy}$$

$$+\frac{h^5}{2!}(a_2^2a_4a_5a_{13}+a_2^2a_5^2a_{13}+a_2^2a_7a_{11}a_{14}+2B^3a_9a_{14}+a_7^3a_9a_{14}+a_2^2a_{11}a_{12}+2a_4a_7a_8a_9$$

$$+a_7^2a_{11}a_{12}+a_2^5a_2^2+a_2^2a_5^2a_8+a_3^2a_5a_6a_8+2a_3a_4a_5a_6a_8+a_4^2a_5a_6a_8+2a_2a_4a_5$$

$$+2a_3a_4a_6+2a_4^2a_6+a_2a_5a_6+a_3a_5a_6+a_4a_5a_6+a_2a_5^2+2a_2a_3a_5a_7+2a_4a_5a_7+2a_2a_4^2a_5a_7$$

$$+a_2^3a_6^2+2a_2a_4a_5a_6a_7+a_2a_3^2+a_2a_4^2a_6a_7+2a_2a_3a_4+a_2a_4^2+a_2^3a_3+a_2^3a_4+2a_2a_3a_4a_7^2$$

$$+2a_2a_4^2a_7^2+a_3^3+3a_3^2a_4+3a_3a_4^2+a_4^3)f^3f_yf_{yy}^2+\frac{h^5}{2!}(a_2a_4^2a_5a_{13}+a_2a_5^3a_{13}+a_2a_4a_7^2+a_2a_{11}^3a_{14}$$

$$+a_{11}^2a_4a_{12}a_{14}+a_7a_{11}^2a_{12}a_{14}+(a_2+B)D+a_4a_9+a_2^2a_9^3+2a_2^2a_{11}a_{14}^2+2a_2a_4a_{12}a_{14}^2+2a_2a_7a_9a_{14}^2$$

$$+D^2+4Da_4a_7+2a_7^2+4a_7a_8+5a_7a_9+2a_2a_{11}a_{12}a_{14}^2+a_4a_{12}^2a_{14}^2+a_5a_{12}^2a_{14}^2+D^2+4Da_2$$

$$+a_2a_{11}a_{13}^2a_{14}+a_4a_{12}a_{13}^2a_{14}+a_7a_{12}a_{13}^2a_{14}+2a_7^2+B^2+a_{13}^3B^2+a_2a_{11}a_{12}^2a_{14}+Ba_{12}^3a_{14}+a_2^2$$

$$+2a_2a_7a_{11}a_{14}^2+2Ba_8a_9a_{14}^2+D^2+2a_2a_4a_{12}a_{13}^2+2a_2a_7a_9a_{13}^2+a_{12}a_{13}^2B^2+2Ba_2a_{12}^2a_{13}+2a_2a_7a_9a_{13}^2$$

$$+a_{12}a_{13}^2B^2+2Ba_2a_{12}^2a_{13}+a_2^2a_{12}^2a_{13}+a_{11}a_{13}^2B^2+a_2a_{11}a_{14}+Ba_{12}a_{14}+3D^2)f^3f_y^2f_{yyy}+\frac{h^4}{2!}(a_2^2a_5^2a_{13}$$

$$+a_{14}(a_2^2a_8^2+2a_2a_8+Ba_9+2a_8(a_2a_4a_5+a_2a_5^2)+a_{13}a_{14}(a_2a_8+a_9B^2)+a_{12}a_{14}(a_2^2a_8+Ba_2a_9)$$

$$+a_{14}^2D(a_2a_{11}+a_{12}B)f^2f_y^3f_{yy}+\frac{h^5}{2!}(a_2^3a_8a_9+3a_2^3a_9^2)f^4f_{yy}f_{yyy}+\frac{h^5}{4!}(a_2^4a_{12}+4(a_2a_{11}^3a_{12}$$

$$+a_{11}^3a_{13}B+a_{11}^3a_{14}D+3a_2a_{11}^2a_{12}^2+3a_{11}^2a_{12}a_{13}(a_2+B)+3a_{11}^2a_{12}a_{14}(a_2+D)+3a_{11}^2a_{13}a_{14}+BD$$

$$+6a_8a_9a_{13}a_{14}((a_2+B)+(D)+3a_8a_9^2a_{13}(2a_2+B)+3a_2a_8a_9^2+3a_8a_9^2a_{14}(2a_2+D+3a_{11}a_{13}a_{14}^2(2D$$

$$+B+3a_8a_9a_{14}^2+(2D)+a_2)+3a_{11}a_{13}^2a_{14}(D(2B))+3a_{11}^2a_{14}^2D+3a_{11}^2a_{13}^2B+3a_{12}^2a_{13}a_{14}(2a_2+a_4$$

$$+2a_7+a_8+a_9)+3a_{12}a_{13}a_{14}^2(2D+(a_2+B)+3a_{12}^2a_{14}^2(a_2+D)+3a_{12}a_{13}^2a_{14}(D)+(a_2+2B))$$

$$+3a_8a_9a_{13}^2(a_2+2B)+3a_{11}a_{14}^3D+3a_{11}a_{13}^3B+a_{12}a_{13}^3(a_2+3B+a_{12}a_{14}^2(a_2+3D)+a_{12}^3a_{13}(3a_2+B)$$

$$+3a_{12}^2a_{13}^2(a_2+B)+a_{12}^3a_{14}(3a_2+D)+3a_{13}^2a_{14}^2(a_4+2a_7+a_8+a_9)+a_{13}^3a_{14}(D(3B))+a_{13}a_{14}^3(3D)$$

$$+B+a_2a_{12}^4+a_{13}^4B+a_{14}^4D)f^4f_yf_{yyyy}+\frac{h^5}{5!}E^5f^5f_{yyyyy}$$

$$K_6 = f + hFff_y + h^2(a_2a_{17} + Ba_{18} + Da_{19} + Ea_{20})ff_y^2 + \frac{h^2}{2!}F^2f^2f_{yy} + h^3(a_2a_5a_{18} + a_2a_8a_{19}$$
$$+ Ba_9a_{19} + a_2a_{12}a_{20} + Ba_{13}a_{20} + Da_{14}a_{20})ff_y^3 + \frac{h^3}{2!}(a_2^2a_{17} + B^2a_{18} + D^2a_{19} + E^2a_{20} + 2a_2a_{17}^2$$
$$+ 2Ba_{18}^2 + 2Da_{19}^2 + 2Ea_{20}^2 + 4a_4a_5a_{18} + 2a_7a_8a_{19} + 2a_7a_9a_{19} + 2a_8a_9a_{19} + 2a_{11}a_{12}a_{20}$$
$$+ 2a_{11}a_{13}a_{20} + 2a_{11}a_{14}a_{20} + 2a_{12}a_{13}a_{20} + 2a_{12}a_{14}a_{20} + 2a_{13}a_{14}a_{20}2a_8a_9a_{19} + 2a_{11}a_{12}a_{20}$$

