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Fuzzification of Some Results on Multigroups

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Abstract

Theory of fuzzy multisets is an extension of multisets which handles uncertainty by allowing several membership values. In this paper, we extend some existing results on multigroup and provide new results arising from the definition of multigroup, submultigroup, normal multigroup and factor multigroupto fuzzy multigroup.

Keywords: Multisets, Fuzzy Multisets, Multigroups, Fuzzy Multigroups

1.0 Introduction

The theory of sets formulated by George Cantor (1845-1918) based on the necessity of providing exact membership values has proved itself to be fundamental and indispensable for the whole of Mathematics. Considering Problems that are not easily handled by classical computing techniques, Lofti Zadeh [1] introduced fuzzy sets as an extension of the classical notion of set, in which the latter admit to partial set membership.

Besides having an object representing an unordered collection of distinct elements, an important generalization of set, known as “multiset”, has emerged by violating a basic underlying set construction. The term “multiset” (mset, for short) was first suggested by N.G. de Bruijn in a private communication to Knuth[2]. A comprehensive account of fundamentals of multiset and its applications in various forms can be found in [2 – 7].

As a generalizationof multisets, Yager[8] introduced the concept of fuzzy multiset (FMS), a mathematical structure possessing both fuzziness and multiplicity.In a fuzzy multiset, an element
of X mayoccur more than once with possibly the same or different membership values.

In [9], concept of fuzzy multigroup was introduced but in this paper we extend the idea and some new results are obtained.

2.0 Preliminaries

In this section, we give basic definitions and additional results required in the subsequent sections of this paper.

Definition 2.1 Let X be a set. A multiset (mset) M drawn from X is represented by a count functionC_M defined as $C_M: X \to \mathbb{N} = \{0,1,2,\dots\}$.

For each $x \in X$, $C_M(x)$ denotes the number of occurrences of the element x in the mset M. The representation of the mset M drawn from $X = \{x_1, x_2, \dots, x_n\}$ will be as $M = [x_1, x_2, \dots, x_n]_{m_1, m_2, \dots, m_n}$ ssuch that $x_i$appears m_i times, $i = 1,2,\dots,n$ in the mset M.

Also, for any positive integer$n$, $[X]^n$ is the set of all msets drawn from$X$ such that no element in the mset occurs more than n times and $[X]^\infty$ is the set of all msets drawn from X such that there is no limit on the number of occurrences of an object in an mset. $[X]^n$ and $[X]^\infty$ are referred to as mset spaces.

Let M_1, $M_2 \in [X]^n$, then we have the following:

(i) $M_1 \subseteq M_2 \Leftrightarrow C_{M_1}(x) \le C_{M_2}(x), \forall x \in X$.

(ii) $M_1 = M_2 \Leftrightarrow C_{M_1}(x) = C_{M_2}(x), \forall x \in X$.

(iii) $M_1 \cap M_2 = C_{M_1}(x) \wedge C_{M_2}(x), \forall x \in X$.

(iv) $M_1 \cup M_2 = C_{M_1}(x) \vee C_{M_2}(x), \forall x \in X$.

Definition 2.2[10]Let X be a group.Then$A$ is called a multigroup over X if the count function A or C_A satisfies the following conditions.

(i) $C_A(xy) \ge [C_A(x) \wedge C_A(y)],\ \forall x, y \in X$;

(ii) $C_A(x^{-1}) \ge C_A(x),\ \forall x \in X$;

(iii) $C_A(e) \ge C_A(x),\ \forall x \in X$.

Although condition (iii) is embedded in conditions (i) and (ii), it is included for easy identification of a multigroup within a multiset space $[X]^n$.

