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An Obscure Theorem of Binmore and a Popular Characterization of Differentiability

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Abstract

We present an obscure theorem of K.G. Binmore found in [3] and employ it to give an analytic proof, the only one known to the author so far, of a popular characterization of differentiation that is an old as Isaac Newton. The hich is as old as Isaac Newton. The age long geometric proof given by George E. Andrews in [1] that this characterization is geometrically evident can now be dispensed with by teachers and students of Elementary Real Analysis.

Keywords: Limit, continuity, differentiability.
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1.0 Introduction

Our language and notations shall be pretty standard as found for example in standard texts of Elementary Real Analysis, [2], [3], [4], [5], [6] and [7]. $\mathbb{R}$ denotes the real numbers and $\mathbb{N}$ the positive integers. If $\delta > 0$ and $a \in \mathbb{R}$, by $N_\delta(a)$ shall be meant the open interval $(a - \delta, a + \delta)$ called the δ-*neighbourhood* of a. $N_\delta^{'}(a) \equiv (a - \delta, a + \delta) - \{a\}$ is called the *deleted* δ-*neighbourhood of a*. Let $\varnothing \neq A \subseteq \mathbb{R}$ and $x_o \in \mathbb{R}$ (not necessarily belonging to A) be a point of accumulation of A. The number L is called the *limit* of the real function $f : A \to \mathbb{R}$ at x_o provided whenever given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$x \in A$ and $0 < |x - x_o| < \delta(\varepsilon) \Rightarrow | f(x) - L| < \varepsilon$ ... (ρ)

And we write $\lim_{x \to x_o} f(x) = L$.

SEQUENTIAL CHARACTERIZATION OF LIMIT Let $\varnothing \neq A \subseteq \mathbb{R}$ and $x_o \in \mathbb{R}$ be a point of accumulation of A. Let $L \in \mathbb{R}$ and $f : A \to \mathbb{R}$. Then, $L = \lim_{x \to x_o} f(x) \Leftrightarrow$ for very sequence $(x_n)_{n=1}^{\infty}$ in $A - \{x_o\}$ converging to x_o, the sequence $(f(x_n))_{n=1}^{\infty}$ converges to L. ///

SEQUENTIAL CHARACTERIZATION OF CONTINUITY

Let $\varnothing \neq A \subseteq \mathbb{R}$, $a \in A$ and $f : I \to \mathbb{R}$. Then,

(i) ***Definition*** f is *continuous at a* if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\left.\begin{array}{l} x \in A, \text{ and} \\ |x - a| < \delta(\varepsilon) \end{array}\right\} \Rightarrow | f(x) - f(a)| < \varepsilon.$$

(ii) ***Sequential Characterization*** f is continuous at a

$\Leftrightarrow$ For EVERY sequence $(x_k)_{k=1}^{\infty}$ in A converging to a, the sequence of values, $(f(x_k))_{k=1}^{\infty}$, converges to $f(a)$. ///

Let I be an interval of $\mathbb{R}$ and $a \in I$. The function $f : I \to \mathbb{R}$ is said to be *differentiable at a* if the function

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$f^{*a} : I - \{a\} \to \mathbb{R}, x \mapsto \dfrac{f(x) - f(a)}{x - a}, x \in I - \{a\}$

has limit at a. If so, $\lim\limits_{x \to a} f^{*a}(x)$ is called the *derivative of f at a* and denoted

$f'(a)$. That is, $\lim\limits_{x \to a} f^{*a}(x) = f'(a)$.

Again, let I be an interval and $a \in I$. Then, clearly, a moment's thought shows that
(i) there exists an interval J_{aI} such that
(ii) $0 \in J_{aI}$, and
(iii) $\kappa : J_{aI} \to \mathbb{R}, h \mapsto a + h, h \in J_{aI}$, is a bijection
See Figure 1below.

