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Collocation Approximation with Legendre Basis Functions to Self-Starting Implicit-Hybrid LMMs For Second Order Stiff Equations

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Abstract

This article presents three-step and four-step implicit hybrid block methods for direct solution of second order stiff differential equations. We achieved this by approach of interpolation and collocation which was applied on one of the orthogonal polynomials precisely Legendre polynomial. Analysis of the methods recovered reveals that they are zero stable, consistent and convergent. The stability domain of the block methods were derived and sketch out. The performances of the methods were tested on numerical examples. The results show that the methods compared creditably well with those of existing methods.

Keywords: Interpolation, Collocation Approximation, Implicit-Hybrid, Legendre polynomial, Orthogonal polynomial, Linear Multistep Methods (LMMs), Stiff equations and Ordinary Differential Equations(ODEs)

1.0 Introduction

In the last decades, there has been a great deal of interest in the research of new methods for the numerical integration of initial value problems (IVP) associated to second-order ordinary differential equations (ODEs) of the form:

$$y'' = f(x, y, y'),\ y(a) = \eta_0,\ \ y'(a) = \eta_1 \qquad (1)$$

where f satisfied Lipstichz condition that guaranteed the existence and the uniqueness of the solution of (1). Equations of these types occur mostly in the modeling of real life situations in areas like physical, chemical, biological and even social sciences. However, the analytical solution of some of the equations is not easily obtainable in real sense hence, the need for a numerical approximation. Earlier attempts in literature employed the technique of reduction of second order ODEs to the system of first order differential equation which has been found to be unsuitable due to some of its drawbacks [1, 2].

The Multi-step Collocation (MC) formula has been studied intensively in [3] and [4]. Moreover, Several Researchers [5-7] adopted Lanczos Tau method combined with the idea of collocation of classes of ODEs thus reproducing linear multistep algorithms which are adequate to handle direct solution of higher order ordinary differential equations. Efforts are now being shifted to the adoption of orthogonal polynomials as basis function without the introduction of perturbation terms in the development of numerical methods.

Moreover, the implicit linear multistep methods were introduced in the quest for more accurate numerical solution for stiff ordinary differential equations and its implementation in predictor-corrector mode are found to be prone to some disadvantages. Among the authors that proposed implicit linear multistep methods [2, 8, 9, 10], there are individually proposed methods which are implemented in predictor-corrector mode, and adopted Taylor series expansion to supply starting value. The setback of the predictor-corrector method is that it is very costly as subroutines are very complicated to write because of the special techniques required to supply starting values and for varying the step size which leads to longer computer time and more human efforts. Above all, the predictors developed are of reducing order hence it affects the overall accuracy of the method. This demerit led to the introduction of block technique which has now become a common choice because it addresses most of the disadvantages of predictor-corrector mode. The idea of the block methods using the MC approach has yielded very efficient numerical methods for the solution of ODEs. This was further investigated in [11-13].

Therefore, this article proposes three-step and four-step implicit-hybrid LMMs by collocation approximation with Legendre basis functions. The block method adopted with the option of Legendre polynomial as the bases functions makes its

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computation be competitive and especially its implicit type formulas can be used in solving stiff ODEs effectively. The stability domain of the block methods was sketched.

Definition 1.1: ([14]):The stability region *R* associated with a multistep formula is defined as the set $R=\{hz: y''=\lambda y,\ y(x_0)=y_0, h>0\}$ produces a sequence $\{y_n\}$ satisfying $y_n \to 0$ as $n\to\infty$.

Definition 1.2: ([15]): A formula is A-stable if the stability region associated with that formula contains the open left half complex place.

Definition 1.3 : ([16]): A convergent linear multistep method is $A(\alpha)$ –stable if,

$$S \supset S_\alpha = \left\{ z: \left|\arg(-z)\right| < \alpha, z\neq 0\right\}, 0<\alpha<\frac{\pi}{2}$$

A method is A(0)-stable if it is $A(\alpha)$-stable for some (sufficiently small) $\alpha > 0$.

The paper is organized as follows: section 1 is of an introductory nature. The mathematical formulation was described in section 2. Stability analysis of the new methods was discussed in section 3. In section 4, some numerical experiments and results showing the relevance of the new methods are discussed. Finally, in section 5, some conclusions are drawn.

2.0 Mathematical Formulation

Assuming an approximate solution to (1) by taking the partial sum of Legendre polynomial of the form:

$$y(x)=\sum_{r=0}^{t+s-1} a_r P_r(x),\ x_n \le x \le x_{n+r} \tag{2}$$

where x can be used only after certain transformation . The second derivative of (2) gives

$$y''(x)=\sum_{r=0}^{t+s-1} a_r P_r''(x) \tag{3}$$

Substituting (3) into (1) gives

$$\sum_{r=0}^{t+s-1} a_r P_r''(x) = f(x, y(x), y'(x)),\ x_n \le x \le x_{n+r} \tag{4}$$

where $P_r(x)$ is the Legendre polynomial of degree r , valid in $x_n \le x \le x_{n+r}$ and a_r's are real unknown parameters to be determined and ($t+s-1$) is the sum number of collocation and interpolation points. The well-known Legendre polynomials are defined on the interval [-1,1].

2.1 Derivation of the Continuous Coefficient Implicit Hybrid Methods

In this section, the continuous formulation of the general linear multistep method $\overline{y}(x)$ of degree $r=t+s-1, t>0, s>0,$ is obtained. Equation (2) and (4) were adopted for the derivation of the proposed schemes. In order to be able to use the polynomial in the $[x_r, x_{n+r}]\subseteq[-1,1]$ some necessary transformation were made. Two cases were considered three-step and four-step methods.