$$+ 2a_{11}a_{13}a_{20} + 2a_{11}a_{14}a_{20} + 2a_{12}a_{13}a_{20} + 2a_{12}a_{14}a_{20} + 2a_{13}a_{14}a_{20} + 2a_2a_{16}a_{17} + 2a_4a_{16}a_{18}$$
$$+ 2Da_{16}a_{19} + 2Ea_{16}a_{20} + 2Ba_{17}a_{18} + 2a_2a_{17}a_{18} + 2Da_{17}a_{19} + 2a_2a_{17}a_{19} + 2Ea_{17}a_{20}$$
$$+ 2a_2a_{17}a_{20} + 2Da_{18}a_{19} + 2Ba_{18}a_{19} + 2Ea_{18}a_{20} + 2Ba_{18}a_{20} + 2Ea_{19}a_{20} + 2Da_{19}a_{20})f^2f_yf_{yy}$$
$$+ \frac{h^3}{3!}F^3f^3f_{yyy} + h^4(a_2a_5a_9a_{19} + a_2a_5a_{13}a_{20} + a_2a_8a_{14}a_{20} + Ba_9a_{14}a_{20})ff_y^4 + 2Ba_{12}a_{20}^2$$
$$+ 2Da_{14}a_{20}^2 + 2a_{11}a_{12}a_{20}^2 + 2a_{11}a_{13}a_{20}^2 + 2Ea_{14}a_{20}^2 + 2a_2a_4a_5a_{18} + 2a_2a_4a_9a_{19} + 2a_2a_7a_8a_{19}$$
$$+ 2Ba_7a_9a_{19} + 2a_5a_8a_9a_{19} + 2a_2a_8a_9a_{19} + 4a_4a_5a_{13}a_{20} + 2a_7a_8a_{14}a_{20} + 2a_7a_9a_{14}a_{20} +$$
$$2a_8a_9a_{14}a_{20} + 2a_2a_{11}a_{12}a_{20} + 2Ba_{11}a_{13}a_{20} + 2Da_{11}a_{14}a_{20} + 2Ba_{12}a_{13}a_{20} + 2a_2a_{12}a_{13}a_{20}$$
$$+ 2Da_{12}a_{14}a_{20} + 2a_2a_{12}a_{14}a_{20} + 2Da_{13}a_{14}a_{20} + 2Ba_{13}a_{14}a_{20} + 2a_2a_5a_{16}a_{18} + 2a_2a_8a_{16}a_{19}$$
$$+ 2Ba_9a_{16}a_{19} + +2a_4a_{12}a_{16}a_{20} + 2Ba_{13}a_{16}a_{20} + 2Da_{14}a_{16}a_{20} + 2a_2a_4a_{17}a_{18} + 4a_2a_8a_{17}a_{19}$$
$$+ 2Ba_9a_{17}a_{19} + 4a_2a_5a_{17}a_{18} + 4a_2a_7a_{17}a_{19} + 4a_2a_{12}a_{17}a_{20} + 2Ba_{13}a_{17}a_{20} + 2Da_{14}a_{17}a_{20}$$
$$+ 2a_2a_{11}a_{17}a_{20} + 4a_2a_{13}a_{11}a_{20} + 2a_2a_{14}a_{17}a_{20} + 2a_2a_8a_{18}a_{19} + 2a_4a_9a_{16}a_{19} + 4a_5a_9a_{18}a_{19}$$
$$+ 2Da_4a_{18}a_{19} + 2a_5a_7a_{18}a_{19} + 2a_5a_8a_{18}a_{20} + 2a_2a_5a_{18}a_{19} + 2a_2a_{12}a_{18}a_{20} + 2a_4a_{13}a_{18}a_{20}$$
$$+ 4a_5a_{13}a_{18}a_{20} + 2Da_{14}a_{18}a_{20} + 2Ba_{11}a_{18}a_{20} + 2a_4a_{12}a_{18}a_{20} + 2a_4a_{13}a_{18}a_{20} + 2Ba_{14}a_{18}a_{20}$$
$$+ 2a_5a_{12}a_{18}a_{20} + 2a_2a_5a_{18}a_{20} + 2a_2a_{12}a_{19}a_{20} + 2Ba_{13}a_{19}a_{20} + 4Ea_7a_{19}a_{20} + 2a_{14}a_{19}a_{20}$$
$$+ 2Ea_8a_{19}a_{20} + 2Ea_9a_{19}a_{20} + 2a_2a_8a_{19}a_{20} + 2Ba_9a_{19}a_{20})f^2f_y^2f_{yy} + \frac{h^4}{3!}(a_2^3a_{17} + B^3a_{18}$$
$$+ \frac{D^3}{4}a_{19} + 4a_{11}^3a_{20} + a_{13}^3a_{20} + a_{14}^3a_{20} + 3a_2a_{17}^3 + 3Ba_{18}^3 + 3Da_{19}^3 + 3Ea_{20}^3 + 3B^3a_{18} + 3Da_7a_8a_{19}$$
$$+ 3a_7^2a_9a_{19} + 3a_8^2a_9a_{19} + 3a_8a_9^2a_{19} + +9a_{12}^2a_{13}a_{20} + \frac{h^4}{3!}(a_2^3a_{17} + B^3a_{18} + \frac{D^3}{4}a_{19} + 4a_{11}^3a_{20}$$
$$+ a_{13}^3a_{20} + a_{14}^3a_{20} + 3a_2a_{17}^3 + 3Ba_{18}^3 + 3Da_{19}^3 + 3Ea_{20}^3 + 3a_4^2a_5a_{18} + 3a_4a_5^2a_{18} + 3a_7^2a_8a_{19}$$
$$+ 3a_7^2a_9a_{19} + 3a_7a_8^2a_{19} + 3a_7a_9^2a_{19} + 3a_8^2a_9a_{19} + 3a_8a_9^2a_{19} + 12a_{11}^2a_{12}a_{20} + 12a_{11}^2a_{13}a_{20}$$
$$+ 6a_{11}^2a_{14}a_{20} + 9a_{12}^2a_{13}a_{20} + 9a_{12}^2a_{14}a_{20} + 3a_{13}^2a_{14}a_{20} + 3a_{13}a_{14}^2a_{20} + 3a_2a_{16}^2a_{17} + 3Ba_{16}^2a_{18}$$
$$+ 3a_7a_{16}^2a_{19} + 3a_9a_{16}^2a_{19} + 3Ea_{16}^2a_{20} + +6a_2a_{16}a_{17}^2 + 12a_5a_{16}a_{18}^2 + 6Da_{16}a_{19}^2 + 6Ea_{16}a_{20}^2$$
$$+ 3a_5a_{17}^2a_{18} + 6a_2a_{17}^2a_{19} + 6a_7a_{17}^2a_{19} + 3a_8a_{17}^2a_{19} + 6a_2a_{17}^2a_{20} + 6a_{11}a_{17}^2a_{20} + 6a_{12}a_{17}^2a_{20}$$
$$+ 3a_2a_{17}a_{18}^2 + 6Ba_{17}a_{18}^2 + 3a_2a_{17}a_{19}^2 + 6a_7a_{17}a_{19}^2 + 6a_8a_{17}a_{19}^2 + 3a_2a_{17}a_{20}^2 + 6Ea_{17}a_{20}^2$$
$$+ 3Da_{18}^2a_{19} + 6Ba_{18}^2a_{19} + 6Ba_{18}^2a_{20} + 3Ea_{18}^2a_{20} + 3Ba_{18}a_{19}^2 + 6a_7a_{18}a_{19}^2 3Ba_{17}^2a_{18}$$

$$+6Da_{18}a_{19}^2+3Ba_{18}a_{20}^2+6Ea_{18}a_{20}^2+6Da_{19}^2a_{20}+3Ea_{19}^2a_{20}+3a_7a_{19}a_{20}^2+3a_8a_{19}a_{20}^2$$
$$+3a_9a_{19}a_{20}^2+6Ea_{19}a_{20}^2+6a_7a_8a_9a_{19}+6a_{11}a_{12}a_{13}a_{20}+6a_{11}a_{12}a_{14}a_{20}+6a_{11}a_{13}a_{14}a_{20}$$
$$+6a_{12}a_{13}a_{14}a_{20}+12a_2a_{16}a_{17}a_{18}+6Ba_{16}a_{17}a_{18}+6Da_{16}a_{17}a_{19}+6a_2a_{16}a_{17}a_{20}+6Ba_{16}a_{18}a_{19}$$
$$+6Da_{16}a_{18}a_{19}+6Ea_{16}a_{17}a_{20}+6a_4a_{16}a_{18}a_{20}+6Ea_{16}a_{18}a_{20}+6a_{14}a_{16}a_{19}a_{20}+6Da_{16}a_{19}a_{20}$$
$$+6Ea_{16}a_{19}a_{20}+6a_6a_{17}a_{18}a_{19}+6a_2a_{17}a_{18}a_{19}+6Da_{17}a_{18}a_{19}+6Ba_{17}a_{18}a_{20}+6a_2a_{17}a_{18}a_{20}$$
$$+6Ea_{17}a_{18}a_{20}+6Ea_{17}a_{19}a_{20}+6Da_{17}a_{19}a_{20}+6a_2a_{17}a_{19}a_{20}+6Ea_{18}a_{19}a_{20}+6Da_{18}a_{19}a_{20}$$
$$+6Ba_{18}a_{19}a_{20})f^3f_yf_{yyy}+\frac{h^4}{2!}(a_2^2a_{17}^2+B^2a_{18}^2+a_7^2a_{19}^2+E^2a_{20}^2+a_2^2a_{16}a_{17}+B^2a_{16}a_{18}$$
$$+D^2a_{16}a_{19}+E^2a_{16}a_{20}+B^2a_{17}a_{18}+a_2^2a_{17}a_{18}+a_7^2a_{17}a_{19}+a_8^2a_{17}a_{19}+2a_2^2a_{17}a_{19}$$
$$+a_2^2a_{17}a_{19}+E^2a_{17}a_{20}+a_2^2a_{17}a_{20}+D^2a_{18}a_{19}+2a_4a_5a_{18}^2++B^2a_{18}a_{19}+E^2a_{18}a_{20}$$
$$+B^2a_{18}a_{20}+2Da_8a_{19}^2+E^2a_{19}a_{20}+D^2a_{19}a_{20}+2a_{11}a_{12}a_{20}^2+2a_{11}a_{13}a_{20}^2+2a_{11}a_{14}a_{20}^2$$
$$+2a_{12}a_{13}a_{20}^2+2a_{12}a_{14}a_{20}^2+2a_{13}a_{18}a_{20}^2+2a_4a_5a_{16}a_{18}+2a_7a_8a_{16}a_{19}+2a_8a_9a_{16}a_{19}$$
$$+2a_{11}a_{12}a_{16}a_{20}+2a_{11}a_{13}a_{16}a_{20}+2a_{12}a_{13}a_{16}a_{20}+2a_{12}a_{14}a_{16}a_{20}+2a_{13}a_{14}a_{16}a_{20}$$
$$+2a_4a_5a_{17}a_{18}+2a_7a_8a_{17}a_{20}+2a_7a_9a_{17}a_{19}+2a_8a_9a_{17}a_{19}+2a_{11}a_{12}a_{17}a_{20}$$
$$+2a_{11}a_{13}a_{17}a_{20}+2a_{11}a_{14}a_{17}a_{20}+2a_{12}a_{13}a_{17}a_{20}+2a_{12}a_{14}a_{17}a_{20}+2a_{13}a_{14}a_{17}a_{20}$$
$$+2a_7a_8a_{18}a_{19}+2a_4a_9a_{18}a_{19}+2a_8a_9a_{18}a_{19}+2a_4a_5a_{18}a_{19}+2a_{11}a_{12}a_{18}a_{20}$$
$$+2a_{11}a_{13}a_{18}a_{20}+2a_{11}a_{14}a_{18}a_{20}+2a_{12}a_{13}a_{18}a_{20}+2a_{12}a_{14}a_{18}a_{20}+2a_{13}a_{14}a_{18}a_{20}$$
$$+2a_{11}a_{12}a_{19}a_{20}+2a_{11}a_{13}a_{19}a_{20}+2a_{11}a_{14}a_{19}a_{20}+2a_{12}a_{13}a_{19}a_{20}+2a_{12}a_{14}a_{19}a_{20}$$
$$+2a_{13}a_{14}a_{19}a_{20}+2a_7a_8a_{19}a_{20}+2a_7a_9a_{19}a_{20}+2a_8a_9a_{19}a_{20})f^3f_{yy}^2+\frac{h^4}{4!}F^4f^4f_{yyyy}$$
$$+\frac{h^5}{5!}F^5f^5f_{yyyyy}.$$