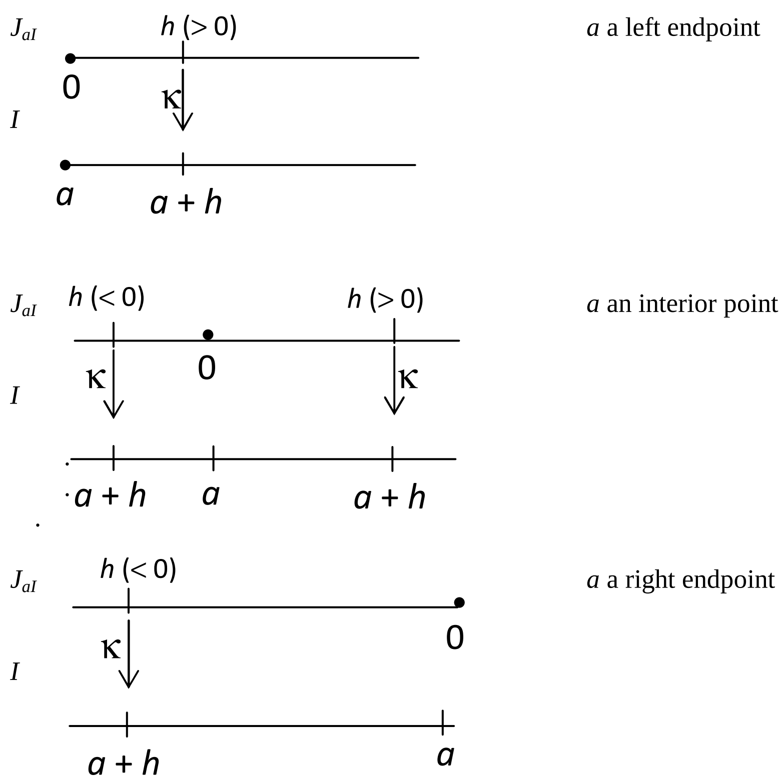


Fig. 1:
Define

$\omega : J_{aI} - \{0\} \to \mathbb{R}, h \mapsto a + h, \ h \in J_{aI} - \{0\}$
Clearly, $\omega = \kappa \mid J_{aI} - \{0\}$. And,
(iv) $\omega(h) \neq a \ \lim\limits_{h \to 0} \omega(h)$ for $h \in J_{aI} - \{0\}$

Now let $f : I \to \mathbb{R}$. Defined
$f^{*J_{aI}} : J_{aI} - \{0\} \to \mathbb{R}$,

$h \mapsto \dfrac{f(a+h) - f(a)}{h}$

Clearly,
$f^{*J_{aI}} = f^{*a} \circ \omega$.

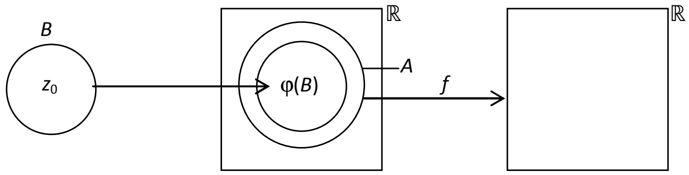
The aim of this paper is to show that f is differentiable at a if and only if $\lim\limits_{h \to 0} f^{*J_{aI}}(h)$ exists, and that if this is so, $f'(a) =$

$\lim_{x\to a} f*^{a}(x) = \lim_{h\to 0} f^{*J_{aI}}(h)$. We achieve this by employing an obscure theorem of K.G. Binmore found in his book [3]

2.0 Binmore's (Obscure) Theorem

Suppose $\varnothing \neq B \subseteq \mathbb{R}$, $\varphi : B \to \mathbb{R}$, $\varphi(B) \subseteq A \subseteq \mathbb{R}$, $z_0 \in \mathbb{R}$ is a point of accumulation of B, $\lim_{z\to z_o} \varphi(z)$ exists, equals $a \in \mathbb{R}$ and a is a point of accumulation of A, and, $f : A \to \mathbb{R}$ has limit equal to ℓ, say, at a. *Note*: z_0 may belong to B or not, just as a may belong to A or not; z_0 and a are just points of accumulation of B and A, respectively.