CASE 1: Three-Step Implicit Hybrid Method (TSIHM)

The method will have the collocation at all the grid points $x=x_{n+t}, t=0\left(\frac{1}{2}\right)3$ and interpolation at $x=x_{n+s}, s=0,\frac{1}{2}$. This must satisfy the equation of conditions (4) and (2) respectively. These yield the following systems of equations expressed in matrix form:

$$MD=U \tag{5}$$

$$M=\begin{bmatrix}
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1\\
1 & -\frac{2}{3} & \frac{1}{6} & \frac{7}{27} & -\frac{22}{648} & \frac{11}{36} & -\frac{67}{3888} & -\frac{1403}{5832} & \frac{29467}{93312}\\
0 & 0 & \frac{4}{3} & -\frac{20}{3} & 20 & -\frac{140}{3} & \frac{280}{3} & -168 & 280\\
0 & 0 & \frac{4}{3} & -\frac{40}{3} & \frac{190}{27} & -\frac{140}{27} & -\frac{455}{162} & \frac{3199}{243} & -\frac{51695}{2916}\\
0 & 0 & \frac{4}{3} & -\frac{20}{9} & -\frac{20}{27} & \frac{140}{27} & -\frac{280}{81} & -\frac{1288}{243} & \frac{7000}{729}\\
0 & 0 & \frac{4}{3} & 0 & -\frac{10}{3} & 0 & \frac{35}{6} & 0 & -\frac{35}{4}\\
0 & 0 & \frac{4}{3} & \frac{20}{9} & -\frac{20}{27} & -\frac{140}{27} & -\frac{280}{81} & \frac{1288}{243} & \frac{7000}{729}\\
0 & 0 & \frac{4}{3} & \frac{40}{9} & \frac{190}{270} & \frac{140}{27} & -\frac{455}{162} & -\frac{3199}{243} & -\frac{51695}{2916}\\
0 & 0 & \frac{4}{3} & \frac{20}{3} & 20 & \frac{140}{3} & \frac{280}{3} & 168 & 280
\end{bmatrix}$$

where, $D = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8]^T$ and $U = [y_n, y_{n+1/2}, f_n, f_{n+1/2}, f_{n+1}, f_{n+3/2}, f_{n+2},, f_{n+5/2},, f_{n+3}]^T$.
Solving (5) using Gaussian Elimination method in Maple soft environment produces the following values of $a_r's$ and substituting into equation (2) and after some manipulation gives the continuous scheme:

$$\begin{aligned}\bar{y}(x) =\ & 1 - (y_n - y_{n+\frac{1}{2}}) + \frac{h^2}{241920}[384\left(\frac{x-x_n}{h}\right)^8 - 5376\left(\frac{x-x_n}{h}\right)^7 + 31360\left(\frac{x-x_n}{h}\right)^6 - 98784\left(\frac{x-x_n}{h}\right)^5 + 181888\left(\frac{x-x_n}{h}\right)^4 \\ & - 197568\left(\frac{x-x_n}{h}\right)^3 + 120960\left(\frac{x-x_n}{h}\right)^2 - 28549\left(\frac{x-x_n}{h}\right)] f_n - \frac{h^2}{40320} 384\left(\frac{x-x_n}{h}\right)^8 - 5120\left(\frac{x-x_n}{h}\right)^7 + 27776\left(\frac{x-x_n}{h}\right)^6 \\ & - 77952\left(\frac{x-x_n}{h}\right)^5 + 116928\left(\frac{x-x_n}{h}\right)^4 - 80640\left(\frac{x-x_n}{h}\right)^3 + 9625\left(\frac{x-x_n}{h}\right)] f_{n+\frac{1}{2}} + \frac{h^2}{80640}[1920\left(\frac{x-x_n}{h}\right)^8 - 24320\left(\frac{x-x_n}{h}\right)^7 \\ & + 122752\left(\frac{x-x_n}{h}\right)^6 - 309792\left(\frac{x-x_n}{h}\right)^5 + 393120\left(\frac{x-x_n}{h}\right)^4 - 201600\left(\frac{x-x_n}{h}\right)^3 + 17151\left(\frac{x-x_n}{h}\right)] f_{n+1} - \frac{h^2}{60480}[1920\left(\frac{x-x_n}{h}\right)^8 \\ & - 23040\left(\frac{x-x_n}{h}\right)^7 + 108416\left(\frac{x-x_n}{h}\right)^6 - 249984\left(\frac{x-x_n}{h}\right)^5 + 284489\left(\frac{x-x_n}{h}\right)^4 - 134400\left(\frac{x-x_n}{h}\right)^3 + 10621\left(\frac{x-x_n}{h}\right)] f_{n+\frac{3}{2}} \\ & + \frac{h^2}{80640}[1920\left(\frac{x-x_n}{h}\right)^8 - 21760\left(\frac{x-x_n}{h}\right)^7 + 95872\left(\frac{x-x_n}{h}\right)^6 - 206304\left(\frac{x-x_n}{h}\right)^5 + 221760\left(\frac{x-x_n}{h}\right)^4 + 100800\left(\frac{x-x_n}{h}\right)^3 \\ & + 7703\left(\frac{x-x_n}{h}\right)] f_{n+2} - \frac{h^2}{40320}[384\left(\frac{x-x_n}{h}\right)^8 - 4096\left(\frac{x-x_n}{h}\right)^7 - 17024\left(\frac{x-x_n}{h}\right)^6 - 34944\left(\frac{x-x_n}{h}\right)^5 + 36288\left(\frac{x-x_n}{h}\right)^4 \\ & - 16128\left(\frac{x-x_n}{h}\right)^3 + 1209\left(\frac{x-x_n}{h}\right)] f_{n+\frac{5}{2}} + \frac{h^2}{241920}[384\left(\frac{x-x_n}{h}\right)^8 - 3840\left(\frac{x-x_n}{h}\right)^7 + 15232\left(\frac{x-x_n}{h}\right)^6 + 30240\left(\frac{x-x_n}{h}\right)^5 \\ & + 30688\left(\frac{x-x_n}{h}\right)^4 - 13440\left(\frac{x-x_n}{h}\right)^3 + 995\left(\frac{x-x_n}{h}\right)] f_{n+3} \qquad (6)\end{aligned}$$