Putting the $K_{j's}$ (j=1, 2, 3, 4, 5, 6) into $y_{n+1}-y_n=h\phi(x_n,y_n;h)$ and collecting like terms, we have that

$$\Rightarrow h\phi(x_n,y_n;h)=h(c_1+c_2+c_3+c_4+c_5+c_6)f \qquad +h^2\left[\left((c_2a_2+c_3B+c_4D+C_5E+C_6F)ff_y\right)\right]$$
$$+h^3\left[\left((C_3a_2a_5+C_4(a_2a_8+Ba_6)+C_5(a_2a_{12}+a_{13}Ba_{14}D+C_6(a_2a_{17}+a_{18}B+a_{19}D+a_{20}E)ff_y^2\right)\right]$$

$$+\frac{h^3}{2!}\left[\left((C_2a_2^2+C_3B^2+C_4D^2+C_5E^2+C_6F^2f^2f_{yy}\right)\right]$$
$$+\frac{h^4}{3!}\left[\left((C_2a_2^3+C_3B^3+C_4D^3c_5E^3+c_6F^3)f^3f_{yyy}+\right)\right]$$
$$+h^4\left[\left(\begin{matrix}(C_4(a_2a_5a_9+C_5(a_2a_5a_{13}+a_2a_8a_{14}+Ba_9a_{14})+C_6(a_2a_5a_{18}+a_2a_8a_{19}+Ba_9a_{19}+a_2a_{12}a_{20}\\+Ba_{13}a_{20}+Da_{14}a_{20})ff_y^3+\end{matrix}\right)\right]$$
$$+\frac{h^4}{2}\left(\begin{matrix}(C_3(a_2^2a_5+2a_2a_4a_5+2a_2a_5^2)+C_4(a_2^2a_8+B^2a_9+2Ba_9^2+2a_4a_5a_9+2a_2a_7a_8\\+2a_4a_7a_9+2Ba_8a_9+2a_2a_8a_9)+C_5G+C_6H)f^2f_yf_{yy}\end{matrix}\right)$$
$$+\frac{h^5}{2!}\left(C_3(a_2{}^2a_5{}^2)+C_5J+C_6K)f^2f_y^2f_{yy}\right)$$

$$+\frac{h^5}{3!}\begin{pmatrix} C_3(a_4^3a_5+3a_4^2a_5+6a_2a_4a_5^2+3a_2a_5{}^3)+C_4(a_2^3a_8+B^3a_9+3a_5^2a_4a_9+3a_5a_4^2a_9+3a_2a_8^3 \\ +3a_4a_9^3+3a_5^3a_9+3a_2a_7^2a_8+6a_2a_7a_8a_9+3a_5a_7^2a_9++6a_4a_7a_8a_9+6a_2a_7a_8^2+6a_4a_7a_8a_9 \\ +6Ba_7a_9^2+3Ba_8^2a_9+3a_4a_7^2a_9+6a_2a_8^2a_9+6Ba_8a_9^2+3a_2a_8a_9^2+)+C_5L+C_6M)f^3f_yf_{yyy} \end{pmatrix}$$

$$+\frac{h^5}{4!}\left[\left(C_2a_2{}^4+C_3B^4+C_4(D^4+C_5E^4+C_6F^4f^4f_{yyyy}\right)\right]$$

$$+h^4\left[\left(C_4(a_2^2a_7a_8a_9)f^2f_y^3\right)\right]$$

$$+h^5\left[\left(C_5(a_2a_5a_9a_{14})+C_6(a_2a_5a_9a_{19}+a_2a_5a_{13}a_{20}+a_2a_8a_{14}a_{20}+Ba_9a_{14}a_{20})ff_y{}^4\right)\right]$$

$$+\frac{h^6}{2!}\begin{pmatrix} C_3a_2^3a_5^2+C_4(a_2^2a_4a_7a_9+a_2^2a_9+a_2^3a_7a_9+2a_2a_4a_5^2a_9+2a_2a_7^3a_9+a_2^3a_8a_9+4a_2a_5a_7a_8a_9 \\ +3a_2a_7^2a_8a_9+a_2a_4^2a_8a_9+a_2^2a_4a_8a_9+a_2^2a_7a_8a_9+a_2^3a_8^2+a_2^3a_9^2+2a_2a_4a_7a_9^2+2a_2a_7^2a_9^2 \\ +(a_4+a_5)^3+C_5N)f^3f_yf_{yy}^2 \end{pmatrix}$$

$$+\frac{h^6}{2!}\left[\begin{pmatrix} C_4(a_2^2a_7^2a_9^2+2a_2a_7a_9B)+C_5(a_2^2a_5^2a_{13}+a_{14}(a_2^2a_8^2+2a_2a_8+a_9B+2a_8(a_2a_4a_5+a_2a_5^2)+ \\ a_{13}a_{14}(a_2a_8+a_9B^2)+a_{12}a_{14}(a_2^2a_8+Ba_2a_9)+a_{14}^2D(a_2a_{11}+a_{12}B)f^2f_y^3f_{yy}+ \end{pmatrix}\right]$$

$$+\frac{h^6}{2!}\left[\begin{pmatrix} C_3(Ba_2^2a_5^2)+C_4(a_2^3a_7a_9+3a_2a_4^2a_7a_9+6a_2a_4a_7^2a_9+6a_2a_7^3a_9+a_2^2a_7a_8^2+4a_2a_7^2a_8a_9 \\ +2a_2a_4a_7a_8a_9+2a_2a_4a_8^2a_9+6a_2a_7a_8^2a_9+2a_2^2a_8^2a_9+4a_2a_7a_8a_9^2+a_4a_8a_9^2+a_7a_8a_9^2 \\ +2a_2a_4a_8a_9^2+a_2^2a_8^3+\frac{1}{3}(a_2a_7a_9^3+2a_9^3B^2)+C_5P)f^3f_y^2f_{yyy} \end{pmatrix}\right]$$

$$+\frac{h^6}{2!}\left[\left(C_3(a_2^3a_4a_5+a_2^3a_5^2+\frac{3}{2}a_2^2a_4^2a_5+\frac{3}{2}a_2^2a_5^2)+C_4Q)f^4f_{yy}f_{yyy}\right)\right]$$

$$+\frac{h^6}{4!}\left[\begin{pmatrix} C_3(a_2{}^4a_4+4a_2a_4{}^3a_5+8a_2a_4{}^2a_5{}^2+4a_2a_5{}^4+12a_2a_4a_5)+C_4R+ \\ C_5T)f^4f_yf_{yyyy} \end{pmatrix}\right]$$

$$+\frac{h^6}{5!}\left[\left(C_2a_2{}^5+C_3B^5+C_4D^5+C_5E^5+C_6F^5f^5f_{yyyyy}\right)\right]$$

The Taylor series expansion for h^6 (y derivatives only) is shown below:

$$y_{n+1}-y_n=h\phi(x,y;h)=hf+\frac{h^2}{2!}ff_y+\frac{h^3}{3!}(ff_y^2+f^2f_{yy})+\frac{h^4}{4!}(4f^2f_yf_{yy}+ff_y^3+f^3f_{yyy})+$$
$$\frac{h^5}{5!}(11f^2f_y^2f_{yy}+f^3f_{yy}^2+7f^3f_yf_{yyy}+ff_y^4+f^4f_{yyyy})+\frac{h^6}{6!}(38f^3f_yf_{yy}^2+32f^3f_y^2f_{yyy}+$$
$$4ff_yf_{yyy}+f^5f_{yyyyy}+26f^2f_y^3f_{yy}+ff_y^5+15f^4f_{yy}f_{yyy}+7f^4f_yf_{yyyy})$$

.Then equating coefficient with Taylor series expansion above , we have:

$$\Rightarrow (C_1+C_2+C_3+C_4+C_5+C_6)=1 \quad (3.2)$$

$$\Rightarrow C_2a_2+C_3B+C_4D+C_5E+C_6F=\frac{1}{2} \quad (3.3)$$

$$\Rightarrow C_3a_2a_5+C_4a_2a_8+C_4a_9B+C_5a_2a_{12}+C_5a_{13}B+C_5a_{14}D+C_6a_2a_{17}+C_6a_{18}B+C_6a_{19}D+C_6a_{20}E=\frac{1}{6}$$ (3.4)

$$\Rightarrow C_2a_2^2+C_3B^2+C_4D^2+C_5E^2+C_6F^2=\frac{1}{3.}$$ (3.5)

$$\Rightarrow C_2a_2^3+C_3B^3+C_4D^3+C_5E^3+C_6F^3=\frac{1}{4}$$ (3.6)

$$\Rightarrow C_4(a_2a_5a_9)+C_5(a_2a_5a_{13}+a_2a_8a_{14}+Ba_9a_{14})+C_6(a_2a_5a_{18}+a_2a_8a_{19}+Ba_9a_{19}$$
$$+a_2a_{12}a_{20}+Ba_{13}a_{20}+Da_{14}a_{20})=\frac{1}{24}$$ (3.7)

$$\Rightarrow(C_3(a_2^2a_5+2a_2a_5B)+C_4(a_2^2a_8+B^2a_9+2Ba_9^2+2a_4a_5a_9+2a_2a_7a_8+2a_4a_7a_9$$
$$+2Ba_8a_9+2a_2a_8a_9)+C_5(a_2^2a_{12}+B^2a_{13}+a_7^2a_{14}+3a_8^2a_{14}+2a_2a_{12}^2+2Ba_{13}^3+2Da_{14}^2$$
$$+2a_7a_8a_{14}^2+2a_7a_8a_{14}+2a_7a_9a_{14}+2a_4a_5a_{13}+2a_8a_9a_{14}+2a_2a_{11}a_{12}+2Ba_{11}a_{13}$$
$$+2Ba_{12}a_{13}+2a_2a_{12}a_{13}+2Da_{12}a_{14}+2a_9a_{12}a_{14}+2a_2a_{12}a_{14}+2a_7a_{13}a_{14}+2a_8a_{13}a_{14})$$
$$+C_6(a_2^2a_{17}+B^2a_{18}+D^2a_{19}+E^2a_{20}++2a_2a_{17}^2+2Ba_{18}^2+2Da_{19}^2+2Ea_{20}^2+4a_4a_5a_{18}$$
$$+2a_7a_8a_{19}+2a_7a_9a_{19}+2a_8a_9a_{19}+2Ea_{12}a_{20}+2a_{12}a_{13}a_{20}+2a_{12}a_{14}a_{20}+2a_{13}a_{14}a_{20}$$
$$+2a_2a_{16}a_{17}+2a_4a_{16}a_{18}+2Da_{16}a_{19}+2Ea_{16}a_{20}+2Ba_{17}a_{18}+2a_2a_{17}a_{18}+2Da_{17}a_{19}$$
$$+2a_2a_{17}a_{19}+2Ea_{17}a_{20}+2a_2a_{17}a_{20}+2Da_{18}a_{19}+2Ba_{18}a_{19}+2Ea_{18}a_{20}+2Ba_{18}a_{20}$$
$$+2Da_{11}a_{14}+2Ea_{19}a_{20}+2Da_{19}a_{20})=\frac{1}{3}$$ (3.8)