Fig. 2:



K.G. Binmore in [3, p.81], with the help of a

Counter Example $\varphi = \kappa_1 : \mathbb{R} \to \mathbb{R}$, $x \mapsto 1$ for all $x \in \mathbb{R}$,

and $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \begin{cases} 3, & \text{if } x = 1 \\ 2, & \text{if } x \neq 1 \end{cases}$,

and $z_0 = 0$,

[| $\lim_{z\to 0} \varphi(z) = 1$, $\lim_{x\to 1} f(x) = 2$, $f\text{o}\varphi(z) = f(1) = 3$, for all $z \in \mathbb{R}$. So, $\lim_{z\to 0} (f\text{o}\varphi)(z) = 3 \neq 2 = \lim_{x\to 1} f(x)$. So, in general $\lim_{x\to a} f(x) \neq \lim_{z\to z_0} (f\text{o}\varphi)(z)$.|]

showed, as the reader verifies easily, that it does not necessarily follow from these hypotheses that the composition $f\,\text{o}\varphi : B \to \mathbb{R}$, $z \mapsto f(\varphi(z))$, $z \in B$

has limit at z_0 equal to $\lim_{x\to a} f(x)$ Actually, a very careful look at the situation here shows that the problem arises from the fact that there may exist in every deleted ε-neighbourhood of z_o, $N_\varepsilon^{'}(z_o)$, points $p \in B$ such that $\varphi(p) = a$ but $f(a) \neq \lim_{x\to a} f(x)$.

Binmore proceeded to give in [3] two sufficient conditions under which $\lim_{z\to z_o} f\,\text{o}\varphi(z)$ exists and equals $\lim_{x\to a} f(x)$. We state them.

Binmore's Theorem [3, Theorem 8.17, p.81] Suppose $\varnothing \neq B \subseteq \mathbb{R}$, $\varphi : B \to \mathbb{R}$, $\varphi(B) \subseteq A \subseteq \mathbb{R}$, $z_0 \in \mathbb{R}$ is a point of accumulation of B, $\lim_{z\to z_o} \varphi(z)$ exists, equals $a \in \mathbb{R}$, and a is a point of accumulation of A, and $f : A \to \mathbb{R}$ has limit, $\lim_{x\to a} f(x)$, at a.

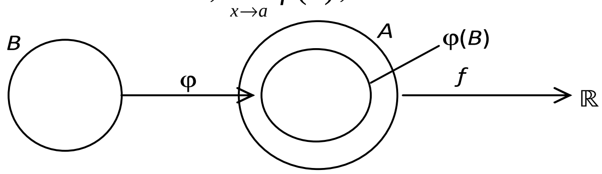


Fig. 3:

Then, either of the two conditions below is sufficient for $\lim_{z\to z_o} f\,\text{o}\varphi(z)$ to exist and be equal to $\lim_{x\to a} f(x)$.

(i) f is continuous at a.(Here $a \in A$)

(ii) There exists a deleted neighbourhood, $N_\delta^{'}(z_0)$, say, of z_0 (for some $\delta > 0$) such that $\varphi(z) \neq \lim_{z\to z_o} \varphi(z)$ for all $z \in N_\delta^{'}(z_0) \cap B$. ///

Observations (a) It is clear from (i) and the sequential characterization of continuity 1.2 that if $\lim_{z\to z_o}(f\,o\varphi)(z)$ exists, it is equal to $\lim_{x\to a} f(x)$. It is also clear from (ii0 and the Sequential Characterization of Limit 1.1 that if $\lim_{z\to z_o}(f\,o\varphi)(z)$ exists, it is equal to $\lim_{x\to a} f(x)$. Or, see Binmore's proof on page 81/82 of [3]. Indeed, [2, Exercise 5.2.6, p.129] says that if (i) is true then $\lim_{z\to z_o}(f\,o\varphi)(z)$ exists and equals $\lim_{x\to a} f(x)$.

(b) The reader should compare this theorem with [4, Exercise 3(a) (4), p.35].

3.0 The Characterization

We state and prove the characterization in question.

THE CHARACTERIZATION Let I be an interval and $a \in I$. A function $f : I \to \mathbb{R}$ is differentiable at $a \Leftrightarrow$ the function

$$\left.\begin{array}{l} f^{*J_{aI}} : :J_{aI} - \{0\} \to \mathbb{R}, \\ h \mapsto \dfrac{f(a+h)-f(a)}{h} \end{array}\right\} \quad \ldots(*)$$

has limit at 0. If (*) holds, then $\lim_{h\to 0} f^{*J_{aI}}(h)$, usually written $\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$, equals $f'(a)$.