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Definition 2.3[1]Let X be a nonempty set. A fuzzy set Adrawn from $X$is defined as
$A = \{(x, \mu_A(x)) : x \in X\}$,where $\mu_A: X \to [0,1]$is the membership function of A and $\mu_A(x)$ is the degree of membership in A of $x \in X$.
The following are basic relations and operations on fuzzy sets:
(i) $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x),\ \forall x \in X$,
(ii) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x),\ \forall x \in X$,
(iii)$\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x)$, where $\mu_{A\cup B}(x)$ is the union of fuzzy sets and $\vee$ is the maximum operation,
(iv) $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$, where $\mu_{A\cap B}(x)$ is the intersection of fuzzy sets and $\wedge$ is the minimum operation.
Definition 2.4 [11] Let X be a group. A fuzzy subset A of a group X is called a fuzzy subgroup of X if
(i) $\mu_A(xy) \ge \mu_A(x) \wedge \mu_A(y),\ \forall x, y \in X$;
(ii) $\mu_A(x^{-1}) \ge \mu_A(x),\ \forall x \in X$.
Definition 2.5 [12]Let$X$be a nonempty set. A fuzzy multiset (FMS),A,drawn from $X$is characterized by a count membership function of A, denoted by CM_A such that $CM_A: X \to Q$ where Q is the set of all crisp multisets drawn from the unit interval [0,1]. Then for any$x \in X$, the value $CM_A(x)$ is a crisp multiset drawn from [0,1]. For each $x \in X$, the membership sequence is defined as a decreasingly ordered sequence of elements in $CM_A(x)$. It is denoted by $\mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^p(x)$,where $\mu_A^1(x) \ge \mu_A^2(x) \ge \cdots \ge \mu_A^p(x)$. A fuzzy multiset$A$ in$X$ is a set of ordered sequence given as$A = \{(x, \mu_1(x),\ \mu_2(x), \mu_3(x), \ldots, \mu_n(x), \ldots) : x \in X\}$, where
$\mu_n(x) : X \to [0, 1]$is the membership function of$A$.
If the sequence of the membership functions have only n-terms (finite number of terms), n is called the "*dimension*"of A. The collection of all finite fuzzy multisets in X is denoted by $FM(X)$.
The length $L(x; A)$ i.e. the length of $\mu_A^j(x)$of a fuzzy multiset A is defined as follows:
$L(x; A) = \max\{j : \mu_A^j(x) \ne 0\}$, and $L(x; A, B) = \max\{L(x; A), L(x; B)\}$.When no ambiguity arises, we write $L(x) = L(x; A, B)$ for simplicity.
Two fuzzy multisets A and B are conformable to fuzzy operations ifthe lengths of the membership sequences $\mu_A^1(x), \mu_A^2(x),\ldots,\mu_A^p(x)$, and $\mu_B^1(x), \mu_B^2(x),\ldots, \mu_B^{p'}(x)$are equal.
The following are basic relations and operations on fuzzy multisets A and B taken from [12]
(i) [Inclusion]

$$A \subseteq B \Leftrightarrow \mu_A^j(x) \le \mu_B^j(x),\ j = 1, 2, \ldots, L(x)\ \forall\, x \in X.$$

(ii) [Equality]

$$A = B \Leftrightarrow \mu_A^j(x) = \mu_B^j(x), \qquad j = 1, 2, \ldots, L(x)\ \forall\, x \in X.$$

(iii) [Union]

$$\mu_{A\cup B}^j(x) = \mu_A^j(x) \vee \mu_B^j(x),\ j = 1,2, \ldots, L(x).$$

(iv) [Intersection]

$$\mu_{A\cap B}^j(x) = \mu_A^j(x) \wedge \mu_B^j(x),\ j = 1,2, \ldots, L(x).$$

Definition 2.6 Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. Then the image $f(A)$ of FMS $A \in FM(X)$ is defined as

$$CM_{f(A)}(y) = \begin{cases} \vee_{f(x)=y}\, CM_A(x), & f^{-1}(y) \ne \emptyset \\ 0, & f^{-1}(y) = \emptyset \end{cases}$$