Evaluating (6) at $x = x_{n+3}$ gives the discrete scheme

$$\begin{aligned} y_{n+3} =\ & 6y_{n+\frac{1}{2}} - 5y_n + \frac{h^2}{16128}\{1375 f_n + 19554 f_{n+\frac{1}{2}} + 13401 f_{n+1} + 15004 f_{n+\frac{3}{2}} \\ & + 6177 f_{n+2} + 4770 f_{n+\frac{5}{2}} + 199 f_{n+3}\} \qquad (7)\end{aligned}$$

The discrete scheme (7) is consistent, zero-stable and of order $p = 7$ with error constant $C_{p+2} = \frac{349}{1542880}$. In order to get additional schemes, we evaluated (6) at the points $x = x_{n+i},\ i = \frac{5}{2},\ 2,\ \frac{3}{2},\ 1$ and obtained the following discrete schemes:

$$y_{n+\frac{5}{2}} = 5y_{n+\frac{1}{2}} - 4y_n + \frac{h^2}{24192}\left(1669 f_n + 23250 f_{n+\frac{1}{2}} + 15207 f_{n+1} + 15004 f_{n+\frac{3}{2}} + 4371 f_{n+2} + 107 f4_{n+\frac{5}{2}} - 95 f_{n+3}\right) \quad (8)$$

$$y_{n+2} = 4y_{n+\frac{1}{2}} - 3y_n + \frac{h^2}{40320}\left(2089 f_n + 28878 f_{n+\frac{1}{2}} + 16383 f_{n+1} + 13828 f_{n+\frac{3}{2}} - 1257 f_{n+2} + 654 f_{n+\frac{5}{2}} - 95 f_{n+3}\right) \quad (9)$$

$$y_{n+\frac{3}{2}} = 3y_{n+\frac{1}{2}} - 2y_n + \frac{h^2}{80640}\left(2803 f_n + 37950 f_{n+\frac{1}{2}} + 14913 f_{n+1} + 7108 f_{n+\frac{3}{2}} - 3147 f_{n+2} + 990 f_{n+\frac{5}{2}} - 137 f_{n+3}\right) \quad (10)$$

$$y_{n+1} = 2y_{n+\frac{1}{2}} - y_n + \frac{h^2}{241920}\left(4315 f_n + 53994 f_{n+\frac{1}{2}} - 2307 f_{n+1} + 7948 f_{n+\frac{3}{2}} - 4827 f_{n+2} + 1578 f_{n+\frac{5}{2}} - 221 f_{n+3}\right) \quad (11)$$

Here, our intention is to have additional schemes, so the first derivative of (6) is found and evaluated at points $x = x_{n+i}, i = 0\left(\frac{1}{2}\right)3$ yields the following derivative schemes:

$$hy'_n = 2y_{n+\frac{1}{2}} - 2y_n + \frac{h^2}{241920}\left(28549 f_n + 57750 f_{n+\frac{1}{2}} - 51453 f_{n+1} + 42484 f_{n+\frac{3}{2}} - 23109 f_{n+2} + 7254 f_{n+\frac{5}{2}} - 995 f_{n+3}\right) \quad (12)$$

$$hy'_{n+\frac{1}{2}} = 2y_{n+\frac{1}{2}} - 2y_n + \frac{h^2}{241920}\left(9625 f_n + 72474 f_{n+\frac{1}{2}} - 41469 f_{n+1} + 32524 f_{n+\frac{3}{2}} - 17313 f_{n+2} + f\,5370_{n+\frac{5}{2}} - 731 f_{n+3}\right) \quad (13)$$

$$hy'_{n+1} = 2y_{n+\frac{1}{2}} - 2y_n + \frac{h^2}{80640}\left(2633 f_n + 40910 f_{n+\frac{1}{2}} + 17503 f_{n+1} + 4 f_{n+\frac{3}{2}} - 905 f_{n+2} + 398 f_{n+\frac{5}{2}} - 63 f_{n+3}\right) \quad (14)$$

$$hy'_{n+\frac{3}{2}} = 2y_{n+\frac{1}{2}} - 2y_n + \frac{h^2}{241920}\left(8441 f_n + 117210 f_{n+\frac{1}{2}} + 114147 f_{n+1} + 75020 f_{n+\frac{3}{2}} - 16257 f_{n+2} + 4410 f_{n+\frac{5}{2}} - 571 f_{5+3}\right) \quad (15)$$

$$hy'_{n+2} = 2y_{n+\frac{1}{2}} - 2y_n + \frac{h^2}{241920}\left(8059 f_n + 120426 f_{n+\frac{1}{2}} + 100605 f_{n+1} + f\,150028_{n+\frac{3}{2}} + 45381 f_{n+2} - 1110 f_{n+\frac{5}{2}} - 29 f_{n+3}\right) \quad (16)$$

$$hy'_{n+\frac{5}{2}} = 2y_{n+\frac{1}{2}} - 2y_n + \frac{h^2}{80640}\left(2867 f_n + 38750 f_{n+\frac{1}{2}} + 38401 f_{n+1} + 39172 f_{n+\frac{3}{2}} + 46453 f_{n+2} + 16382 f_{n+\frac{5}{2}} - 585 f_{n+3}\right) \quad (17)$$

$$hy'_{n+3} = 2y_{n+\frac{1}{2}} - 2y_n + \frac{h^2}{241920}\left(6875 f_n + 128874 f_{n+\frac{1}{2}} + 74781 f_{n+1} + 192524 f_{n+\frac{3}{2}} + 46437 f_{n+2} + 179370 f_{n+\frac{5}{2}} + 36419 f_{n+3}\right) \quad (18)$$