Proof We first establish the implication $\Leftarrow$. So, suppose $\lim_{h\to 0} f^{*J_{aI}}(h)$ exists. Clearly,

$$f^{*J_{aI}} : :J_{aI} - \{0\} \to \mathbb{R},$$

$$h \mapsto \frac{f(a+h)-f(a)}{h}$$

and

$$f^{*J_{aI}} = f^{*a}\ \mathrm{o}\ \omega \qquad \ldots(\Sigma)$$

Suppose the sequence $(x_n)_{n=1}^{\infty}$ in $I - \{a\}$ converges to a. Then, $x_n = a + h_n$, $h_n \in J_{aI} - \{0\}$. By the Algebra of Limits $(x_n - a)_{n=N}^{\infty} = (h_n)_{n=1}^{\infty}$ $h_n \neq 0$ for all no converges to 0. By the assumption that $\lim_{h\to 0} f^{*J_{aI}}(h)$ exists it follows from the Sequential Characterization of Limit 1.1, that $\lim_{h\to 0} f^{*J_{aI}}(h) = \lim_{n\to\infty} f^{*J_{aI}}(h_n)$ exists and equals $\lim_{n\to\infty}(f^{*a}\mathrm{o}\omega)(h_n) = \lim_{n\to\infty}(f^{*a}(\omega)(h_n) = \lim_{n\to\infty}(f^{*a}(a+h_n) = \lim_{n\to\infty} f^{*a}(x_n)$; and so, $\lim_{n\to\infty} f^{*a}(x_n)$ exists. By the Sequential Characterization of Limit 1.1, again, $\lim_{n\to\infty} f^{*a}(x)$ exists; and equals $\lim_{h\to 0} f^{*J_{aI}}(h)$ This concludes the proof the implication $\Leftarrow$.

$\Rightarrow$: Suppose f is differentiable at a with derivative $f'(a)$. From (Σ),

$$f^{*J_{aI}} = f^{*a}\ \mathrm{o}\ \omega$$

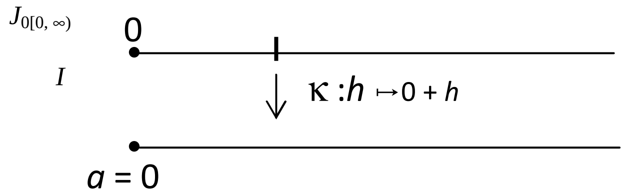
$$J_{aI} - \{0\} \xrightarrow{\ \omega\ } I - \{a\} \xrightarrow{\ f^{*a}\ } \mathbb{R}.$$

Clearly, replacing φ by ω and f by f^{*a} in Binmore's Theorem above, and noting that $\lim_{h\to 0}\omega(h) = a \neq \omega(h)$ for all $h \in J_{aI} - \{0\}$,one sees easily that condition (ii) of the theorem is met by f^{*a} and ω. Hence, $\lim_{h\to 0} f^{*J_{aI}}(h) = \lim_{n\to\infty}(f^{*a}\mathrm{o}\omega)(h)$ exists, and equals $\lim_{x\to a} f^{*a}(x) = f'(a)$. ///

4.0 Examples

We illustrate with some examples.

Example 1 Let $f : [0, \infty) \to \mathbb{R}$, $x \mapsto |x|$, $x \in [0, \infty)$. We show that f is differentiable at 0 with derivative 1. *Proof* : Here $I = [0, \infty)$ and $a = 0 \in [0, \infty) = I$. Clearly, (See Figure 4 below)

Fig. 4:


$J_{aI} = J_{0[0,\infty)} = [0, \infty)$
and

$$f^{*J_{0[0,\infty)}} = f^{*[0,\infty]} : [0, \infty) - \{0\} \to \mathbb{R},\ h \mapsto \frac{f(0+h) - f(0)}{h}$$

That is,

$$f^{*J_{0[0,\infty)}}(h) = \frac{|0+h| - |0|}{h} \text{ for all } h \in [0, \infty) - \{0\} = (0, \infty)$$