Example 2.1 Let $X = \{a, b, c, d\}$ and $Y = \{u, v, w, z\}$.
Define $f : X \to Y$ by $f(a) = w,\ f(b) = w,\ f(c) = u,\ f(d) = u$.
LetA $= \{(1,0.5,0.5)/a, (0.6,0.4,0.1)/b, (0.9,0.7)/c, (0.7,0.5,0.1)/d\}$. Then A is a fuzzy multiset of X, since

$$\begin{aligned} CM_{f(A)}(u) &= \vee\{CM_A(x) : f(x) = u\} = \vee\{CM_A(c),\ CM_A(d)\} \\ &= \vee\{(0.9,0.7), (0.7,0.5,0.1)\} \\ &= (0.9,0.7,0.1) \end{aligned}$$

$CM_{f(A)}(v) = 0,$ since $f^{-1}(v) = \emptyset$

$$\begin{aligned} CM_{f(A)}(w) &= \vee\{CM_A(x) : f(x) = w\} = \vee\{CM_A(a),\ CM_A(b)\} \\ &= \vee\{(1,0.5,0.5), (0.6,0.4,0.1)\} \\ &= (1, 0.5,0.5) = CM_A(a) \end{aligned}$$

$CM_{f(A)}(z) = 0,$ since $f^{-1}(z) = \emptyset$
Therefore, $f(A) = \{(0.9,0.7,0.1), (1,0.5,0.5)\}$ is the image of A under f and $f(A)$ is a fuzzy multiset of Y.
Definition 2.7 Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. Thenthe inverse image $f^{-1}(B)$of FMS $B \in FM(X)$ is defined as $CM_{f^{-1}(B)}(x) = CM_B(f(x))$.
Example 2.2 Let $X = \{a, b, c, d\}$ and $Y = \{u, v, w, z\}$.

Define $f : X \to Y$ by $f(a) = v,\ f(b) = u, f(c) = w$ and $f(d) = w$.
Consider $B = \{(1,0.6,0.5)/u, (0.9,0.8)/v, (0.7,0.6)/w, (0.5)/z\}$, a fuzzy multiset of Y.
Now, $CM_{f^{-1}(B)}(a) = CM_B\big(f(a)\big) = CM_B(v) = (0.9,0.8)$

$$CM_{f^{-1}(B)}(b) = CM_B\big(f(b)\big) = CM_B(u) = (1,0.6,0.5)$$
$$CM_{f^{-1}(B)}(c) = CM_B\big(f(c)\big) = CM_B(w) = (0.7,0.6)$$
$$CM_{f^{-1}(B)}(d) = CM_B\big(f(d)\big) = CM_B(w) = (0.7,0.6)$$

Therefore, $f^{-1}(B) = \{(0.9,0.8), (1,0.6,0.5), (0.7,0.6), (0.7,0.6)\}$ is a fuzzy multiset of X.

3.0 Fuzzy Multigroups

Definition 3.1 Let X be a group. A fuzzy multiset A over X is a fuzzy multigroup over X if the count (count membership) of A satisfies the following conditions:
(i) $CM_A(xy) \geq [CM_A(x) \wedge CM_A(y)],\ \forall\, x, y \in X$,
(ii) $CM_A(x^{-1}) = CM_A(x),\ \forall\, x \in X$,
(iii) $CM_A(e) \geq CM_A(x),\ \forall\, x \in X$.
We include condition (iii) for easy identification of a fuzzy multigroup within $FM(X)$. Condition (iii) is embedded in conditions (i) and (ii), since

$$CM_A(e) = CM_A(xx^{-1}) \geq CM_A(x) \wedge CM_A(x^{-1}) = CM_A(x), \qquad \forall\, x \in X.$$