$$\Rightarrow C_3(a_2{}^2a_4a_5+a_2{}^2a_5^2)+C_4(a_2^2a_7a_8+a_4^2a_7a_8+4B^2a_7a_9+a_2^2a_8^2+a_4^2a_9^2+a_5^2a_8a_9$$
$$+a_2^2a_8a_9+4a_4a_5a_9^2)+C_5(a_2^2a_{12}^2+B^2a_{13}^2+D^2a_{14}^2+a_2^2a_{11}a_{12}+B^2a_{11}a_{13}+D^2a_{11}a_{14}$$
$$+a_5^2a_{13}a_{14}+2a_7a_8a_{14}^2+2a_7a_9a_{14}^2+2a_8a_9a_{14}^2+2a_4a_5a_{11}a_{13}+2a_7a_8a_{11}a_{14}+2a_7a_9a_{11}a_{14}$$
$$+2a_8a_9a_{11}a_{14}+2a_4a_5a_{12}a_{13}+2a_7a_8a_{12}a_{14}+2a_7a_9a_{12}a_{14}+2a_8a_9a_{12}a_{14}+2a_7a_8a_{13}a_{14}$$
$$+2a_7a_9a_{13}a_{14}+2a_8a_9a_{13}a_{14}+2a_4a_5a_{13}a_{14})+C_6(a_2^2a_{17}^2+B^2a_{18}^2+a_7^2a_{19}^2+E^2a_{20}^2+a_2^2a_{16}a_{17}$$
$$+B^2a_{16}a_{18}+D^2a_{16}a_{19}+B^2a_{16}a_{18}+D^2a_{16}a_{19}+E^2a_{16}a_{20}+B^2a_{17}a_{18}+a_2^2a_{17}a_{18}+D^2a_{17}a_{19}$$
$$+a_2^2a_{17}a_{19}+E^2a_{17}a_{20}+a_2^2a_{17}a_{20}+D^2a_{18}a_{19}+2a_4a_5a_{18}^2+B^2a_{18}a_{19}+E^2a_{18}a_{20}$$
$$+B^2a_{18}a_{20}+2a_7a_8a_{19}^2+2a_7a_9a_{19}^2+2a_8a_9a_{19}^2+E^2a_{18}a_{19}+D^2a_{19}a_{20}+2a_{11}a_{12}a_{20}^2$$
$$+2a_{11}a_{13}a_{20}^2+2a_{11}a_{14}a_{20}^2+2a_{12}a_{13}a_{20}^2+2a_{12}a_{14}a_{20}^2+2a_{13}a_{18}a_{20}^2+2a_4a_5a_{16}a_{18}$$
$$+2a_7a_8a_{16}a_{19}+2a_8a_9a_{16}a_{19}+2a_{11}a_{12}a_{16}a_{20}+2a_{11}a_{13}a_{16}a_{20}+2a_{12}a_{13}a_{16}a_{20}$$
$$+2a_{12}a_{14}a_{16}a_{20}+2a_{13}a_{14}a_{16}a_{20}+2a_4a_5a_{17}a_{18}+2a_7a_8a_{17}a_{20}+2a_7a_9a_{17}a_{19}$$
$$+2a_8a_9a_{17}a_{19}+2a_{11}a_{12}a_{17}a_{20}+2a_{11}a_{13}a_{17}a_{20}+2a_{11}a_{14}a_{17}a_{20}+2a_{12}a_{13}a_{17}a_{20}$$
$$+2a_{12}a_{14}a_{17}a_{20}+2a_{13}a_{14}a_{17}a_{20}+2a_7a_8a_{18}a_{19}+2a_4a_9a_{18}a_{19}+2a_8a_9a_{18}a_{19}$$
$$+2a_4a_5a_{18}a_{19}+2a_{11}a_{12}a_{18}a_{20}+2a_{11}a_{13}a_{18}a_{20}+2a_{11}a_{14}a_{18}a_{20}+2a_{12}a_{13}a_{18}a_{20}$$
$$+2a_{12}a_{14}a_{18}a_{20}+2a_{13}a_{14}a_{18}a_{20}+2a_{11}a_{12}a_{19}a_{20}+2a_{11}a_{13}a_{19}a_{20}+2a_{11}a_{14}a_{19}a_{20}$$
$$+2a_{12}a_{13}a_{19}a_{20}+2a_{12}a_{14}a_{19}a_{20}+2a_{13}a_{14}a_{19}a_{20}+2a_7a_8a_{19}a_{20}+2a_7a_9a_{19}a_{20}$$
$$+2a_8a_9a_{19}a_{20})=\frac{1}{60}$$ (3.9)

$$\Rightarrow C_2 a_2{}^4 + C_3 B^4 + C_4 D^4 + C_5 E^4 + C_6 F^4 = \frac{1}{5} \tag{3.10}$$

$$\Rightarrow C_5(a_2 a_5 a_9 a_{14}) + C_6(a_2 a_5 a_9 a_{19} + a_2 a_5 a_{13} a_{20} + a_2 a_8 a_{14} a_{20} + B a_9 a_{14} a_{20}) = \frac{1}{120} \tag{3.11}$$

$$\Rightarrow C_4(a_2^2 a_7^2 a_9^2 + 2a_2 a_7 a_9 B) + C_5(a_2^2 a_5^2 a_{13} + a_{14}(a_2^2 a_8^2 + 2a_2 a_8 + a_9 B + 2a_8(a_2 a_4 a_5$$
$$+ a_2 a_5^2) + a_{13} a_{14}(a_2 a_8 + a_9 B^2) + a_{12} a_{14}(a_2^2 a_8 + B a_2 a_9) + a_{14}^2 D + (a_2 a_{11} + a_{12} B) = \frac{13}{180} \tag{3.12}.$$

$$\Rightarrow C_2 a_2{}^5 + C_3 B^5 + C_4 D^5 + C_5 E^5 + C_6 F^5 = \frac{1}{6} \tag{3.13}$$

By setting $C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = \frac{1}{6}.$ (3.14)

From (2,3), we have $\frac{a_2}{6} + \frac{B}{6} + \frac{D}{6} + \frac{E}{6} + \frac{F}{6} = \frac{1}{2}$

From (2.5), we have : $\frac{a_2^2}{6} + \frac{B^2}{6} + \frac{D^2}{6} + \frac{E^2}{6} + \frac{F^2}{6} = \frac{1}{3}$

From (2.6), we have : $\frac{a_2^3}{6} + \frac{B^3}{6} + \frac{D^3}{6} + \frac{E^3}{6} + \frac{F^3}{6} = \frac{1}{4}$

$$\Rightarrow \quad a_2 + B + D + E + F = 3$$
$$a_2^2 + B^2 + D^2 + E^2 + F^2 = 2$$
$$a_2^3 + B^3 + D^3 + E^3 + F^3 = \frac{3}{2} \tag{3.15}$$

Hence equation (2.15) borders on undetermined co-efficient, therefore to solve the equations we find values that best solve the equations since the equations are linear, quadratic and cubic. choosing the best values that best solve the equations first we consider (1.6) for these values and then (1.7) and these values are:

$$a_2 = B = D = E = \frac{1}{2} \text{ and } F = 1 \tag{3.16}$$

From (2.4), we have: $\frac{a_5}{12} + \frac{a_8}{12} + \frac{a_9}{12} + \frac{a_{12}}{12} + \frac{a_{13}}{12} + \frac{a_{14}}{12} + \frac{a_{17}}{12} + \frac{a_{18}}{12} + \frac{a_{19}}{12} + \frac{a_{20}}{12} = \frac{1}{6}$

Hence: $a_5 + a_8 + a_9 + a_{12} + a_{13} + a_{14} + a_{17} + a_{18} + a_{19} + a_{20} = 2$ (3.17)

From (2.7), we have: $a_5 a_9 + a_5 a_{13} + a_8 a_{14} + a_9 a_{14} + a_5 a_{18} + a_8 a_{19} + a_9 a_{19} + a_{12} a_{20} + a_{13} a_{20} + a_{14} a_{20} = \frac{1}{2}$

(3.18)

Since (2.17) and (2.18) cannot be solve simultaneously, we resolve them as follows:

$$a_2 = \frac{1}{2}, a_4 = \frac{1}{4}, a_5 = \frac{1}{4}, a_7 = \frac{1}{8}, a_8 = \frac{1}{8}, a_9 = \frac{1}{4}, a_{11} = \frac{-1}{8}, a_{12} = \frac{1}{8}, a_{13} = \frac{1}{4}, a_{14} = \frac{1}{4}, a_{17} = \frac{1}{8},$$
$$a_{18} = \frac{1}{8}, a_{19} = \frac{1}{4}, a_{20} = \frac{1}{4}$$