That is, for $h \in (0, \infty)$,

$$f^{*J_{0[0,\infty)}}(h) = \frac{|0+h| - |0|}{h} = \frac{|h| - |0|}{h} = \frac{|h|}{h},$$

which since $h \in (0, \infty)$,

$$= \frac{h}{h} = 1$$

That is, for $h \in (0, \infty)$,

$$f^{*J_{0[0,\infty)}}(h) = 1.$$

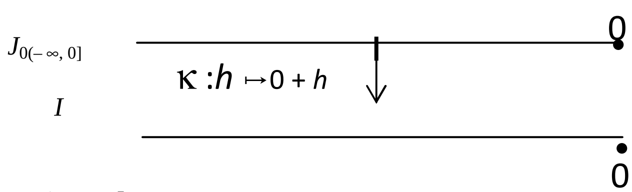
And so, by the Algebra of Limits $\lim_{h\to 0} f^{*J_{0[0,\infty)}}(h)$ exists and equals $\lim_{h\to 0} 1 = 1$.

Hence, f is differentiable at 0 with $f'(0) = 1$.

Example 2: Define

$g : (-\infty, 0] \to \mathbb{R},\ x \mapsto |x|,\ x \in (-\infty, 0]$.

We show that g is differentiable at 0 with derivative -1. *Proof* : Here, $I = (-\infty, 0]$, $a = 0 \in (-\infty, 0] = I$, (See Figure 5 below)

Fig. 5:


$J_{aI} = J_{0(-\infty, 0]} = (-\infty, 0]$,
and

$$f^{*J_{0(-\infty,0)}} : (-\infty, 0] - \{0\} \to \mathbb{R},\ h \mapsto \frac{f(0+h) - f(0)}{h},\ h \in (-\infty, 0) - \{0\}.$$

That is,

$$f^{*J_{0(-\infty,0)}} : (-\infty, 0] \to \mathbb{R},\ h \mapsto \frac{|0+h| - |0|}{h}$$

That is

$$f^{*J_{0(-\infty,0)}} : (-\infty, 0] \to \mathbb{R},\ h \mapsto \frac{|0+h| - |0|}{h}$$

That is, for $h \in (-\infty, 0]$,

$$f^{*J_{0(-\infty,0)}}(h) = \frac{|h| - |0|}{h}$$

$= \dfrac{-h-0}{h} \ -1.$

That is, for $h \in (-\infty, 0)$,

$f^{*J_{0(-\infty,0)}}(h) = -1,$

And so by the Algebra of Limits, $\lim\limits_{h\to 0} f^{*J_{0(-\infty,0)}}(h)$ exists, and $\lim\limits_{h\to 0} f^{*J_{0(-\infty,0)}}(h) = -1.$

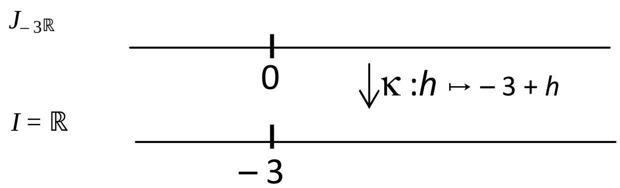
Example 3 Define the square function on $\mathbb{R}$,

$sq_{\mathbb{R}}: \mathbb{R} \to \mathbb{R},\ x \mapsto x^2, x \in \mathbb{R}.$

We show that $sq_{\mathbb{R}}$ is differentiable at -3, and compute its derivative there.

Proof : Here $I = \mathbb{R}$, $a = -3 \in \mathbb{R} = I$. See Figure 6 below.

Fig. 6:



$J_{aI} = J_{-3\mathbb{R}} = \mathbb{R},$

and

$sq_R^{*J_{-3R}} : J_{-3\mathbb{R}} - \{0\} = \mathbb{R} - \{0\} \to \mathbb{R},$

$h \mapsto \dfrac{sq_R(-3+h) - sq_R(-3)}{h}$

That is, for $h \in \mathbb{R} - \{0\}$,

$$sq_R^{*J_{-3R}} = sq_R^{*R}(h) = \frac{(-3+h)^2-(-3)^2}{h} = \frac{9+h^2-6h-9}{h} = \frac{h^2-6h}{h} = h-6$$

That is, for $h \in \mathbb{R} - \{0\}$,

$sq_R^{*R}(h) = h - 6,$

and so by the Algebra of Limits $\lim\limits_{h\to 0} sq_R^{*J_{-3R}}(h)$ exists and $\lim\limits_{h\to 0} sq_R^{*J_{-3R}}(h) = -6.$

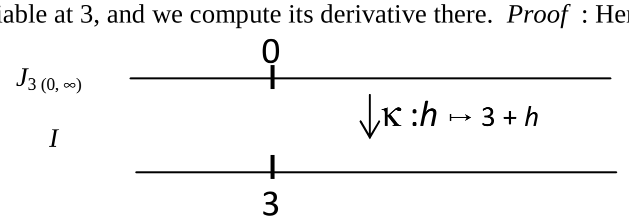
Hence, $sq_{\mathbb{R}}'(-3) = -6 = 2(-3)^{2+1}$.