We denote the set of all fuzzy multigroups over X by $FMG(X)$.
Example 3.1 Let $X = (V_4, .) = \{1, a, b, c\}$ be a klein's 4-group and
$A = \{(1,0.7,0.6,0.5,0.5)/1, (0.6,0,4,0.2)/a, (0.7,0.6,0.5,0.4)/b, (0.6,0.4,0.2)/c\}$ be a fuzzy multiset over X. Now
$CM_A(1.a) = CM_A(a) = (0.6,0.4,0.2) \geq [CM_A(1) \wedge CM_A(a)]$
$CM_A(1.b) = CM_A(b) = (0.7,0.6,0.5,0.4) \geq [CM_A(1) \wedge CM_A(b)]$
$CM_A(1.c) = CM_A(c) = (0.6,0.4,0.2) \geq [CM_A(1) \wedge CM_A(c)]$
$CM_A(a.b) = CM_A(c) = (0.6,0.4,0.2) \geq [CM_A(a) \wedge CM_A(b)]$
$CM_A(b.c) = CM_A(a) = (0.6,0.4,0.2) \geq [CM_A(b) \wedge CM_A(c)]$
$CM_A(c.a) = CM_A(b) = (0.7,0.6,0.5,0.4) \geq [CM_A(c) \wedge CM_A(a)]$
$CM_A(1^2) = CM_A(1) = (1,0.7,0.6,0.5,0.5) \geq [CM_A(1) \wedge CM_A(1)]$
$CM_A(a^2) = CM_A(1) = (1,0.7,0.6,0.5,0.5) \geq [CM_A(a) \wedge CM_A(a)]$
$CM_A(b^2) = CM_A(1) = (1,0.7,0.6,0.5,0.5) \geq [CM_A(b) \wedge CM_A(b)]$
$CM_A(c^2) = CM_A(1) = (1,0.7,0.6,0.5,0.5) \geq [CM_A(c) \wedge CM_A(c)]$
$CM_A(1^{-1}) = CM_A(1) = (1,0.7,0.6,0.5,0.5)$, $\quad CM_A(a^{-1}) = CM_A(a) = (0.6,0.4,0.2)$
$CM_A(b^{-1}) = CM_A(b) = (0.7,0.6,0.5,0.4)$, $\quad CM_A(c^{-1}) = CM_A(c) = (0.6,0.4,0.2)$
Therefore, A is a fuzzy multigroup over X.
Proposition 3.1[9] Let $A \in FMG(X)$. Then $CM_A(x^n) \geq CM_A(x),\ \forall\, x \in X$.
Proposition 3.2[9] Let $A \in FMG(X)$ and $CM_A(x^{-1}) \geq CM_A(x)$. Then $CM_A(x^{-1}) = CM_A(x)$.
Proof. *Straightforward*.
Proposition 3.3 Let $A \in FMG(X)$. Then
(i) $CM_A(xy)^{-1} \geq CM_A(x) \wedge CM_A(y),\ \forall\, x, y \in X$,
(ii) $CM_A(xy)^n \geq CM_A(xy),\ \forall\, x, y \in X$.
Proof. *Straightforward.*
Proposition 3.4 Let $A \in FMG(X)$. If $CM_A(x) < CM_A(y)$ for some $x, y \in X$, then
$CM_A(xy) = CM_A(x) = CM_A(yx)$.
Proof
Let $CM_A(x) < CM_A(y)$.
Now $CM_A(xy) \geq CM_A(x) \wedge CM_A(y) = CM_A(x)$
Also, $CM_A(x) = CM_A(xyy^{-1}) \geq CM_A(xy) \wedge CM_A(y) = CM_A(xy)$, since $CM_A(x) < CM_A(y)$,
$CM_A(xy) < CM_A(y)$
Therefore, $CM_A(xy) = CM_A(x)$.
Similarly, $CM_A(yx) = CM_A(x)$.
Hence, the proof.
Proposition 3.5 Let $A \in FMG(X)$. Then $CM_A(xy^{-1}) = CM_A(e)$ implies $CM_A(x) = CM_A(y)$.
Proof
Given $A \in FMG(X)$ and $CM_A(xy^{-1}) = CM_A(e)\ \ \forall\, x, y \in X$. Then

