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A Remark on the Definition of the Partial Derivative

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Abstract

This short note simply furnishes the puritanical definition of the partial derivative, and offers two clarifications on this definition.

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1.0 Introduction

Our language and notations shall be pretty standard as found for example in [2, 3, 4, 7, 8]. By $\mathbb{R}$ we shall mean the *real numbers* and by $\mathbb{N}$ the *natural numbers* 1, 2, ……., If $n \in \mathbb{N}$ and $n \geq 2$, by $\mathbb{R}^n$ we shall mean the *Cartesian space* $\mathbb{R}$x$\mathbb{R}$x...x$\mathbb{R}$, (n factors). $\mathbb{R}^n$ with the *Euclidean norm* $\|\ \|$[8, p.206] is called the *Euclidean n-space* [7, 2.19, p. 51]. If $I_1, I_2, \ldots, I_n$ are intervals in $\mathbb{R}$, the Cartesian product

$$I_1 \text{x}\, I_2 \text{x} \ldots \text{x}\, I_n\ (\subseteq \mathbb{R}^n) \qquad \ldots(\Delta)$$

is called an *interval* in $\mathbb{R}^n$; an interval (Δ) in which the *sides* $I_1, I_2, \ldots, I_n$ are finite intervals in $\mathbb{R}$ is called a *cell*[8, First paragraph p.52] with I_k , $k = 1, 2, \ldots, n$, called the kth *side* of the cell. An *open interval* /*open cell* is one with all its sides open intervals of $\mathbb{R}$.

Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. By the *Euclidean norm* of a, $\|a\|$, is meant $\sqrt{a_1^2 + a_2^2 + \ldots.. + a_n^2}$ [8, p. 206] and if $r \in \mathbb{R}$, $r > 0$, by a *ball of radius r centered on a*, $B(a, r)$, is meant the set $\{x \in \mathbb{R}^n : \|x - a\| < r\}$[2, 3.3, p.49] referred to in [4, Definition 59.1, p. 211] as an *r-neighbourhood of* a. If $\varnothing \neq A \subseteq \mathbb{R}^n$ and $a \in A$, a is called an *interior point of A*[9, Definition 59.2, p.211][2, Definition 3.5, p.49] if there exists $r > 0$ such that $B(a, r) \subseteq A$.
If $\delta > 0$ and $a \in \mathbb{R}$, by $N_\delta(a)$ shall be meant the open interval $(a - \delta, a + \delta)$ called the δ-*neighbourhood* of a. $N_\delta^{'}(a) = (a - \delta, a + \delta) - \{a\}$, is called the *deleted* δ-*neighbourhood* of a.

2.0 Sequential Characterization of Limit

Let $\varnothing \neq A \subseteq \mathbb{R}$ and $x_o \in \mathbb{R}$ (not necessarily belonging to A) be a point of accumulation of A. The number L is called the *limit* of the real function $f : A \rightarrow \mathbb{R}$ at x_o provided whenever given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that
$x \in A$ and $0 < |x - x_o| < \delta(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$ ... (ρ)
And we write $\lim_{x \to x_o} f(x) = L$.

Let $\varnothing \neq A \subseteq \mathbb{R}$ and $x_o \in \mathbb{R}$ be a point of accumulation of A. Let $L \in \mathbb{R}$ and $f : A \rightarrow \mathbb{R}$. Then, $L = \lim_{x \to x_o} f(x) \Leftrightarrow$ for every sequence $(x_n)_{n=1}^{\infty}$ in $A - \{x_o\}$ converging to x_o, the sequence $\left(f(x_n)\right)_{n=1}^{\infty}$ converges to L. ///

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THEOREM 1 (i) Let $\varnothing \neq B \subseteq A \subseteq \mathbb{R}$ and suppose $x_o \in \mathbb{R}$ is a point of accumulation of B, and that $f : A \to \mathbb{R}$ has limit L at x_o. Then, the restriction

$f \mid B : B \to \mathbb{R},\ x \mapsto f(x),\ x \in B$
of f to B also has limit L at x_o.