2.2 Implementation of the Three-step Implicit Hybrid Method

In this section, the implementation strategy of this work is discussed. Following [17] and [18], the general discrete block formula is given as :

$$A^0 Y_m = e y_n + h^{\mu} dF(Y_m) + h^{\mu} BF(Y_m) \quad (19)$$

where e, d are vectors, B are RxR matrix and A^0 identity matrix, μ is the order of differential equation . Expressing equation (7) - (11) and (12) in form of (19) and then solve using matrix inversion gives:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & 1 \\ 1 & \frac{3}{2} \\ 1 & 2 \\ 1 & \frac{5}{2} \\ 1 & 3 \end{bmatrix} \begin{bmatrix} y_n \\ hy'_n \end{bmatrix} + \begin{bmatrix} \frac{28549}{483840} \\ \frac{1027}{7560} \\ \frac{759}{3584} \\ \frac{272}{945} \\ \frac{35225}{96768} \\ \frac{123}{280} \end{bmatrix} \begin{bmatrix} h^2 f_n \end{bmatrix} + \begin{bmatrix} \frac{275}{2304} & \frac{-5717}{53760} & \frac{10621}{120960} & \frac{-7703}{161280} & \frac{403}{26880} & \frac{-199}{96768} \\ \frac{97}{210} & \frac{-2}{9} & \frac{197}{945} & \frac{-97}{840} & \frac{23}{630} & \frac{-19}{3780} \\ \frac{1485}{1792} & \frac{-2403}{17920} & \frac{45}{128} & \frac{-3267}{17920} & \frac{513}{8960} & \frac{-141}{17920} \\ \frac{376}{315} & \frac{-2}{105} & \frac{656}{945} & \frac{-2}{9} & \frac{8}{105} & \frac{-2}{189} \\ \frac{8375}{5376} & \frac{3125}{32256} & \frac{25625}{24192} & \frac{-625}{10752} & \frac{275}{2304} & \frac{-1375}{96768} \\ \frac{27}{14} & \frac{27}{140} & \frac{51}{35} & \frac{27}{280} & \frac{27}{70} & 0 \end{bmatrix} \begin{bmatrix} h^2 f_{n+\frac{1}{2}} \\ h^2 f_{n+1} \\ h^2 f_{n+\frac{3}{2}} \\ h^2 f_{n+2} \\ h^2 f_{n+\frac{5}{2}} \\ h^2 f_{n+3} \end{bmatrix} \quad (20)$$

Writing (20) explicitly

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{2} y'_n + \frac{h^2}{483840}\left(28549 f_n + 57750 f_{n+\frac{1}{2}} - 51453 f_{n+1} + 42484 f_{n+\frac{3}{2}} - 23109 f_{n+2} + 7254 f_{n+\frac{5}{2}} - 995 f_{n+3}\right) \quad (21)$$

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{7560}\left(1027 f_n + 3492 f_{n+\frac{1}{2}} - 1680 f_{n+1} + 1576 f_{n+\frac{3}{2}} - 873 f_{n+2} + 276 f_{n+\frac{5}{2}} - 38 f_{n+3}\right) \quad (22)$$

$$y_{n+\frac{3}{2}} = y_n + \frac{3h}{2} y'_n + \frac{3h^2}{17920}\left(1265 f_n + 4950 f_{n+\frac{1}{2}} - 801 f_{n+1} + 2100 f_{n+\frac{3}{2}} - 1089 f_{n+2} + 342 f_{n+\frac{5}{2}} - 47 f_{n+3}\right) \quad (23)$$

$$y_{n+2} = y_n + 2hy'_n + \frac{2h^2}{945}\left(136 f_n + 564 f_{n+\frac{1}{2}} - 9 f_{n+1} + 328 f_{n+\frac{3}{2}} - 105 f_{n+2} + 36 f_{n+\frac{5}{2}} - 5 f_{n+3}\right) \quad (24)$$

$$y_{n+\frac{5}{2}} = y_n + \frac{5h}{2} y'_n + \frac{h^2}{96768}\left(1409 f_n + 6030 f_{n+\frac{1}{2}} + 375 f_{n+1} + 4100 f_{n+\frac{3}{2}} - 225 f_{n+2} + 462 f_{n+\frac{5}{2}} - 55 f_{n+3}\right) \quad (25)$$

$$y_{n+3} = y_n + 3hy'_n + \frac{3h^2}{280}\left(41f_n + 180f_{n+\frac{1}{2}} + 18f_{n+1} + 136f_{n+\frac{3}{2}} + 9f_{n+2} + 36f_{n+\frac{5}{2}}\right) \qquad (26)$$

The order of the block method is $p = (7,\ 7,\ 7,\ 7, 7, 7)^T$ with error constant

$C_{p+2} = \left(\frac{6031}{464486400}, \frac{233}{7257600}, \frac{9}{179200}, \frac{31}{453600}, \frac{1625}{18579456}, \frac{9}{89600}\right)^T$.

Substituting (21) into (13)-(18) yields

$$y'_{n+\frac{1}{2}} = y'_n + \frac{h}{120960}\left(19087f_n + 65112f_{n+\frac{1}{2}} - 46461f_{n+1} + 37504f_{n+\frac{3}{2}} - 20211f_{n+2} + 6312f_{n+\frac{5}{2}} - 863f_{n+3}\right) \qquad (27)$$

$$y'_{n+1} = y'_n + \frac{h}{7560}\left(1139f_n + 5640f_{n+\frac{1}{2}} + 33f_{n+1} + 1328f_{n+\frac{3}{2}} - 807f_{n+2} + 264f_{n+\frac{5}{2}} - 37f_{n+3}\right) \qquad (28)$$

$$y'_{n+\frac{3}{2}} = y'_n + \frac{h}{4480}\left(685f_n + 3240f_{n+\frac{1}{2}} + 1161f_{n+1} + 2176f_{n+\frac{3}{2}} - 729f_{n+2} + 216f_{n+\frac{5}{2}} - 29f_{n+3}\right) \qquad (29)$$

$$y'_{n+2} = y'_n + \frac{h}{945}\left(143f_n + 696f_{n+\frac{1}{2}} + 192f_{n+1} + 752f_{n+\frac{3}{2}} + 87f_{n+2} + 24f_{n+\frac{5}{2}} - 4f_{n+3}\right) \qquad (30)$$

$$y'_{n+\frac{5}{2}} = y'_n + \frac{h}{24192}\left(743f_n + 3480f_{n+\frac{1}{2}} + 1275f_{n+1} + 3200f_{n+\frac{3}{2}} + 2325f_{n+2} + 1128f_{n+\frac{5}{2}} - 55f_{n+3}\right) \qquad (31)$$

$$y'_{n+3} = y'_n + \frac{h}{280}\left(41f_n + 216f_{n+\frac{1}{2}} + 27f_{n+1} + 272f_{n+\frac{3}{2}} + 27f_{n+2} + 216f_{n+\frac{5}{2}} - 41f_{n+3}\right) \qquad (32)$$

Equations (21)-(32) are then applied in block form as simultaneous numerical integrators to solve (1).