$$CM_A(x) = CM_A(x(y^{-1}y))$$
$$= CM_A((xy^{-1})y)$$

$$\geq CM_A(xy^{-1}) \wedge CM_A(y)$$
$$= CM_A(e) \wedge CM_A(y)$$
$$= CM_A(y)$$

That is, $CM_A(x) \geq CM_A(y)$
Now, $CM_A(y) = CM_A(y^{-1})$,since $A \in FMG(X)$

$$= CM_A(ey^{-1})$$
$$= CM_A((x^{-1}x)y^{-1})$$
$$\geq CM_A(x^{-1}) \wedge CM_A(xy^{-1})$$
$$= CM_A(x) \wedge CM_A(e)$$
$$= CM_A(x)$$

That is, $CM_A(y) \geq CM_A(x)$
Hence, $CM_A(x) = CM_A(y)$.
Definition 3.2 Let $A, B \in FMG(X)$, we have the following definitions:
(i)$CM_{A\circ B}(x) = \vee\{CM_A(y) \wedge CM_B(z) : y, z \in X,\ yz = x\}$,
(ii)$CM_{A^{-1}}(x) = CM_A(x^{-1})$.
We call $A \circ B$ the product of A and B, and A^{-1} the inverse of A.
Example 3.2 Let $X = \{1, -1\}$ be a group with multiplication, $A = \{(1,0.6,0.5)/1, (0.5,0.3)/-1\}$ and $B = \{(0.9,0.6,0.3)/1, (0.7,0.5,0.2)/-1\}$. Now

$$CM_{A\circ B}(1) = \vee_{\substack{1\cdot 1=1 \\ -1\cdot -1=1}} \{CM_A(\pm 1) \wedge CM_B(\pm 1)\}$$
$$= \vee\{(0.9,0.6,0.3), (0.5,0.3)\} = (0.9,0.6,0.3)$$
$$CM_{A\circ B}(-1) = \vee_{\substack{1\cdot -1=-1 \\ -1\cdot 1=-1}} \{CM_A(\pm 1) \wedge CM_B(\mp 1)\}$$
$$= \vee\{(0.7,0.5,0.2), (0.5,0.3)\} = (0.7,0.5,0.2)$$

$\Longrightarrow A \circ B = \{(0.9,0.6,0.3)/1, (0.7,0.5,0.2)/-1\}$.
Since $X = \{-1,1\}$ is a group and $A = \{(1,0.6,0.5)/1, (0.5,0.3)/-1\}$, then
$CM_A(1) = (1,0.6,0.5) = CM_A(1^{-1}) = CM_{A^{-1}}(1)$.
Proposition 3.6[9] Let $A, B, C, A_i \in FMG(X)$, then the following hold:
(i) $CM_{A\circ B}(x) = \vee_{y\in X}[CM_A(y) \wedge CM_B(y^{-1}x)] = \vee_{y\in X}[CM_A(xy^{-1}) \wedge CM_B(y)],\ \forall x \in X$;
(ii) $A^{-1} = A$,
(iii) $(A^{-1})^{-1} = A$,
(iv) $A \subseteq B \Longrightarrow A^{-1} \subseteq B^{-1}$,
(v) $(\cup_{i=1}^{n} A_i)^{-1} = \cup_{i=1}^{n} ({A_i}^{-1})$,
(vi) $(\cap_{i=1}^{n} A_i)^{-1} = \cap_{i=1}^{n} ({A_i}^{-1})$,
(vii) $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$,
(viii) $(A \circ B) \circ C = A \circ (B \circ C)$.
Proof. *Straightforward*.
Proposition 3.7 Let $A, B \in FMG(X)$. Then $A \circ B = B \circ A$.
Proof
For all $x \in X$, we have

$$CM_{A\circ B}(x) = \vee\{CM_A(y) \wedge CM_B(z) : yz = x, \quad y, z \in X\}$$
$$= \vee_{y\in X}\{CM_A(xy^{-1}) \wedge CM_B(y) : (xy^{-1})y = x\}$$
$$= \vee_{y\in X}\{CM_B(y) \wedge CM_A(y^{-1}x) :\ y(y^{-1}x) = x\}$$