(ii) Let $\varnothing \neq A \subseteq \mathbb{R}$ and suppose x_o is a point of accumulation of A. If $f : A \to \mathbb{R}$ has limit at x_o, then x_o is still a point of accumulation of $A - \{x_o\}$ and the restriction $f^* : A - \{x_o\} \to$ $\mathbb{R}$, $f^*(x) = f(x),\ x \in A - \{x_o\}$, of f to $A - \{x_o\}$, also has limit at x_o, and
$\lim\limits_{x \to x_o} f^*(x) = \lim\limits_{x \to x_o} f(x)$.

Proof (i) is immediate from (ρ) above while (ii) follows from (i). ///

The converse of THEOREM 1 is false as one shows easily by simple examples. That is, $\lim\limits_{x \to x_o} (f \mid B)(x)$ exists does not necessarily imply that $\lim\limits_{x \to x_o} f(x)$ exists. For an instance,

Example 2 Consider the function

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

Let $\mathbb{Q}$ be the rationals. By the Density Theorems of Elementary Real Analysis, 5 is a point of accumulation of $\mathbb{Q}$. Clearly, $\lim\limits_{x \to 5} (f \mid \mathbb{Q})(x)$ exists, since $f \mid \mathbb{Q}$ is the constant function

$\kappa_0 : \mathbb{Q} \to \mathbb{R},\ x \mapsto 0$ for all $x \in \mathbb{Q}$,
and we know that $\lim\limits_{x \to 5} \kappa_0(x)$ exists and = 0. But, $\lim\limits_{x \to 5} f(x)$ does not exist.

We furnish here a converse of THEOREM 1.

A CONVERSE (Sunday Oluyemi[6, Theorem B p.108]) Suppose $\varnothing \neq B \subseteq A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $x_o \in \mathbb{R}$ a point of accumulation of B. If there exists $\delta^* > 0$ such that $B \supseteq N'_{\delta^*}(x_o) \cap A$, then if $\lim\limits_{x \to x_o} (f \mid B)(x)$ exists so does $\lim\limits_{x \to x_o} f(x)$, and $\lim\limits_{x \to x_o} f(x) = \lim\limits_{x \to x_o} (f \mid B)(x)$.

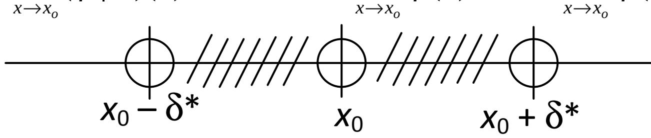

Fig. 1

Proof Let $\varepsilon > 0$. By hypothesis, there exists $\delta'(\varepsilon) > 0$ such that

$x \in N'_{\delta'(\varepsilon)}(x_o) \cap B \Rightarrow |(f \mid B)(x) - L| < \varepsilon$ ...(∇)

where $L = \lim\limits_{x \to x_o} (f \mid B)(x)$. Let $\delta(\varepsilon) = \min\{\delta'(\varepsilon), \delta^*\}$.

Then,

$B \supseteq N'_{\delta^*}(x_o) \cap A \supseteq N'_{\delta(\varepsilon)}(x_o) \cap A$ ...(Δ)

and

$N'_{\delta'(\varepsilon)}(x_o) \supseteq N'_{\delta(\varepsilon)}(x_o)$ ...(ΔΔ)

and so by (Δ) and (ΔΔ),

$x \in N'_{\delta(\varepsilon)}(x_o) \cap A \Rightarrow x \in N'_{\delta'(\varepsilon)}(x_o) \cap B$ …(∇∇)

(∇) and (∇∇) now give

$x \in N'_{\delta(\varepsilon)}(x_o) \cap A \Rightarrow |f(x) - L| < \varepsilon$

and so $\lim_{x \to x_o} f(x)$ exists and $\lim_{x \to x_o} f(x) = \lim_{x \to x_o} (f \mid B)(x)$. ///

A careful application of the Sequential Characterization of Limit furnishes another proof. ///

COROLLARY Suppose $I \subseteq \mathbb{R}$ is an interval, $x_o \in I$ and $f : I \to \mathbb{R}$. Suppose the restriction to $I - \{x_o\}$ of f,

$f \mid I - \{x_o\} : I - \{x_o\} \to \mathbb{R},\ x \mapsto f(x)$ for all $x \in I - \{x_o\}$,
has limit L at x_o. Then, f has limit L at x_o.