With $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = \frac{1}{6}$

These results lead us to our sixth stage sixth-order Runge-Kutta method below

$$y_{n+1} - y_n = \frac{h}{6}(k_1 + k_2 + k_3 + k_4 + k_5 + k_6)$$

Where

$$k_1 = f(y_n)$$

$$k_2 = f(y_n + \frac{h}{2}k_1)$$

$$k_3 = f(y_n + \frac{h}{4}(k_1 + k_2))$$

$$k_4 = f(y_n + \frac{h}{8}(k_1 + k_2 + 2k_3))$$

$$k_5 = f(y_n + \frac{h}{8}(-k_1 + k_2 + 2k_3 + 2k_4))$$

$$k_6 = f(y_n + \frac{h}{8}(2k_1 + k_2 + k_3 + 2k_4 + 2k_5))$$

$B = (a_4 + a_5)$,

$D = (a_7 + a_8 + a_9)$,

$E = (a_{11} + a_{12} + a_{13} + a_{14})$ *and*

$F = (a_{16} + a_{17} + a_{18} + a_{19} + a_{20})$

G =

$$\begin{aligned}&a_2^2a_{12} + a^2a_{13} + a_5^2a_{13} + a_7^2a_{14} + 3a_8^2a_{14} + 2a_2a_{12}^2 + \\ &2a_4a_{13}^3 + 2a_5a_{13}^2 + 2a_7a_{13}^2 + 2a_8a_{14}^2 + 2a_9a_{14}^2 + 2a_7a_8a_{14}^2 + 2a_7a_8a_{14} + 2a_7a_9a_{14} + 2a_4a_5a_{13} + \\ &2a_8a_9a_{14} + 2a_2a_{11}a_{12} + 2a_4a_{11}a_{13} + 2a_5a_{11}a_{13} + 2a_7a_{11}a_{14} + 4a_8a_{11}a_{14} + 2a_9a_{11}a_{14} + 2a_4a_{12}a_{13} + \\ &2a_5a_{12}a_{13} + 2a_2a_{12}a_{13} + 2a_7a_{12}a_{14} + 2a_8a_{12}a_{14} + 4a_9a_{12}a_{14} + 2a_2a_{12}a_{14} + 2a_7a_{13}a_{14} + 2a_8a_{13}a_{14}\end{aligned}$$

H =

$$\begin{aligned}&a_2^2a_{17} + a_4^2a_{18} + a_5^2a_{18} + a_7^2a_{19} + a_8^2a_{19} + a_9^2a_{19} + a_{11}^2a_{20} + a_{12}^2a_{20} + a_{13}^2a_{20} + a_{14}^2a_{20} + 2a_2a_{17}^2 + \\ &2a_4a_{18}^2 + 2a_5a_{18}^2 + 2a_7a_{19}^2 + 2a_5a_{19}^2 + 2a_9a_{19}^2 + 2a_{11}a_{20}^2 + 2a_{12}a_{20}^2 + 2a_{13}a_{20}^2 + 2a_{14}a_{20}^2 + 2a_4a_5a_{18} + \\ &2a_4a_5a_{18} + 2a_7a_8a_{19} + 2a_7a_9a_{19} + 2a_8a_9a_{19} + 2a_{11}a_{12}a_{20} + 2a_{11}a_{13}a_{20} + 2a_{11}a_{14}a_{20} + 2a_{12}a_{13}a_{20} + \\ &2a_{12}a_{14}a_{20} + 2a_{13}a_{14}a_{20} + 2a_2a_{16}a_{17} + 2a_4a_{16}a_{18} + 2a_7a_{16}a_{19} + 2a_8a_{16}a_{19} + 2a_9a_{16}a_{19} + 2a_{11}a_{16}a_{20} \\ &+ 2a_{12}a_{16}a_{20} + 2a_{13}a_{16}a_{20} + 2a_{14}a_{16}a_{20} + 2a_4a_{17}a_{18} + 2a_5a_{17}a_{18} + 2a_2a_{17}a_{18} + 2a_7a_{17}a_{19} + 2a_8a_{17}a_{19} \\ &+ 2a_9a_{17}a_{19} + 2a_2a_{17}a_{19} + 2a_{11}a_{17}a_{20} + 2a_{12}a_{17}a_{20} + 2a_{13}a_{17}a_{20} + 2a_{14}a_{17}a_{20} + 2a_2a_{17}a_{20} + 2a_7a_{18}a_{19} \\ &+ 2a_8a_{18}a_{19} + 2a_6a_{18}a_{19} + 2a_4a_{18}a_{19} + 2a_5a_{18}a_{19} + 2a_{11}a_{18}a_{20} + 2a_{12}a_{18}a_{20} + 2a_{13}a_{18}a_{20} + 2a_{14}a_{18}a_{20} \\ &+ 2a_4a_{18}a_{20} + 2a_5a_{18}a_{20} + 2a_{11}a_{19}a_{20} + 2a_{12}a_{19}a_{20} + 2a_{13}a_{19}a_{20} + 2a_{14}a_{19}a_{20} + 2a_7a_{19}a_{20} + 2a_8a_{19}a_{20} \\ &+ 2a_9a_{19}a_{20}\end{aligned}$$

J =

$$\begin{aligned}&a_2^2a_5a_{13} + 2a_2a_5^2a_{13} + a_2^2a_{12}^2 + a_5^2a_{13}^2 = a_2^2a_8a_{14} + a_4^2a_9a_{14} + a_5^2a_9a_{14} + a_2a_8^2a_{14} + a_4a_9^2a_{14} \\ &+ 2a_5a_9^2a_{14} + a_4a_9^2a_{14} + 2a_2a_5a_{13}^2 + a_4a_5a_{13}^2 + a_7^2a_{14}^2 + a_8^2a_{14}^2 + a_9^2a_{14}^2 + 2a_2a_8a_{14}^2 + 2a_4a_9a_{14}^2 + 2a_5a_9a_{14}^2 \\ &+ 2a_7a_8a_{14}^2 + 2a_7a_9a_{14}^2 + 2a_8a_9a_{14}^2 + 2a_4a_5a_{13} + 2a_4a_5a_9a_{14} + 2a_2a_7a_8a_{14} + 2a_4a_7a_9a_{14} + 2a_5a_7a_9a_{14} + \\ &2a_4a_8a_9a_{14} + 2a_5a_8a_9a_{14} + 2a_2a_8a_9a_{14} + 2a_2a_5a_{11}a_{13} + 2a_2a_8a_{11}a_{14} + 2a_4a_9a_{11}a_{14} + 2a_5a_9a_{11}a_{14} + \\ &4a_2a_5a_{12}a_{13} + 2a_2a_4a_{12}a_{13} + 4a_2a_8a_{12}a_{14} + 2a_4a_9a_{12}a_{14} + 2a_5a_9a_{12}a_{14} + 2a_2a_7a_{12}a_{14} + 2a_2a_9a_{12}a_{14} + \\ &2a_2a_8a_{13}a_{14} + 2a_4a_9a_{13}a_{14} + 4a_5a_9a_{13}a_{14} + 2a_4a_7a_{13}a_{14} + 2a_4a_8a_{13}a_{14} + 2a_4a_9a_{13}a_{14} + 2a_5a_7a_{13}a_{14} + \\ &2a_5a_8a_{13}a_{14} + 2a_2a_5a_{13}a_{14}\end{aligned}$$

K = $a_2^2a_5a_{18} + a_2^2a^2{}_{17} + 2a_2a_5^2a_{18} + a_2^2a_8a_{19} + a_4^2a_9a_{19} + a_4^2a_9a_{19} +$
$+ a_5^2a_9a_{19} + 2a_2a_8^2a_{19} + 2a_4a_8^2a_{19} + 3a_4a_9^2a_{19} + 2a_5a_9^2a_{19} + a_2^2a_{12}a_{20} + a_4^2a_{18}^2 + a_4^2a_{13}a_{20} + a_5^2a_{13}a_{20} + a_7^2a_{14}a_{20}$
$+ a_8^2a_{14}a_{20} + a_9^2a_{14}a_{20} + 2a_2a_{12}^2a_{20} + 2a_4a_{13}^2a_{20} + 2a_5a_{13}^2a_{20} + 2a_7a_{14}^2a_{20} + 2a_{11}a_{14}^2a_{20} + 2a_9a_{14}^2a_{20} + a_5^2a_{18}^2 + a_7^2a_{19}^2$
$+ 2a_2a_5a_{18}^2 + 2a_4a_5a_{18}^2 + a_8^2a_{19}^2 + a_9^2a_{19}^2 + a_{11}^2a_{20}^2 + a_{12}^2a_{20}^2 + a_{13}^2a_{20}^2 + a_{14}^2a_{20}^2 + 2a_2a_8a_{19}^2 + 2a_4a_9a_{19}^2 + 4a_5a_9a_{19}^2$
$+ 2a_7a_8a_{19}^2 + 2a_7a_9a_{19}^2 + 2a_2a_{12}a_{20}^2 + 2a_8a_9a_{19}^2 + 2a_4a_{12}a_{20}^2 + 2a_5a_{13}a_{20}^2 + 2a_7a_{14}a_{20}^2 + 4a_8a_{14}a_{20}^2 + 4a_9a_{14}a_{20}^2$
$+ 2a_{11}a_{12}a_{20}^2 + 2a_{11}a_{13}a_{20}^2 + 2a_{11}a_{14}a_{20}^2 + 2a_{12}a_{13}a_{20}^2 + 2a_{12}a_{14}a_{20}^2 + 2a_{13}a_{14}a_{20}^2 + 2a_2a_4a_5a_{18} + 2a_2a_4a_9a_{19}$
$+ 2a_2a_7a_8a_{19} + 2a_4a_7a_9a_{19} + 2a_5a_7a_9a_{19} + 2a_5a_8a_9a_{19} + 2a_2a_8a_9a_{19} + 2a_4a_5a_{13}a_{20} + 2a_4a_5a_{13}a_{20} + 2a_7a_8a_{14}a_{20}$
$+ 2a_7a_9a_{14}a_{20} + 2a_8a_9a_{14}a_{20} + 2a_2a_{11}a_{12}a_{20} + 2a_4a_{11}a_{13}a_{20} + 2a_5a_{11}a_{13}a_{20} + 2a_7a_{11}a_{14}a_{20} + 2a_8a_{11}a_{14}a_{20}$
$+ 2a_9a_{11}a_{14}a_{20} + 2a_4a_{12}a_{13}a_{20} + 2a_5a_{12}a_{13}a_{20} + 2a_2a_{12}a_{13}a_{20} + 2a_7a_{12}a_{14}a_{20} + 2a_8a_{12}a_{14}a_{20} + 2a_9a_{12}a_{14}a_{20}$
$+ 2a_2a_{12}a_{14}a_{20} + 2a_7a_{13}a_{14}a_{20} + 2a_8a_{13}a_{14}a_{20} + 2a_9a_{13}a_{14}a_{20} + 2a_4a_{13}a_{14}a_{20} + 2a_5a_{13}a_{18}a_{20} + 2a_2a_5a_{16}a_{18}$
$+ 2a_2a_8a1_6a_{19} + 2a_4a_9a_{16}a_{19} + 2a_5a_9a_{16}a_{19} + 2a_4a_{12}a_{16}a_{20} + 2a_4a_{13}a_{16}a_{20} + 2a_5a_{13}a_{16}a_{20} + 2a_2a_{14}a_{16}a_{20}$
$+ 2a_8a_{14}a_{16}a_{20} + 2a_9a_{14}a_{16}a_{20} + 2a_2a_4a_{17}a_{18} + 4a_2a_8a_{17}a_{19} + 2a_4a_9a_{17}a_{19} + 2a_5a_9a_{17}a_{19} + 4a_2a_5a_{17}a_{18}$
$+ 2a_2a_7a_{17}a_{19} + 2a_2a_7a_{17}a_{19} + 4a_2a_{12}a_{17}a_{20} + 2a_4a_{13}a_{17}a_{20} + 2a_5a_{13}a_{17}a_{20} + 2a_7a_{14}a_{17}a_{20} + 2a_8a_{14}a_{17}a_{20}$
$+ 2a_9a_{14}a_{18}a_{20} + 2a_2a_{11}a_{17}a_{20} + 4a_2a_{13}a_{11}a_{20} + 2a_2a_{14}a_{17}a_{20} + 2a_2a_8a_{18}a_{19} + 2a_4a_9a_{16}a_{19} + 4a_5a_9a_{18}a_{19}$
$+ 2a_4a_7a_{18}a_{19} + 2a_4a_8a_{18}a_{19} + 2a_4a_9a_{18}a_{19} + 2a_5a_7a_{18}a_{19} + 2a_5a_8a_{18}a_{20} + 2a_2a_5a_{18}a_{19} + 2a_2a_{12}a_{18}a_{20}$
$+ 2a_4a_{13}a_{18}a_{20} + 4a_5a_{13}a_{18}a_{20} + 2a_7a_{14}a_{18}a_{20} + 2a_8a_{14}a_{18}a_{20} + 2a_9a_{11}a_{18}a_{20} + 2a_4a_{11}a_{18}a_{20} + 2a_4a_{12}a_{18}a_{20}$
$+ 2a_4a_{13}a_{18}a_{20} + 2a_4a_{14}a_{18}a_{20} + 2a_5a_{11}a_{18}a_{20} + 2a_5a_{12}a_{18}a_{20} + 2a_5a_{14}a_{18}a_{20} + 2a_2a_5a_{18}a_{20} + 2a_2a_{12}a_{19}a_{20}$
$+ 2a_4a_{13}a_{19}a_{20} + 2a_5a_{13}a_{19}a_{20} + 4a_7a_{14}a_{19}a_{20} + 2a_8a_{14}a_{19}a_{20} + 2a_7a_{11}a_{19}a_{20} + 2a_7a_{12}a_{19}a_{20} + 2a_7a_{13}a_{19}a_{20}$
$+ 2a_8a_{11}a_{19}a_{20} + 2a_8a_{12}a_{19}a_{20} + 2a_8a_{13}a_{19}a_{20} + 2a_8a_{14}a_{19}a_{20} + 2a_9a_{11}a_{19}a_{20} + 2a_9a_{12}a_{19}a_{20} + 2a_9a_{13}a_{19}a_{20}$
$+ 2a_9a_{14}a_{19}a_{20} + 2a_2a_8a_{19}a_{20} + 2a_4a_9a_{19}a_{20} + 2a_5a_9a_{19}a_{20}$