$= CM_{B\circ A}(x)$.
Therefore, $A \circ B = B \circ A$ **.**
Remark 3.1 If$A, B \in FMG(X)$, then $CM_{A\circ B}(x^{-1}) = CM_{A\circ B}(x)$.
Proposition 3.8 Let $A, B, C, D \in FMG(X)$. If $A \subseteq B$ and $C \subseteq D$, then $A \circ C \subseteq B \circ D$.
Proof
Since $A \subseteq B$ and $C \subseteq D$, it follows that $CM_A(x) \leq CM_B(x),\ \forall x \in X$ and $CM_C(x) \leq CM_D(x),\ \forall x \in X$. So, $CM_{A\circ C}(x) = \vee\{CM_A(y) \wedge CM_C(z) : y, z \in X,\ yz = x\}$

$$\leq \vee\{CM_B(y) \wedge CM_D(z) : y, z \in X, \quad yz = x\} = CM_{B\circ D}(x)$$

Hence, $A \circ C \subseteq B \circ D$.
Proposition3.9[9]Let $A \in FM(X)$. Then $A \in FMG(X)$iff$CM_A(xy^{-1}) \geq [CM_A(x) \wedge CM_A(y)],\ \forall x, y \in X$.
Proposition 3.10 Let $A \in FM(X)$. Then $A \in FMG(X)$ iff $A \circ A \leq A$ and $A^{-1} = A$.
Proof
Let $x, y \in X$. Since $A \in FMG(X)$, then $CM_A(xy) \geq CM_A(x) \wedge CM_A(y)$.

$\Longrightarrow CM_{A\circ A}(z) = \vee_{z=xy}\{CM_A(x)\wedge CM_A(y)\}$

$$\leq \vee_{z=xy}\{CM_A(xy)\} = CM_A(z)$$

Hence, $A\circ A \leq A$.
On the other hand, $A \in FMG(X) \Longrightarrow CM_A(x^{-1}) = CM_A(x),\ \forall\, x\in X$.
But $CM_A(x^{-1}) = CM_{A^{-1}}(x)$. Therefore, $A^{-1} = A$.
Conversely, if $A\circ A = A$ and $A^{-1} = A$, then it is sufficient to prove that $A \in FMG(X)$.
Now, $CM_{A\circ A}(z) = \vee_{z=xy}\{CM_A(x)\wedge CM_A(y)\}$

$$\geq CM_A(x)\wedge CM_A(y), \qquad \forall\, x,y\in X$$

$$\Longrightarrow CM_A(xy) \geq CM_A(x)\wedge CM_A(y), xy = z$$

Since $CM_A(x) = CM_{A^{-1}}(x)$ and $CM_{A^{-1}}(x) = CM_A(x^{-1})$, it follows that
$CM_A(x^{-1}) = CM_A(x),\ \forall\, x\in X$.
Therefore, $A \in FMG(X)$.

Proposition 3.11[9] Let $A, B \in FMG(X)$. Then $A\cap B \in FMG(X)$.

Remark 3.2[9] If $\{A_i\}_{i\in I}$ is a family of FMG over X, then $\cap_{i\in I} A_i$ is also a FMG over X.

Remark 3.3[9]If $\{A_i\}_{i\in I}$is a family of FMG over X, then $\cup_{i\in I} A_i$ need not be a FMG over X.

Proposition 3.12 Let $A, B \in FMG(X)$ and $A \subseteq B$ or $B\subseteq A$. Then $A\cup B \in FMG(X)$.