Proof Immediate. ///

Let I be an interval of $\mathbb{R}$ and $a \in I$. Clearly, a is a point of accumulation of $I - \{a\}$. The real function $f : I \to \mathbb{R}$ is said to be *differentiable at a* if the function

$$f^{*a} : I - \{a\} \to \mathbb{R},\ x \mapsto \frac{f(x) - f(a)}{x - a},\quad x \in I - \{a\}$$

has limit at a [2, 3]. If so, $\lim_{x \to a} f^{*a}(x)$ is called the *derivative of f at a* and denoted $f'(a)$. That is,

$$\lim_{x \to a} f^{*a}(x) = f'(a).$$

We shall use /// to signify the end or absence of a proof.

3.0 A Characterization of Differenti-Ability

Before stating the characterization of differentiability of this section, we give some language and notations found in its statement.

So, let I be an interval and $a \in I$. Then, clearly, a moment's thought shows that

(i) there exists an interval J_{aI} such that

(ii) $0 \in J_{aI}$, and

(iii) $\kappa : J_{aI} \to I,\ h \mapsto a + h,\ h \in J_{aI}$ is a bijection see Figure 1.

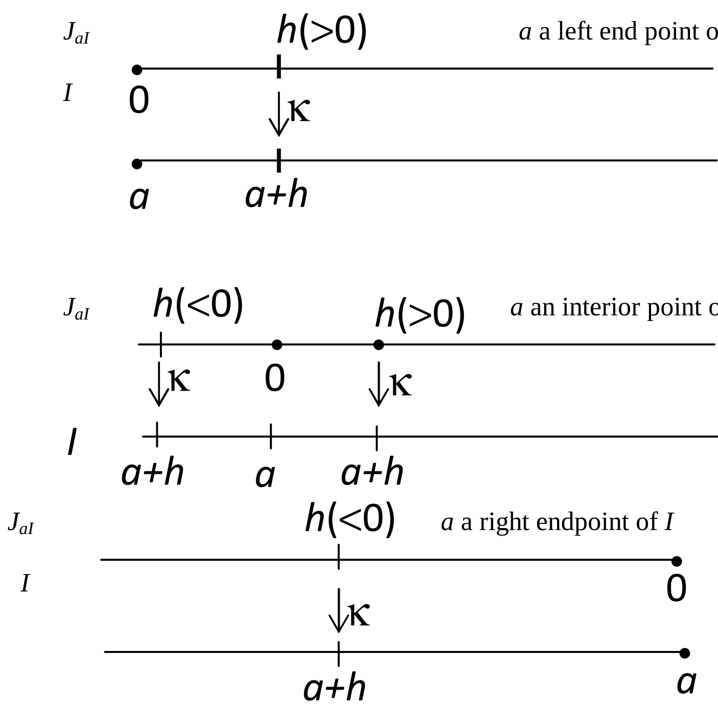

Fig. 1

Let $f : I \to \mathbb{R}$, and define the function

$$f^{*J_{aI}} : J_{aI} - \{0\} \to \mathbb{R},$$

$$h \mapsto \frac{f(a+h) - f(a)}{h}$$

Now to the characterization.