CASE 2: Four-step Implicit Hybrid Method (FSIHM)

Similarly, collocating (4) at points, $x = x_{n+t}, t = 0,\ \frac{1}{2}\ \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, 4$ and interpolating (2) at points $x = x_{n+s}, s = 0,\ \frac{1}{2}$ lead to a system of equations of form (5) where,

$$M = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-3}{4} & \frac{11}{32} & \frac{9}{128} & \frac{-717}{2048} & \frac{3411}{8192} & \frac{-18401}{65536} & \frac{8961}{262144} \\ 0 & 0 & \frac{3}{4} & \frac{-15}{4} & \frac{45}{4} & \frac{-105}{4} & \frac{105}{2} & \frac{-189}{2} \\ 0 & 0 & \frac{3}{4} & \frac{-45}{16} & \frac{705}{128} & \frac{-3465}{512} & \frac{35385}{8192} & \frac{78813}{32768} \\ 0 & 0 & \frac{3}{4} & \frac{-15}{16} & \frac{-135}{128} & \frac{1365}{512} & \frac{105}{8192} & \frac{-140049}{32768} \\ 0 & 0 & \frac{3}{4} & \frac{15}{16} & \frac{-135}{128} & \frac{-1365}{512} & \frac{105}{8192} & \frac{140049}{32768} \\ 0 & 0 & \frac{3}{4} & \frac{45}{16} & \frac{705}{128} & \frac{3465}{512} & \frac{35385}{8192} & \frac{-78813}{32768} \\ 0 & 0 & \frac{3}{4} & \frac{15}{4} & \frac{45}{4} & \frac{105}{4} & \frac{105}{2} & \frac{189}{2} \end{pmatrix}$$

$D = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7]^T$ and $U = [y_n, y_{n+1/2}, f_n, f_{n+1/2}, f_{n+3/2}, f_{n+5/2}, f_{n+7/2}, f_{n+4}]^T$. The continuous scheme is as follows:

$$\bar{y}(x)=y_n-2\left[\left(\frac{x-x_n}{h}\right)y_n+\left(\frac{x-x_n}{h}\right)y_{n+\frac{1}{2}}\right]+\left[\frac{48281}{352800}\left(\frac{x-x_n}{h}\right)+\frac{1}{2}\left(\frac{x-x_n}{h}\right)^2-\frac{1513}{2520}\left(\frac{x-x_n}{h}\right)^3+\frac{12}{35}\left(\frac{x-x_n}{h}\right)^4\right.$$
$$\left.-\frac{107}{1050}\left(\frac{x-x_n}{h}\right)^5\right]+\frac{8}{525}\left(\frac{x-x_n}{h}\right)^6-\frac{2}{2205}\left(\frac{x-x_n}{h}\right)^7]h^2f_n+\left[-\frac{1217}{8820}\left(\frac{x-x_n}{h}\right)+\frac{5}{6}\left(\frac{x-x_n}{h}\right)^3-\frac{673}{1008}\left(\frac{x-x_n}{h}\right)^4\right.$$
$$\left.+\frac{191}{840}\left(\frac{x-x_n}{h}\right)^5-\frac{23}{630}\left(\frac{x-x_n}{h}\right)^6+\frac{1}{441}\left(\frac{x-x_n}{h}\right)^7\right]h^2f_{n+\frac{1}{2}}+\left[\frac{4051}{100800}\left(\frac{x-x_n}{h}\right)-\frac{7}{18}\left(\frac{x-x_n}{h}\right)^3+\frac{137}{240}\left(\frac{x-x_n}{h}\right)^4\right.$$
$$\left.-\frac{151}{600}\left(\frac{x-x_n}{h}\right)^5+\frac{7}{150}\left(\frac{x-x_n}{h}\right)^7-\frac{1}{315}\left(\frac{x-x_n}{h}\right)^7\right]h^2f_{n+\frac{3}{2}}+\left[-\frac{1147}{50400}\left(\frac{x-x_n}{h}\right)-\frac{7}{30}\left(\frac{x-x_n}{h}\right)^3+\frac{269}{720}\left(\frac{x-x_n}{h}\right)^4\right.$$
$$\left.+\frac{119}{600}\left(\frac{x-x_n}{h}\right)^5-\frac{19}{450}\left(\frac{x-x_n}{h}\right)^6+\frac{1}{315}\left(\frac{x-x_n}{h}\right)^7\right]h^2f_{n+\frac{5}{2}}+\left[\frac{1601}{141120}\left(\frac{x-x_n}{h}\right)-\frac{5}{42}\left(\frac{x-x_n}{h}\right)^3+\frac{199}{1008}\left(\frac{x-x_n}{h}\right)^4\right.$$
$$\left.-\frac{19}{168}\left(\frac{x-x_n}{h}\right)^5+\frac{17}{630}\left(\frac{x-x_n}{h}\right)^6-\frac{1}{441}\left(\frac{x-x_n}{h}\right)^7\right]h^2f_{n+\frac{7}{2}}+\left[-\frac{1391}{352800}\left(\frac{x-x_n}{h}\right)+\frac{1}{24}\left(\frac{x-x_n}{h}\right)^3-\frac{22}{315}\left(\frac{x-x_n}{h}\right)^4\right.$$
$$\left.+\frac{43}{1050}\left(\frac{x-x_n}{h}\right)^5-\frac{16}{1575}\left(\frac{x-x_n}{h}\right)^6+\frac{2}{2205}\left(\frac{x-x_n}{h}\right)^7\right]h^2f_{n+4} \qquad (33)$$