Proof

Suppose $A\subseteq B$. Then $CM_{A\cup B}(x) = CM_A(x)\vee CM_B(x) = CM_B(x),\ \forall\, x\in X$.
Let $x, y\in X$. Then $CM_{A\cup B}(xy) = CM_A(xy)\vee CM_B(xy)$

$$= CM_B(xy) \geq CM_B(x)\wedge CM_B(y) \qquad (3.1)$$

$$CM_{A\cup B}(x)\wedge CM_{A\cup B}(y) = [CM_A(x)\vee CM_B(x)]\wedge[CM_A(y)\vee CM_B(y)]$$

$$= CM_B(x)\wedge CM_B(y) \leq CM_B(xy) \qquad (3.2)$$

From (3.1) and (3.2)
$CM_{A\cup B}(xy) = CM_{A\cup B}(x)\vee CM_{A\cup B}(y)$
Again, $CM_{A\cup B}(x^{-1}) = CM_A(x^{-1})\vee CM_B(x^{-1}) = CM_B(x^{-1}) = CM_B(x)$

$$= CM_A(x)\vee CM_B(x) = CM_B(x) = CM_{A\cup B}(x)$$

Therefore, $A\cup B\in FMG(X)$.

Proposition 3.13 Let $A\in FMG(X)$ and $x\in X$. Then $CM_A(xy) = CM_A(y)\ \forall\, y\in X$ iff $CM_A(x) = CM_A(e)$.

Proof

Let $CM_A(xy) = CM_A(y),\ \forall\, y\in X$.
$\Longrightarrow CM_A(xe) = CM_A(e)$,since $e\in X$
$\Longrightarrow CM_A(x) = CM_A(e)$,since $xe = x\in X$ as X is a group
Conversely, let $CM_A(x) = CM_A(e)$.
But $CM_A(e)\geq CM_A(y)\ \ \forall\, y\in X$

$$\Longrightarrow CM_A(y)\geq CM_A(x) \qquad (3.3)$$

Now, $CM_A(xy)\geq CM_A(x)\wedge CM_A(y) = CM_A(e)\wedge CM_A(y) = CM_A(y)$
$\Longrightarrow CM_A(xy)\geq CM_A(y), \forall\, y\in X$
But $CM_A(y) = CM_A(x^{-1}xy)\geq CM_A(x)\wedge CM_A(xy)$.
Since $CM_A(x)\geq CM_A(xy), \forall\, y\in X$, then $CM_A(x)\wedge CM_A(xy) = CM_A(xy)\leq CM_A(y),\ \forall\, y\in X$.

$$\Longrightarrow CM_A(y)\geq CM_A(xy), \qquad \forall\, y\in X \qquad (3.4)$$

Hence, $C_A(xy) = C_A(y)\ \ \forall\, y\in X$ from (3.3) and (3.4).

Proposition 3.14[9] If $A\in FMG(X)$ and $H\leq X$, then the restriction $A|H\in MG(H)$.

Proposition 3.15[9] Let $A\in FMG(X)$. Then $A_{[\alpha,n]}$ are subgroups of X.

Proposition 3.16[9] Let $A\in FMG(X)$.Then A_* is a subgroup of X.

Propositon 3.17[9]Let $A\in FMG(X)$. Then A^j is a subgroup of X iff $\mu_A^{j+1}(xy^{-1}) = 0\ \ \forall\, x,y\in A^j$.

Proposition 3.18 Let $A\in FMG(X)$. Then the following assertions are equivalent:
(a) $CM_A(xy) = CM_A(yx),\ \forall\, x,y\in X$,
(b) $CM_A(xyx^{-1}) = CM_A(y),\ \forall\, x,y\in X$,
(c) $CM_A(xyx^{-1}) \geq CM_A(y),\ \forall\, x,y\in X$,
(d) $CM_A(xyx^{-1}) \leq CM_A(y),\ \forall\, x,y\in X$.
Proof. *Straightforward*.

Definition 3.3 Let $A\in FMG(X)$. Then A is called an abelian fuzzy multigroup over X if $CM_A(xy) = CM_A(yx),\ \forall\, x,y\in X$. The set of all abelian fuzzy multigroups is denoted by $AFMG(X)$.