Example 4: We show that the reciprocal function

$\text{rcp} : (0, \infty) \to \mathbb{R},\ x \mapsto \dfrac{1}{x}, x \in (0, \infty)$

is differentiable at 3, and we compute its derivative there. *Proof* : Here, $I = (0, \infty)$, $a = 3 \in (0, \infty) = I$.

Fig. 7:



$J_{3(0,\infty)} = (-3, \infty)$, and

$rcp^{*J_{3(0,\infty)}} : J_{3(0,\infty)} - \{6\} = (-3, \infty) - \{0\} \to \mathbb{R},$

$h \mapsto \dfrac{rcp(3+h) - rcp(3)}{h}$

That is,

$$rcp^{*J_{3(0,\infty)}}(h) = \frac{\frac{1}{3+h} - \frac{1}{3}}{h}, \; h \in (-3\,,\,\infty) - \{0\}.$$

That is, for $h \in (-3\,,\,\infty) - \{0\}$

$$rcp^{*J_{3(0,\infty)}}(h) = \frac{3-(3+h)}{(3+h)3\cdot h} = \frac{-h}{9h+3h^2} = -\frac{1}{9+3h}$$

That is, for $h \in (-3\,,\,\infty) - \{0\}$

$$rcp^{*J_{3(0,\infty)}}(h) = -\frac{1}{9+3h},$$

and so, by the Algebra of Limits $\lim_{h\to 0} rcp^{*J_{3(0,\infty)}}(h)$ exists and $= \lim_{h\to 0} -\frac{1}{9+3h}$

$$= -\frac{1}{9} = -\frac{1}{3^2} = -\frac{1}{a^2}.$$

REMARK The literature is almost completely silent on Binmore's Theorem, and so, indeed, the befitting adjective *obscure*. However, this obscure theorem has been used here to furnish the first known (at least, to the author) *analytic* proof of the *most* popular of all characterization of differentiability (Not less than four (4) other characterizations are recorded in the author's forthcoming book *Classical Real Analysis* 2). And so, henceforth, we do not have to feign *it is geometrically evident* as a proof of this characterization.

5.0 A Corollary

We furnish a proof of *Differentiability* $\Rightarrow$ *Continuity* using characterization. First, we remind the reader of some

Language* ε-δ *Definition of Continuity Let $\varnothing \neq A \subseteq \mathbb{R}$, $a \in A$ and $f : A \to \mathbb{R}$. f is said to be *continuous at a* provided whenever given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\left.\begin{array}{l} x \in A, \text{ and} \\ x \in N_{\delta(\varepsilon)}(a) \end{array}\right\} \Rightarrow |f(x) - f(a)| < \varepsilon. \;///$$

Again let $\varnothing \neq A \subseteq \mathbb{R}$, and $a \in A$. The element a may be a point of accumulation of A as well; otherwise a is called an *isolated* point of A.

ISOLATED POINT-CONTINUITY THEOREM If $a \in A \subseteq \mathbb{R}$ is an isolated point of A and $f : A \to \mathbb{R}$, then, f is continuous at a. ///

LIMIT-CONTINUITY THEOREM Let $\varnothing \neq A \subseteq \mathbb{R}$, **and** $a \in A$ a point of accumulation of A. Then, f is continuous at $a \Leftrightarrow \lim_{x\to 0} f(x)$ exists and equals $f(a)$. ///

Differentiability ⇒ Continuity Let I be an interval, $a \in I$ and $f : I \to \mathbb{R}$. Then, f is differentiable at $a \Rightarrow f$ is continuous at a.

Proof First, a

Notation : If $\varnothing \neq A \subseteq \mathbb{R}$, by

$i_A : A \to \mathbb{R},\; x \mapsto x,\; x \in A$

we mean the *insertion* of A into $\mathbb{R}$.