The discrete scheme is obtained as:

$$y_{n+4}=8y_{n+\frac{1}{2}}-7y_n+\frac{h^2}{8400}\left(22f_n+19440f_{n+\frac{1}{2}}+23793f_{n+\frac{3}{2}}+11438f_{n+\frac{5}{2}}+4465f_{n+\frac{7}{2}}-358f_{n+4}\right) \qquad (34)$$

The discrete scheme (34) is consistent, zero-stable and the order $p=6$, with the error constant $C_{p+2}=\frac{11477}{2764800}$.

Evaluating (33) at the points $x=x_{n+i}, i=\frac{7}{2},\frac{5}{2},\frac{3}{2}$, we obtained the following discrete schemes:

$$y_{n+\frac{7}{2}}=7y_{n+\frac{1}{2}}-6y_n+\frac{h^2}{67200}\left(512f_n+131615f_{n+\frac{1}{2}}+155113f_{n+\frac{3}{2}}+56273f_{n+\frac{5}{2}}+11815f_{n+\frac{7}{2}}-2528f_{n+4}\right) \qquad (35)$$

$$y_{n+\frac{5}{2}}=5y_{n+\frac{1}{2}}-4y_n+\frac{h^2}{6720}\left(64f_n+8545f_{n+\frac{1}{2}}+8071f_{n+\frac{3}{2}}-49f_{n+\frac{5}{2}}+265f_{n+\frac{7}{2}}-96f_{n+4}\right) \qquad (36)$$

$$y_{n+\frac{3}{2}}=3y_{n+\frac{1}{2}}-2y_n+\frac{h^2}{67200}\left(896f_n+38645f_{n+\frac{1}{2}}+13699f_{n+\frac{3}{2}}-3941f_{n+\frac{5}{2}}+1645f_{n+\frac{7}{2}}-544f_{n+4}\right) \qquad (37)$$

Evaluating the derivative of (33) at points $x=x_{n+i}, i=0,\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2},4$ yield the following derivative schemes:

$$hy'_n=2y_{n+\frac{1}{2}}-2y_n-\frac{h^2}{705600}(96562f_n+97360f_{n+\frac{1}{2}}-28357f_{n+\frac{3}{2}}+16058f_{n+\frac{5}{2}}-8005f_{n+\frac{7}{2}}+2782f_{n+4}) \qquad (38)$$

$$hy'_{n+\frac{1}{2}}=2y_{n+\frac{1}{2}}-2y_n-\frac{h^2}{1411200}(77888f_n+307145f_{n+\frac{1}{2}}-51233f_{n+\frac{3}{2}}+28007f_{n+\frac{5}{2}}-13775f_{n+\frac{7}{2}}+4768f_{n+4}) \qquad (39)$$

$$hy'_{n+\frac{3}{2}}=2y_{n+\frac{1}{2}}-2y_n-\frac{h^2}{1411200}(26048f_n-1053625f_{n+\frac{1}{2}}+880943f_{n+\frac{3}{2}}+200137f_{n+\frac{5}{2}}-83105f_{n+\frac{7}{2}}+27488f_{n+4}) \qquad (40)$$

$$hy'_{n+\frac{5}{2}}=2y_{n+\frac{1}{2}}-2y_n-\frac{h^2}{1411200}(13376f_n+920905f_{n+\frac{1}{2}}+1679838f_{n+\frac{3}{2}}+598759f_{n+\frac{5}{2}}-49615f_{n+\frac{7}{2}}+11936f_{n+4}) \qquad (41)$$

$$hy'_{n+\frac{7}{2}}=2y_{n+\frac{1}{2}}-2y_n-\frac{h^2}{282240}(3776f_n-203557f_{n+\frac{1}{2}}290339f_{n+\frac{3}{2}}-306187f_{n+\frac{5}{2}}-139373f_{n+\frac{7}{2}}+18400f_{n+4}) \qquad (42)$$

$$hy'_{n+4}=2y_{n+\frac{1}{2}}-2y_n-\frac{h^2}{705600}(4274f_n-494000f_{n+\frac{1}{2}}-755909f_{n+\frac{3}{2}}-711494f_{n+\frac{5}{2}}-599365f_{n+\frac{7}{2}}-89506f_{n+4}) \qquad (43)$$

2.3 Implementation of the Block Four-Step Method

Expressing (34)-(37) and (38) in form of (19) and solving yield the block method of uniform order $p=\left(6,\ 6,\ 6,\ 6,\ 6\right)^T$ with the error constant $C_{p+2} = \left(-\frac{42451}{154828800}, \frac{87}{5734400}, \frac{3125}{6193152}, \frac{26411}{22118400}, \frac{37}{18900}\right)^T$. The block method is solved simultaneously with (39) - (43) to obtain