L=
$a_2^3 + a_4^3a_{13} + a_5^3a_{13} + a_7^3a_{14} + a_8^3a_{14} + a_9^3a_{14} + 3a_2a_{12}^3 + 3a_4a_{13}^3 + 3a_4a_{13}^3 + 3a_7a_{14}^3 + 3a_8a_{14}^3 + 3a_4^2a_5a_{13}$
$+ 3a_4a_5^2a_{13} + 3a_7^2a_8a_{14} + 3a_7^2a_9a_{14} + 3a_7a_8^2a_{14} + 3a_7a_9^2a_{14} + 3a_8^2a_9a_{14} + 3a_8a_9^2a_{14} + 3a_2a_{11}^2a_{12} + 3a_4a_{11}^2a_{13}$
$+ 3a_5a_{11}^2a_{13} + 3a_7a_{11}^2a_{14} + 3a_8a_{11}^2a_{14} + 3a_9a_{11}^2a_{14} + 6a_2a_{11}a_{12}^2 + 6a_4a_{11}a_{13}^2 + 6a_5a_{11}a_{13}^2 + 6a_5a_{11}a_{14}^2 + 6a_8a_{11}a_{14}^2$
$+ 6a_9a_{11}a_{14}^2 + 3a_4a_{12}^2a_{13} + 3a_5a_{12}^2a_{13} + 6a_2a_{12}^2a_{13} + 3a_7a_{12}^2a_{14} + 3a_8a_{12}^2a_{14} + 3a_9a_{12}^2a_{14} + 6a_2a_{12}^2a_{14} + 6a_4a_{12}a_{13}^2$
$+ 6a_5a_{12}a_{13}^2 + 3a_2a_{12}a_{13}^2 + 6a_7a_{12}a_{14}^2 + 6a_8a_{12}a_{14}^2 + 6a_9a_{12}a_{14}^2 + 3a_2a_{12}a_{14}^2 + 3a_7a_{13}^2a_{14} + 3a_8a_{13}^2a_{14} + 3a_9a_{13}^2a_{14}$
$+ 6a_4a_{13}^2a_{14} + 6a_5a_{13}^2a_{15} + 6a_7a_{13}a_{14}^2 + 6a_8a_{13}a_{14}^2 + 6a_9a_{13}a_{14}^2 + 3a_4a_{13}a_{14}^2 + 3a_5a_{13}a_{14}^2 + 6a_7a_8a_9a_{14} + 6a_2a_{11}a_{12}a_{13}$
$+ 6a_4a_{11}a_{12}a_{13} + 6a_5a_{11}a_{12}a_{13} + 6a_2a_{11}a_{12}a_{14} + 6a_7a_{11}a_{12}a_{14} + 6a_8a_{11}a_{12}a_{14} + 6a_9a_{11}a_{12}a_{14} + 6a_4a_{11}a_{13}a_{14}$
$+ 6a_5a_{11}a_{13}a_{14} + 6a_7a_{11}a_{13}a_{14} + 6a_8a_{11}a_{13}a_{14} + 6a_7a_{11}a_{13}a_{14} + 6a_2a_{12}a_{13}a_{14} + 6a_4a_{12}a_{13}a_{14} + 6a_5a_{12}a_{13}a_{14}$
$+ 6a_7a_{12}a_{13}a_{14} + 6a_8a_{12}a_{13}a_{14} + 6a_9a_{12}a_{13}a_{14}$