Now, let I be an interval, $a \in I$ and $f : A \to \mathbb{R}$ differentiable at a. And so, the function

$$f^{*J_{aI}} : J_{aI} - \{0\} \to \mathbb{R},$$

$$h \mapsto \frac{f(a+h) - f(a)}{h}$$

has a limit at a. Then, for $h \in J_{aI} - \{0\}$,

$$f^{*J_{aI}}(h) = \frac{f(a+h) - f(a)}{h}$$

from which follows that

$$(i_{J_{aI} - \{0\}} f^{*J_{aI}})(h) = f(a+h) - f(a)$$

for all $h \in J_{aI} - \{0\}$.
That is,

$$i_{J_{aI} - \{0\}} f^{*J_{aI}} = f\text{o}\omega - f(a) \qquad \ldots(\rho)$$

By the hypothesis of differentiability of f at a, our characterization of differentiability and the Algebra of Limits, it follows that $\lim_{h\to 0} (i_{J_{aI} - \{0\}} f^{*J_{aI}})(h)$ exists and so by (ρ) equals $\lim_{h\to 0} (f\text{o}\omega - f(a))(h)$.But by the (ii) of Binmore's

Theorem, $\lim_{h\to 0} (f\text{o}\omega)(h)$ exists, and

$$\lim_{h\to 0} (f\text{o}\omega)(h) = \lim_{h\to 0} f(x),$$

and so, by the Algebra of Limits,

$$\lim_{h\to 0} (f\text{o}\omega - f(a))(h) = \lim_{h\to 0} (f\text{o}\omega)(h) - f(a) = \lim_{x\to a} f(x) - f(a).$$

This, coming down from (ρ), we have shown that

$$\lim_{h\to 0} (i_{J_{aI} - \{0\}} f^{*J_{aI}})(h) = \lim_{h\to 0} (f\text{o}\omega - f(a))(h) = (\lim_{x\to a} f(x)) - f(a).$$

That is,

$$\lim_{h\to 0} (i_{J_{aI} - \{0\}} f^{*J_{aI}})(h) = (\lim_{x\to a} f(x)) - f(a) \qquad \ldots(\rho\rho)$$

But $\lim_{h\to 0} (i_{J_{aI} - \{0\}} f^{*J_{aI}})(h) = 0$, and so by the Algebra of Limits,

$$\lim_{h\to 0} (i_{J_{aI} - \{0\}} f^{*J_{aI}})(h) = \lim_{h\to 0} i_{J_{aI} - \{0\}}(h) \cdot \lim_{h\to 0} f^{*J_{aI}}(h) = 0 \cdot \lim_{h\to 0} f^{*J_{aI}}(h) = 0.$$

And so, by (ρρ),

$$0 = (\lim_{x\to a} f(x)) - f(a).$$

That is,

$$\lim_{x\to a} f(x) = f(a) \qquad \ldots. (\rho\rho\rho)$$

By the Limit Point-Continuity Theorem and (ρρρ) therefore, f is continuous at a.

13.0 References

1 George E. Andrews, *The geometric series in calculus*, The Amer. Math. MONTHLY, 105 (1998), 36 – 40.
2 Robert G. Bartle and Donald R. Sherbert, *Introduction to Real An- alysis*, Third Edition, John Wiley & Sons, Inc. 2000.
3 K.G. Binmore, *Mathematical Analysis, a straightforward appro- ach*, Cambridge University Press, 1977.
4 J.C. Burkill and H. Burkill, *A Second Course in Mathematical Analysis*, Vikas Publishing House, PVT Ltd, Cambridge University Press, Student Edition, First Published 1980.
5 Richard Courant and Fritz John, *Introduction to Calculus and Analysis*, Volume One, Wiley Interscience Publishers, 1965.
6 Adegoke Olubummo, *Introduction to Real Analysis*, Heinemann Educational Book (Nig.) Limited, 1979.
7 Sunday Oluyemi, *Two Characterizations of Differentiability at a Point*, Science Focus, 11(2) (2006), 52 – 55.
8 Maxwell Rosenlicht, *Introduction to Analysis*, Scott, Foresman and Company, 1968