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{2}y_n' + \frac{h^2}{1411200}(96562 f_n + 97360 f_{n+\frac{1}{2}} - 28357 f_{n+\frac{3}{2}} + 16058 f_{n+\frac{5}{2}} - 8005 f_{n+\frac{7}{2}} + 2782 f_{n+4}) \quad (44)$$

$$y_{n+\frac{3}{2}} = y_n + \frac{3h}{2}y_n' + \frac{3h^2}{156800}(11426 f_n + 40875 f_{n+\frac{1}{2}} + 7504 f_{n+\frac{3}{2}} 1281 f_{n+\frac{5}{2}} + 390 f_{n+\frac{7}{2}} - 114 f_{n+4}) \quad (45)$$

$$y_{n+\frac{5}{2}} = y_n + \frac{5h}{2}y_n' + \frac{25h^2}{56448}(794 f_n + 3650 f_{n+\frac{1}{2}} + 2485_{n+\frac{3}{2}} + 112 f_{n+\frac{5}{2}} + 25 f_{n+\frac{7}{2}} - 10 f_{n+4}) \quad (46)$$

$$y_{n+\frac{7}{2}} = y_n + \frac{7h}{2}y_n' + \frac{49h^2}{28800}(286 f_n + 1435 f_{n+\frac{1}{2}} + 1274 f_{n+\frac{3}{2}} + 539 f_{n+\frac{5}{2}} + 80 f_{n+\frac{7}{2}} - 14 f_{n+4}) \quad (47)$$

$$y_{n+4} = y_n + 4hy_n' + \frac{8h^2}{11025}(758 f_n + 3950 f_{n+\frac{1}{2}} + 3682 f_{n+\frac{3}{2}} + 2002 f_{n+\frac{5}{2}} + 670 f_{n+\frac{7}{2}} - 37 f_{n+4}) \quad (48)$$

$$y'_{n+\frac{1}{2}} = y_n' + \frac{h}{201600}(38716 f_n + 71695 f_{n+\frac{1}{2}} - 15421 f_{n+\frac{3}{2}} + 8589 f_{n+\frac{5}{2}} - 4255 f_{n+\frac{7}{2}} + 1476 f_{n+4}) \quad (49)$$

$$y'_{n+\frac{3}{2}} = y_n' + \frac{3h}{22400}(884 f_n + 6605 f_{n+\frac{1}{2}} + 4361 f_{n+\frac{3}{2}} - 889 f_{n+\frac{5}{2}} + 355 f_{n+\frac{7}{2}} - 116 f_{n+4}) \quad (50)$$

$$y'_{n+\frac{5}{2}} = y_n' + \frac{5h}{8064}(236 f_n + 1275 f_{n+\frac{1}{2}} + 1855 f_{n+\frac{3}{2}} + 721 f_{n+\frac{5}{2}} - 75 f_{n+\frac{7}{2}} + 20 f_{n+4}) \quad (51)$$

$$y'_{n+\frac{7}{2}} = y_n' + \frac{7h}{28800}(508 f_n + 3535 f_{n+\frac{1}{2}} + 4067 f_{n+\frac{3}{2}} + 4557 f_{n+\frac{5}{2}} + 1985 f_{n+\frac{7}{2}} - 252 f_{n+4}) \quad (52)$$

$$y'_{n+4} = y_n' + \frac{2h}{1575}(103 f_n + 660 f_{n+\frac{1}{2}} + 812 f_{n+\frac{3}{2}} + 812 f_{n+\frac{5}{2}} + 660 f_{n+\frac{7}{2}} + 103 f_{n+4}) \quad (53)$$

Equations (44)-(53) are applied in block form as simultaneous numerical integrators to (1).

3.0 Stability Analysis

According to [12], the main difficulty associated with stiff equations is that even though the component of the true solutions corresponding to some eigenvalues that may becoming negligible, the restriction on the step-size imposed by the numerical stability of the method requires that $|hz|$ remain small throughout the range of integration. So a suitable formula for stiff equations would be the one that requires that $|hz|$ remains small. The concept of A-stability was introduced by [15]. There are linear multistep methods which are nearly A-stable. Many important classes of practical problems do not require stability on the entire left half-plane C^- (see definitions 1.1, 1. 2 and 1.3).

Below are the graphical representations of stability domain of the block of TSIHM and FSIHM respectively.

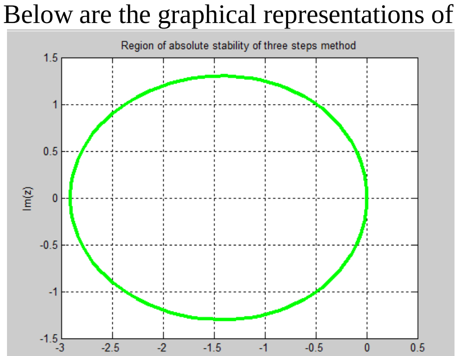

Figure 1: Stability Domain of Block of TSIHM which is A (α)-stable by Definition 1.3.

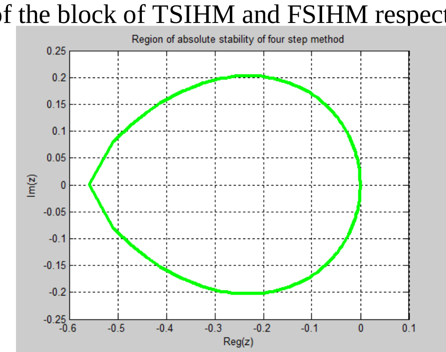

Figure 2: Stability Domain of Block of FSIHM which is A (α)-stable by Definition 1.3.

4.0 Numerical Experiments and Results

In this section, we implement the new methods as a block and applied them on three problems. The absolute errors computed are compared with those obtained in [19] and [1] which are implemented in predictor-corrector mode. The results for the problems are shown in tables 1, 2 and 3. The third problem is an inhomogeneous equation.

Problem 1

$$y'' - \frac{6}{x}y' + \frac{4}{x^2}y = 0 \qquad y(1)=1 \qquad y'(1)=1, \quad x \geq 0, \quad h=\frac{0.1}{32}$$

Exact Solution: $y(x) = \frac{5}{3x} - \frac{2}{3x^4}$

Problem 2

$$y'' - x(y')^2 = 0 \qquad y(0)=1 \qquad y'(0)=\frac{1}{2}, \quad h=\frac{0.1}{32}$$

Exact Solution: $y(x) = 1 + \frac{1}{2} In\left(\frac{2+x}{2-x}\right)$

Problem 3

$$y'' = -100y + 99\sin x, \quad y(0)=1, \quad y(0)=11, \quad h=0.01$$

Exact solution : $y(x) = \cos(10x) + \sin(10x) + \sin x$

This equation has been solved numerically for $0 \leq x \leq 40\pi$ using the new methods.