M=

$$a_2^3a_{17}+a_4^3a_{18}+a_5^3a_{18}+a_7^3a_{19}$$
$$+a_8^3a_{19}+4a_9^3a_{19}+4a_{11}^3a_{20}+a_{13}^3a_{20}+a_{14}^3a_{20}+3a_2a_{17}^3+3a_4a_{18}^3+3a_5a_{18}^3+3a_7a_{19}^3+3a_8a_{19}^3+3a_9a_{19}^3+3a_{11}a_{20}^3$$
$$+3a_{12}a_{20}^3+3a_{13}a_{20}^3+3a_{14}a_{20}^3+3a_4^2a_5a_{18}+3a_4a_5^2a_{18}+3a_7^2a_8a_{19}+3a_7^2a_9a_{19}+3a_7a_8^2a_{19}+3a_7a_9^2a_{19}+3a_8^2a_9a_{19}$$
$$+3a_8a_9^2a_{19}+6a_{11}^2a_{12}a_{20}+6a_{11}^2a_{13}a_{20}+3a_{11}^2a_{14}a_{20}+9a_{12}^2a_{13}a_{20}+9a_{12}^2a_{14}a_{20}+3a_{13}^2a_{14}a_{20}+3a_{13}a_{14}^2a_{20}$$
$$+3a_2a_{16}^2a_{17}+3a_4a_{16}^2a_{18}+3a_5a_{16}^2a_{18}+3a_7a_{16}^2a_{19}+3a_9a_{16}^2a_{19}+3a_{11}a_{16}^2a_{20}+3a_{12}a_{16}^2a_{20}+6a_{13}a^2{}_{16}a_{20}+6a_{14}a_{16}^2a_{20}$$
$$+6a_2a_{16}a_{17}^2+6a_5a_{16}a_{18}^2+6a_5a_{16}a_{18}^2+6a_7a_{16}a_{19}^2+6a_8a_{16}a_{19}^2+6a_9a_{16}a_{19}^2+6a_{11}a_{16}a_{20}^2+6a_{12}a_{16}a_{20}^2+6a_{13}a_{16}a_{20}^2$$
$$+6a_{14}a_{16}a_{20}^2+3a_4a_{17}^2a_{18}+6a_2a_{17}^2a_{18}+3a_5a_{17}^2a_{18}+6a_2a_{17}^2a_{19}+3a_7a_{17}^2a_{19}+3a_8a_{17}^2a_{19}+3a_7a_{17}^2a_{19}+6a_2a_{17}^2a_{20}$$
$$+3a_{11}a_{17}^2a_{20}+3a_{12}a_{17}^2a_{20}+3a_{12}a_{17}^2a_{20}+3a_{14}a_{17}^2a_{20}+3a_2a_{17}a_{18}^2+6a_4a_{17}a_{18}^2+6a_5a_{17}a_{18}^2+3a_2a_{17}a_{19}^2+6a_7a_{17}a_{19}^2$$
$$+6a_8a_{17}a_{19}^2+3a_2a_{17}a_{20}^2+6a_{11}a_{17}a_{20}^2+6a_{12}a_{17}a_{20}^2+6a_{13}a_{17}a_{20}^2+6a_{14}a_{17}a_{20}^2+3a_7a_{18}^2a_{19}+3a_8a_{18}^2a_{19}$$
$$+3a_9a_{18}^2a_{19}+6a_4a_{18}^2a_{19}+6a_5a_{18}^2a_{19}+6a_4a_{18}^2a_{20}+6a_5a_{18}^2a_{20}+3a_{11}a_{18}^2a_{20}+3a_{12}a_{18}^2a_{20}+3a_{13}a_{18}^2a_{20}+3a_{14}a_{18}^2a_{20}$$
$$+3a_4a_{18}a_{19}^2+9a_5a_{18}a_{19}^2+6a_7a_{18}a_{19}^2+6a_8a_{18}a_{19}^2+6a_{11}a_{18}a_{19}^2+3a_4a_{18}a_{20}^2+3a_5a_{18}a_{20}^2+6a_9a_{18}a_{19}^2+6a_{12}a_{18}a_{20}^2$$
$$+6a_{13}a_{18}a_{20}^2+6a_{14}a_{18}a_{20}^2+6a_7a_{19}^2a_{20}+9a_8a_{19}^2a_{20}+9a_9a_{19}^2a_{20}+3a_{11}a_{19}^2a_{20}+3a_{12}a_{19}^2a_{20}++3a_{13}a_{19}^2a_{20}+3a_{14}a_{19}^2a_{20}$$
$$+3a_7a_{19}a_{20}^2+3a_8a_{19}a_{20}^2+3a_9a_{19}a_{20}^2+6a_{11}a_{19}a_{20}^2+6a_{12}a_{19}a_{20}^2+6a_{13}a_{19}a_{20}^2+6a_{14}a_{19}a_{20}^2+6a_7a_8a_9a_{19}$$
$$+6a_{11}a_{12}a_{13}a_{20}+6a_{11}a_{12}a_{14}a_{20}+6a_{11}a_{13}a_{14}a_{20}+6a_{12}a_{13}a_{14}a_{20}+6a_2a_{16}a_{17}a_{18}+6a_4a_{16}a_{17}a_{18}+6a_5a_{16}a_{17}a_{18}$$
$$+6a_2a_{16}a_{17}a_{18}+6a_7a_{16}a_{17}a_{19}+6a_8a_{16}a_{17}a_{19}+6a_9a_{16}a_{17}a_{19}+6a_2a_{16}a_{17}a_{20}+6a_4a_{16}a_{18}a_{19}+6a_5a_{16}a_{18}a_{19}$$
$$+6a_7a_{16}a_{18}a_{19}+6a_8a_{16}a_{18}a_{19}+6a_9a_{16}a_{18}a_{19}+6a_{11}a_{16}a_{17}a_{20}+6a_{12}a_{16}a_{17}a_{20}+6a_{13}a_{16}a_{17}a_{20}+6a_{14}a_{16}a_{17}a_{20}$$
$$+6a_4a_{16}a_{18}a_{20}+6a_{11}a_{16}a_{18}a_{20}+6a_{12}a_{16}a_{18}a_{20}+6a_{13}a_{16}a_{18}a_{20}+6a_{14}a_{16}a_{18}a_{20}+6a_{14}a_{16}a_{19}a_{20}+6a_7a_{16}a_{19}a_{20}$$
$$+6a_8a_{16}a_{19}a_{20}+6a_9a_{16}a_{19}a_{20}+6a_{11}a_{16}a_{19}a_{20}+6a_{12}a_{16}a_{19}a_{20}+6a_{13}a_{16}a_{19}a_{20}+6a_{14}a_{16}a_{19}a_{20}+6a_6a_{17}a_{18}a_{19}$$
$$+6a_2a_{17}a_{18}a_{19}+6a_7a_{17}a_{18}a_{19}+6a_8a_{17}a_{18}a_{19}+6a_9a_{17}a_{18}a_{19}+6a_4a_{17}a_{18}a_{20}+6a_5a_{17}a_{18}a_{20}+6a_2a_{17}a_{18}a_{20}$$
$$+6a_{11}a_{17}a_{18}a_{20}+6a_{12}a_{17}a_{18}a_{20}+6a_{13}a_{17}a_{18}a_{20}+6a_{14}a_{17}a_{18}a_{20}+6a_{11}a_{17}a_{19}a_{20}+6a_{12}a_{17}a_{19}a_{20}+6a_{13}a_{17}a_{19}a_{20}$$
$$+6a_{14}a_{17}a_{19}a_{20}+6a_7a_{17}a_{19}a_{20}+6a_8a_{17}a_{19}a_{20}+6a_9a_{17}a_{19}a_{20}+6a_2a_{17}a_{19}a_{20}+6a_{11}a_{18}a_{19}a_{20}+6a_{12}a_{18}a_{19}a_{20}$$
$$+6a_{13}a_{18}a_{19}a_{20}+6a_{14}a_{18}a_{19}a_{20}+6a_7a_{18}a_{19}a_{20}+6a_8a_{18}a_{19}a_{20}+6a_9a_{18}a_{19}a_{20}+6a_4a_{18}a_{19}a_{20}++6a_5a_{18}a_{19}a_{20}$$

N=

$$a_2^2a_4a_5a_{13}+a_2^2a_5^2a_{13}+a_2^2a_7a_{11}a_{14}+a_4^2a_5a_9a_{14}+2a_4a_5^2a_9a_{14}+a_7^3a_9a_{14}+a_2^2a_{11}a_{12}+2a_4a_7a_8a_9+a_7^2a_{11}a_{12}$$
$$+a_2^5a_2^2+a_2^2a_5^2a_8+a_3^2a_5a_6a_8+2a_3a_4a_5a_6a_8+a_4^2a_5a_6a_8+2a_2a_4a_5+2a_3a_4a_6+2a_4^2a_6+a_2a_5a_6+a_3a_5a_6$$
$$+a_4a_5a_6+a_2a_5^2+2a_2a_3a_5a_7+2a_4a_5a_7+2a_2a_4^2a_5a_7+a_2^3a_6^2+2a_2a_4a_5a_6a_7+a_2a_3^2+a_2a_4^2a_6a_7+2a_2a_3a_4$$
$$+a_2a_4^2+a_2^3a_3+a_2^3a_4+2a_2a_3a_4a_7^2+2a_2a_4^2a_7^2+a_3^3+3a_3^2a_4+3a_3a_4^2+a_4^3$$

$$
\begin{aligned}
\mathbf{P} = {} & a_2a_4^2a_5a_{13}+a_2a_5^3a_{13}+a_2a_4a_7^2+a_2a_{11}^3a_{14}+a_{11}^2a_4a_{12}a_{14}+a_7a_{11}^2a_{12}a_{14}+(a_2+a_4+a_5)(a_7+a_8+a_9)\\
&+a_4a_9+a_2^2a_9^3+2a_2^2a_{11}a_{14}^2+2a_2a_4a_{12}a_{14}^2+2a_2a_7a_9a_{14}^2+(a_7+a_8+a_9)^2+4a_4a_5+4a_4a_8+4a_4a_9\\
&+2a_7^2+4a_7a_8+5a_7a_9+2a_2a_{11}a_{12}a_{14}^2+a_4a_{12}^2a_{14}^2+a_5a_{12}^2a_{14}^2+(a_7+a_8+a_9)^2+4a_2a_5+5a_2a_8+4a_2a_9\\
&+a_2a_{11}a_{13}^2a_{14}+a_4a_{12}a_{13}^2a_{14}+a_7a_{12}a_{13}^2a_{14}+2a_7^2+(a_4+a_5)^2+a_{13}^3(a_4+a_5)^2+a_2a_{11}a_{12}^2a_{14}+a_4a_{12}^3a_{14}\\
&+a_5a_{12}^3a_{14}+a_2^2+2a_2a_7a_{11}a_{14}^2+2a_4a_8a_9a_{14}^2+2a_5a_8a_9a_{14}^2+(a_7+a_8+a_9)^2+2a_2a_4a_{12}a_{13}^2\\
&+2a_2a_7a_9a_{13}^2+a_{12}a_{13}^2(a_4+a_5)^2+2a_2a_4a_{12}^2a_{13}+2a_2a_5a_{12}^2a_{13}+a_2^2a_{12}^2a_{13}+a_{11}a_{13}^2(a_4+a_5)^2+a_2a_{11}a_{14}\\
&+a_4a_{12}a_{14}+a_5a_{12}a_{14}+3(a_7+a_8+a_9)^2
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Q} = {} & 2a_2^3a_7a_8+2a_4^3a_7a_9+6a_4a_7^3a_9+2a_7^4a_9+2a_2^3a_8a_9+6a_4^2a_7a_8a_9+6a_4a_7^2a_8a_9+3a_4^2a_8^2a_9+2a_2^3a_8^2\\
&+a_4^3a_9^2+3a_2^2a_7^2a_8+6a_2^2a_7a_8^2+9a_4^2a_7^2a_9+2a_4a_7^3a_9+a_7^4a_9^3+12a_4^2a_7a_9^2+18a_4a_7^2a_9^2+8a_7^3a_9^2\\
&+6a_2^2a_7a_8a_9+6a_4^2a_7a_8a_9+8a_7^3a_8a_9+12a_4a_7^2a_8a_9+6a_4a_8^3a_9+3a_7^2a_8^2a_9+6a_2^2a_8^2a_9+3a_2^2a_8a_9^2\\
&+3a_4^2a_8a_9^2+6a_4a_7a_8a_9^2+3a_7^2a_8a_9^2+3a_2^2a_8^3+a_3^2a_9^3+2a_4a_7a_9^3+a_7^2a_9^3
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{R} = {} & a_2^4a_8+a_4^4a_9+4a_4^3a_7a_9+4a_4^2a_7^2a_9+4a_4a_7^3a_9+a_7^4a_9+4a_2a_7^3a_8+12a_2a_7^2a_8^2+12a_7^2a_8a_9(a_3+a_4\\
&+a_5)+12a_2a_7a_8^3+4a_7^3a_9(a_4+a_5)+12a_7^2a_9^2(a_4+a_5)+24a_7a_8^2a_9(2a_3+a_4+a_5)+12a_7a_9^3(a_4\\
&+a_5)+4a_8^3a_9(3a_3+a_4+a_5)+12a_8^2a_9^2(a_3+a_4+a_5)+24a_7a_8a_9^2(a_3+2a_4+2a_5)+4a_2a_8^4+4a_4a_9^4\\
&+4a_8a_9^3(a_3+3a_4+3a_5)+4a_7a_9^4
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{T} = {} & a_2^4a_{12}+4(a_2a_{11}^3a_{12}+a_{11}^3a_{13}(a_4+a_5)+a_{11}^3a_{14}(a_7+a_8+a_9)+3a_2a_{11}^2a_{12}^2+3a_{11}^2a_{12}a_{13}(a_2+a_4\\
&+a_5)+3a_{11}^2a_{12}a_{14}(a_2+a_7+a_8+a_9)+3a_{11}^2a_{13}a_{14}+(a_4+a_5)(a_7+a_8+a_9)+6a_8a_9a_{13}a_{14}((a_2+a_4\\
&+a_5)+(a_7+a_8+a_9))+3a_8a_9^2a_{13}(2a_2+a_4+a_5)+3a_2a_8a_9^2+3a_8a_9^2a_{14}(2a_2+(a_7+a_8+a_9)\\
&+3a_{11}a_{13}a_{14}^2(2(a_7+a_8+a_9)+(a_4+a_5)+3a_8a_9a_{14}^2(2(a_7+a_8+a_9)+a_2)+3a_{11}a_{13}^2a_{14}((a_7\\
&+a_8+a_9)(2a_4+2a_5))+3a_{11}^2a_{14}^2(a_7+a_8+a_9)+3a_{11}^2a_{13}^2(a_4+a_5)+3a_{12}^2a_{13}a_{14}(2a_2+a_4+2a_7\\
&+a_8+a_9)+3a_{12}a_{13}a_{14}^2(2(a_7+a_8+a_9)+(a_2+a_4+a_5))+3a_{12}^2a_{14}^2(a_2+a_7+a_8+a_9)+3a_{12}a_{13}^2a_{14}\\
&+(a_7+a_8+a_9)+(a_2+2a_4+2a_5))+3a_8a_9a_{13}^2(a_2+2a_4+2a_5)+3a_{11}a_{14}^3(a_7+a_8+a_9)+3a_{11}a_{13}^3(a_4\\
&+a_5)+a_{12}a_{13}^3(a_2+3a_4+3a_5)+a_{12}a_{14}^2(a_2+3a_7+3a_8+3a_9)+a_{12}^3a_{13}(3a_2+a_4+a_5)+3a_{12}^2a^2(a_2\\
&+a_4+a_5)+a_{12}^3a_{14}(3a_2+a_7+a_8+a_9)+3a_{13}^2a_{14}^2(a_4+2a_7+a_8+a_9)+a_{13}^3a_{14}((a_7+a_8+a_9)(3a_4\\
&+3a_5))+a_{13}a_{14}^3(3a_7+3a_8+3a_9)+(a_4+a_5)+a_2a_{12}^4+a_{13}^4(a_4+a_5)+a_{14}^4(a_7+a_8+a_9)
\end{aligned}
$$