The Characterization [7] Let I be an interval and $a \in I$. A function $f : I \to \mathbb{R}$ is defifferentiable at $a \Leftrightarrow$ the function

$$\left.\begin{array}{ll} f^{*J_{aI}} : & J_{aI} - \{0\} \to \mathbb{R}, \\ h & \mapsto \dfrac{f(a+h)-f(a)}{h} \\ \text{has limit at } 0. & \end{array}\right\} \qquad \text{....(*)}$$

If (*) holds, then $\lim_{h\to 2} f^{*J_{aI}}(h)$ usually written $\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ equals $f'(a)$. ///

4.0 The Partial Derivative

This section furnishes the *two clarifications* advertised in the abstract.

I *Puritanical Definition of the Partial Derivative* The derivative $f'(a)$ of the real function $f : I \to \mathbb{R}, \varnothing \neq I \subseteq \mathbb{R}$, is defined only at points a of a domain I **which is an interval** [3, Definition 6.1.1, p.158][2, Definition 5.1, p.104]

just as solutions of ordinary differential equations are sought over intervals of $\mathbb{R}$ and not over arbitrary subsets of $\mathbb{R}$. Precisely, from Section 2, f is differentiable at a provided $\lim_{x\to a} f^{*a}(x)$ exists where

$$\begin{array}{ll} f^{*a} : I - \{a\} \to \mathbb{R}, & \\ x & \mapsto \dfrac{f(x)-f(a)}{x-a} \end{array}$$

And, we define

$$f'(a) = \lim_{x\to a} f^{*a}(x) \qquad \text{...(*)}$$

This informs why, for the definition of, say, the *first partial derivative*, $D_1 f(a)$, at a of $f : A \to \mathbb{R}, \varnothing \neq A \subseteq \mathbb{R}^n$, $n \geq 2$, $a = (a_1, a_2, \ldots, a_n) \in A$ **to make sense, there is the need for the existence of an interval** I_1, say, of $\mathbb{R}$ such that $a_1 \in I_1$ and $I_1 \text{x}\{a_2\}\text{x}\{a_3\}\text{x}\ldots\text{x}\{a_n\} \subseteq A$. Then, following (*) we define

$f^{p1} : I_1 \to \mathbb{R}, x \mapsto f(x, a_2, \ldots, a_n), x \in I_1$

and

$$\begin{array}{ll} f^{p1*a_1} : & I_1 - \{a_1\} \to \mathbb{R}, \\ x & \mapsto \dfrac{f^{p1}(x)-f^{p1}(a_1)}{x-a_1} = \dfrac{f(x,a_2,\ldots,a_n)-f(a_1,a_2,\ldots,a_n)}{x-a_1} \end{array}$$

And, then, define

$$D_1 f(a) \equiv \lim_{x\to a_1} f^{p_1*a_1}(x)$$

Similarly, to define $D_2 f(a)$ there is the need to have an interval I_2 in $\mathbb{R}$ such that $a_2 \in I_2$ and $\{a_1\}\text{x}I_2\text{x}\{a_3\}\text{x}\ldots\text{x}\{a_n\} \subseteq A$. And then define

$f^{p_2} : I_2 \to \mathbb{R}, x \mapsto f(a_1, x, a_3, \ldots, a_n), x \in I_2$

and

$$\begin{array}{ll} f^{p_2*a_2} : & I_2 - \{a_2\} \to \mathbb{R}, \\ x & \mapsto \dfrac{f^{p2}(x)-f^{p2}(a_2)}{x-a_2} = \dfrac{f(a_1,x,a_3,\ldots,a_n)-f(a_1,a_2,\ldots,a_n)}{x-a_2} \end{array}$$

And, then, define the second partial derivative, $D_2 f(a)$, of f at a, by

$$D_2 f(a) \equiv \lim_{x\to a_2} f^{p_2*a_2}(x)$$

The requirements for and the definitions of $D_3 f(a), \ldots, D_n\ f(a)$, are now clear. These are the puritanical definitions.
If $a \in A$ is interior, then IMMEDIATE 1(i) of [5] provides an open interval $I = I_1 x I_2 x \ldots x\ I_n$, say, indeed, an open cell, such that

$a = (a_1, a_2, ..., a_n) \in I = I_1 \text{ x } I_2 \text{ x } \ldots \text{ x } I_n \subseteq A.$