Proposition 3.19 Let $A\in AFMG(X)$. Then the subgroups A_*, $A^j$and A_n; $n\in\mathbb{N}, \alpha\in[0,1]$ of X are normal subgroups of X.

Proof

(i)Let $x \in X$ and $y \in A_*$. Then $CM_A(y) = CM_A(e)$.
Since$A \in AFMG(X)$, then $CM_A(xy) = CM_A(yx)\ \ \forall\, x, y \in X$.
By proposition 3.18,$CM_A(xyx^{-1}) = CM_A(y) = CM_A(e)$.
Thus,$xyx^{-1} \in A_*$.
Hence,A_* is a normal subgroup of X.

(ii) Let $x \in X$ and $y \in A^j$. Then $\mu_A^j(y) > 0$ and $\mu_A^{j+1}(y) = 0$.
Since $A \in AFMG(X)$,then $CM_A(xy) = CM_A(yx)\ \ \forall\, x, y \in X$.
By proposition 3.18,$CM_A(xyx^{-1}) = CM_A(y)$
$\Rightarrow \mu_A^j(xyx^{-1}) = \mu_A^j(y) > 0$ and $\mu_A^j(xyx^{-1}) = \mu_A^j(y) = 0$.
Thus,$xyx^{-1} \in A^j$.
Hence, A^j is a normal subgroup of X.

(iii) Let $x \in X$ and $y \in A_{[\alpha,n]}$. Then $\mu_A^j(y) \geq \alpha;\ j \geq n$.
Since $A \in AFMG(X)$, then $CM_A(xy) = CM_A(yx)\ \ \forall\, x, y \in X$.
By proposition 3.18, $CM_A(xyx^{-1}) = CM_A(y)$
$\Rightarrow \mu_A^j(xyx^{-1}) = \mu_A^j(y) \geq \alpha$. Thus,$xyx^{-1} \in A_{[\alpha,n]}$.
Hence, $A_{[\alpha,n]}$is a normal subgroup of X.

Definition 3.4 Let $H \in FMG(X)$. For any $x \in X$, xH and Hx defined by $CM_{xH}(y) = CM_H(x^{-1}y)$ and $CM_{Hx}(y) = CM_H(yx^{-1})$,$\forall\, y \in X$ are called the left and right fuzzy multicosets of H in X.

Remark 3.4 If $H \in AFMG(X)$, then $xH = Hx,\ \forall\, x \in X$.

Proposition 3.20 Let $H \in FMG(X)$, then $xH = yH$ iff $x^{-1}y \in H_*$.

Proof

Suppose $xH = yH$. Then $CM_H(x^{-1}y) = CM_{xH}(y) = CM_{yH}(y) = CM_H(y^{-1}y) = CM_H(e)$,
$\Rightarrow x^{-1}y \in H_*$.
Conversely, suppose that $x^{-1}y \in H_*$. It follows that $CM_H(x^{-1}y) = CM_H(e)$, then

$$\begin{aligned} CM_{xH}(z) = CM_H(x^{-1}z) = CM_H(x^{-1}yy^{-1}z) &\geq CM_H(x^{-1}y) \wedge CM_H(y^{-1}z) \\ &= CM_H(e) \wedge CM_H(y^{-1}z) \\ &= CM_H(y^{-1}z) \\ &= CM_{yH}(z), \qquad \forall\, z \in X \end{aligned}$$

$\Rightarrow CM_{xH}(z) \geq CM_{yH}(z),\ \forall\, z \in X$.
Similarly, we have $CM_{yH}(z) \geq CM_{xH}(z),\ \forall\, z \in X$.
Hence,$CM_{xH}(z) = CM_{yH}(z),\ \forall\, z \in X$.
Therefore, $xH = yH$.

4.0 Conclusion

Some existing results in multigroup were extended to fuzzy multigroup and subsequently provide new results in fuzzy multigroup arising from the definitions of multigroup, submultigroup, normal multigroup and factor multigroup.

5.0 References

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