4.0 Numerical Computation, Results and Implementation

For the purpose of testing the performance of our derived method, the following IVPs are selected for implementation

1. $y' = -y$, $y_{(0)} = 1$, $0 \le x \le 1$

   Whose theoretical solution is? $y(x) = e^{-x}$

2. $y' = y$, $y_{(0)} = 1$, $0 \le x \le 1$

   Whose theoretical solution is? $y(x) = e^{x}$

3. $y' = 1 + y^2$, $y_{(0)} = 1$, $0 \le x \le 1$

   Whose theoretical solution is? $y(x) = \tan(x + \frac{\pi}{4})$

4.1 Tables of Results

Table 1: Test for Error Level on Problem 1 with the Derived Formulae

XN	K1	K2	K3	K4	K5	K6	YN	TSOL	ERROR
.1E+00	-.1000E+01	-.9500E+00	-.9513E+00	-.9518E+00	-.9530E+00	-.9036E+00	0.9048E+00	0.9048E+00	0.2370E-07
.2E+00	-.9048E+00	-.8596E+00	-.8607E+00	-.8613E+00	-.8624E+00	-.8176E+00	0.8187E+00	0.8187E+00	0.4288E-07
.3E+00	-.8187E+00	-.7778E+00	-.7788E+00	-.7793E+00	-.7803E+00	-.7398E+00	0.7408E+00	0.7408E+00	0.5820E-07
.4E+00	-.7408E+00	-.7038E+00	-.7047E+00	-.7051E+00	-.7060E+00	-.6694E+00	0.6703E+00	0.6703E+00	0.7022E-07
.5E+00	-.6703E+00	-.6368E+00	-.6376E+00	-.6380E+00	-.6388E+00	-.6057E+00	0.6065E+00	0.6065E+00	0.7942E-07
.6E+00	-.6065E+00	-.5762E+00	-.5770E+00	-.5773E+00	-.5781E+00	-.5481E+00	0.5488E+00	0.5488E+00	0.8623E-07
.7E+00	-.5488E+00	-.5214E+00	-.5221E+00	-.5224E+00	-.5230E+00	-.4959E+00	0.4966E+00	0.4966E+00	0.9103E-07
.8E+00	-.4966E+00	-.4718E+00	-.4724E+00	-.4727E+00	-.4733E+00	-.4487E+00	0.4493E+00	0.4493E+00	0.9414E-07
.9E+00	-.4493E+00	-.4269E+00	-.4274E+00	-.4277E+00	-.4282E+00	-.4060E+00	0.4066E+00	0.4066E+00	0.9582E-07
.1E+01	-.4066E+00	-.3862E+00	-.3867E+00	-.3870E+00	-.3875E+00	-.3674E+00	0.3679E+00	0.3679E+00	0.9634E-07

Table 2: Test for Error Level on Problem 2 with the Derived Formulae

.1E+00	0.1000E+01	0.1050E+01	0.1051E+01	0.1052E+01	0.1053E+01	0.1104E+01	0.1105E+01	0.1105E+01	0.2582E-07
.2E+00	0.1105E+01	0.1160E+01	0.1162E+01	0.1163E+01	0.1164E+01	0.1220E+01	0.1221E+01	0.1221E+01	0.5708E-07
.3E+00	0.1221E+01	0.1282E+01	0.1284E+01	0.1285E+01	0.1286E+01	0.1348E+01	0.1350E+01	0.1350E+01	0.9462E-07
.4E+00	0.1350E+01	0.1417E+01	0.1419E+01	0.1420E+01	0.1422E+01	0.1490E+01	0.1492E+01	0.1492E+01	0.1394E-06
.5E+00	0.1492E+01	0.1566E+01	0.1568E+01	0.1569E+01	0.1571E+01	0.1647E+01	0.1649E+01	0.1649E+01	0.1926E-06
.6E+00	0.1649E+01	0.1731E+01	0.1733E+01	0.1734E+01	0.1736E+01	0.1820E+01	0.1822E+01	0.1822E+01	0.2554E-06
.7E+00	0.1822E+01	0.1913E+01	0.1916E+01	0.1917E+01	0.1919E+01	0.2011E+01	0.2014E+01	0.2014E+01	0.3294E-06
.8E+00	0.2014E+01	0.2114E+01	0.2117E+01	0.2118E+01	0.2121E+01	0.2223E+01	0.2226E+01	0.2226E+01	0.4160E-06
.9E+00	0.2226E+01	0.2337E+01	0.2340E+01	0.2341E+01	0.2344E+01	0.2457E+01	0.2460E+01	0.2460E+01	0.5172E-06
.1E+01	0.2460E+01	0.2583E+01	0.2586E+01	0.2587E+01	0.2590E+01	0.2715E+01	0.2718E+01	0.2718E+01	0.6351E-06

Table 3: Test for Error Level on Problem 3 with the Derived Formulae

XN	K1	K2	K3	K4	K5	K6	YN	TSOL	ERROR
.1E+00	0.2000E+01	0.2210E+01	0.2222E+01	0.2228E+01	0.2241E+01	0.2481E+01	0.1223E+01	0.1223E+01	0.2113E-04
.2E+00	0.2496E+01	0.2817E+01	0.2838E+01	0.2851E+01	0.2875E+01	0.3248E+01	0.1508E+01	0.1508E+01	0.7008E-04
.3E+00	0.3275E+01	0.3796E+01	0.3840E+01	0.3866E+01	0.3916E+01	0.4536E+01	0.1896E+01	0.1896E+01	0.1951E-03
.4E+00	0.4593E+01	0.5517E+01	0.5615E+01	0.5676E+01	0.5793E+01	0.6935E+01	0.2464E+01	0.2465E+01	0.5673E-03
.5E+00	0.7073E+01	0.8941E+01	0.9207E+01	0.9380E+01	0.9717E+01	0.1219E+02	0.3406E+01	0.3408E+01	0.1992E-02
.6E+00	0.1260E+02	0.1729E+02	0.1825E+02	0.1895E+02	0.2031E+02	0.2749E+02	0.5321E+01	0.5332E+01	0.1064E-01
.7E+00	0.2932E+02	0.4706E+02	0.5328E+02	0.5888E+02	0.7068E+02	0.1122E+03	0.1151E+02	0.1168E+02	0.1691E+00
.8E+00	0.1335E+03	0.3318E+03	0.5368E+03	0.9465E+03	0.2609E+04	0.1314E+05	0.3064E+03	-.6848E+02	0.3749E+03
.9E+00	0.9388E+05	0.2501E+08	0.3941E+12	0.9708E+20	0.5891E+37	0.2169E+71	0.3614E+69	-.8688E+01	0.3614E+69
.1E+01	0.1306+138	0.4267+272	0.1#IOE+01	0.1#IOE+01	0.1#IOE+01	0.1#IOE+01	0.1#IOE+01	-.4588E+01	0.1#IOE+01

5.0 Conclusion

Finally, it has been established from our tables of results above that the work carried out is at high value and will go a long way in reducing the rigor in the solution of IVPs in ODEs. We are encouraged by the research done by many reputable researchers in this area. This discovery made by us has now shown that no area of research is ever exhausted depending on the area of interest.

6.0 References

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