Table 1: Accuracy comparison of three-step method with the method in [19] for problem 1.

x	y-Exact	New Block method k = 3	Error in New Block method	Error in k=3 of [19]
1.00313	1.003076526	1.003076526	5.5238E-12	8.300E(−08)
1.00625	1.006057503	1.006057503	1.75532E-10	1.160E(−06)
1.00938	1.008944995	1.008944994	1.32364E-09	6.638E(−06)
1.0125	1.011741018	1.011741022	3.92429E-09	9.491E(−06)
1.01563	1.014447543	1.014447553	1.05764E-08	1.9535E(−06)
1.01875	1.017066494	1.017066513	1.85685E-08	9.416E(−06)
1.02188	1.019599755	1.019599788	3.3339E-08	4.6505E(−05)
1.025	1.022049164	1.022049208	4.44086E-08	4.7122E(−05)
1.02813	1.024416519	1.024416571	5.23834E-08	1.86926E(−04)
1.03125	1.026703578	1.026703644	6.27336E-08	4.43321E(−04)

It could be seen from tables 1 that the maximum absolute error of the new three-step is 5.5238E (-12) is higher (more accurate) than that of maximum absolute error of [19] which is 8.300E(−08) for problems 1.

Table 2: Accuracy comparison of new four-step method with the method in with [19] for problem 1.

x	y-exact	New Block method k = 4	Error in New block method k = 4	Error in k = 4 of [19]
1.00313	1.003076526	1.003076526	4.13363E-10	3.8354E(−05)
1.00625	1.006057503	1.006057504	1.18824E-09	7.5004E(−05)
1.00938	1.008944995	1.008944997	1.5042E-09	1.05926E(−04)
1.0125	1.011741018	1.011741018	6.81366E-12	1.35476E(−04)
1.01563	1.014447543	1.014447544	1.37081E-09	1.55567E(−04)
1.01875	1.017066494	1.017066504	9.61295E-09	1.863726E(−04)
1.02188	1.019599755	1.019599774	1.91258E-08	1.96055E(−04)
1.025	1.022049164	1.022049193	2.98501E-08	2.21045E(−04)
1.02813	1.024416519	1.02441656	4.17286E-08	2.0562E(−04)
1.03125	1.026703578	1.026703637	5.94066E-08	2.77908E(−04)

From tables 2 that the maximum absolute error of the new four-step is 4.13363E-10 is higher (more accurate) than that of maximum absolute error of [19] which is 3.8354E(−05) for problems 1

Table 3: Accuracy comparison of new four-step method and k=4 of [1] for problem 2.

x	y-exact solution	y-computed k=4	Error in y-computed	Error in [1] k = 4
0.1	1.050041729	1.050041729	1.62228E-11	2.60753E-10
0.2	1.100335348	1.100335348	1.371E-10	1.98167E-09
0.3	1.151140436	1.151140435	4.82274E-10	6.50741E-09
0.4	1.202732554	1.202732553	1.20319E-09	1.55924E-08
0.5	1.255412812	1.255412809	2.50692E-09	3.15045E-08
0.6	1.309519604	1.30951959	4.69185E-09	5.63746E-08
0.7	1.365443754	1.365443746	8.2047E-09	9.6164E-08
0.8	1.423648930	1.423648916	1.3736E-08	1.56868E-07
0.9	1.484700279	1.484700256	2.23869E-08	2.48698E-07
1	1.549306144	1.549306108	3.59735E-08	3.87984E-07

In tables 3 the maximum absolute error of the new three and four-step are higher (more accurate) than those of [1] for problems 2.

Table 4: The y-exact, y-approximate and error in problem 3 using three-step method.

x	y-exact solution	y-computed for k=3	Error in k=3
0.010	1.104837415259	1.104837397330	1.792933E-08
0.020	1.198734575330	1.198734491364	8.396563E-08
0.030	1.280852195989	1.280851954669	2.413206E-07
0.040	1.350468670498	1.350468336749	3.337492E-07
0.050	1.406987269765	1.406986767540	5.022252E-07
0.060	1.449942094784	1.449941406238	6.885458E-07
0.070	1.479002721860	1.479001838104	8.837561E-07
0.080	1.493977494216	1.493976415561	1.078655E-06
0.090	1.494815427096	1.494814163135	1.263961E-06
0.100	1.481606707323	1.481605276848	1.430475E-06

Table 5: The y-exact , y-approximate and error in problem 3 using four-step method.

x	y-exact solution	y-computed for k=4	Error in k=4
0.010	1.104837415259	1.104837397329	1.792933E-08
0.020	1.198734575330	1.198734491364	8.396563E-08
0.030	1.280852195989	1.280852004323	1.916665E-07
0.040	1.350468670498	1.350467930986	7.395127E-07
0.050	1.406987269765	1.406986767540	5.022252E-07
0.060	1.449942094784	1.449941406238	6.885458E-07
0.070	1.479002721860	1.479001838104	8.837561E-07
0.080	1.493977494216	1.493976415561	1.078655E-06
0.090	1.494815427096	1.494814163135	1.263961E-06
0.100	1.481606707323	1.481605276848	1.430475E-06

5.0 Conclusion

The newly derived collocation approximation with Legendre basis functions approach to self-starting implicit-hybrid LMMs for stiff equations were implemented in block mode. The block methods have the advantages of being self starting, are uniformly of the same order of accuracy and do not need predictors, having good accuracy as shown on numerical results of tables 1, 2, 3, 4 and 5. It should be noted that accuracy and efficiency rate of a method is dependent on the implementation strategies. If economical computation is required, then the new block methods are the better choice. The block method is recommended for general purposed use. Finally, the stability domain of the block methods of three-step method and four-step method were presented in figure 1 and 2. They are found to be A(α)-stable. This type of stability region of a numerical method is especially important in the choice of methods suitable for solving stiff equations. Maple and Matlab software package were employed to generate the schemes and results.

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