And then, $D_1 f(a), D_2 f(a), \ldots, D_n f(a)$, are all simultaneously definable since the requirement that the domains $I_1, I_2, \ldots, I_n$ of $f^{p1}, f^{p2}, \ldots, f^{pn}$ respectively, be intervals of $\mathbb{R}$ are now simultane- ously met. And the OBCT of [5] here comes up with the first of the advertised clarifications — *the need and existence of an interval I_k, say, in $\mathbb{R}$, on which* f^{pk} is defined, with $a_k \in I_k$, $a = (a_1, a_2, \ldots, a_n)$.

II *The Directional Derivative Route* Ironically, many a good author, if not all, do not employ the puritanical definition of the partial derivative in **I** to define the partial derivative. The need for and the existence of the interval I_k, $k = 1, 2, \ldots, n$, **are not known to the literature**. The route to the literature's definition of the partial derivative has always been through the *directional derivative*, defining the *kth partial derivative* as the directional derivative in the direction of the *k*th fundamental vector $e_k = (0, 0,$
$\ldots, 1, \ldots 0)$[The 1 is in the *k*th position]. Our task here in this section is to show that the definition of the partial derivative arrived at through this route is equivalent to the puritanical definition in **I.** So, we first give a brief description of the directional derivative. The reader is assumed to be familiar with the rudiments of *Calculus in* $\mathbb{R}^n$, as found, for example, in [9].

Consider $\mathbb{R}^n$, $n \geq 2$. The elements $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, …., $e_n = (0, 0, \ldots, 0, 1)$, of $\mathbb{R}^n$ are called the *fundamental vectors* of $\mathbb{R}^n$. $\theta = (0, 0, \ldots, 0) \in \mathbb{R}^n$ is called the *zero vector* of $\mathbb{R}^n$. Clearly, $e_k \neq \theta$ for all $k = 1, 2, \ldots, n$.

Let $\varnothing \neq A \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, and $a = (a_1, a_2, \ldots, a_n) \in A$ interior to A. By the IMMEDIATE 1(i) of [5] there exists a finite open interval $I = I_1 \text{ x } I_2 \text{ x } \ldots \text{ x } I_n$, say, about a and $I \subseteq A$ [*about a* means $a = (a_1, a_2, \ldots, a_n) \in I_1 \text{ x } I_2 \text{ x } \ldots \text{ x } I_n$]. Suppose $I_1 = (x_1, y_1)$, $I_2 = (x_2, y_2)$, …, $I_n = (x_n, y_n)$, where $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in \mathbb{R}$, $x_1 < y_1$, $x_2 < y_2$, …, $x_n < y_n$, and so,
$x_1 < a_1 < y_1$, $x_2 < a_2 < y_2$, …, $x_n < a_n < y_n$.
Let

$$\delta = \min\{ a_1 - x_1,\ y_1 - a_1,\ a_2 - x_2,\ y_2 - a_2, \ldots, a_n - x_n,\ y_n - a_n\} > 0 .$$

Suppose $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, $u \neq \theta$, and so $\| u \| > 0$. Therefore, if $t \in \mathbb{R}$ and

$$| t | < \frac{\delta/2}{\| u \|}$$

we shall have

$$x_k < a_k - \frac{\delta}{2} < a_k - | t | \| u \| \leq a_k - | t ||u_k |$$

$$\leq \begin{Bmatrix} a_k + tu_k \\ a_k - tu_k \end{Bmatrix} \leq a_k + | t ||u_k|$$

$$\leq a_k + | t |\|u\| < a_k + \frac{\delta}{2} < y_k,$$

$k = 1, 2, \ldots, n$. Let $\dfrac{\delta/2}{\| u \|} = \varepsilon$. We have thus shown that

THEOREM Suppose $\varnothing \neq A \subseteq \mathbb{R}^n$, a an interior point of A, $u \in \mathbb{R}^n$, $u \neq \theta$ and so $\| u \| > 0$. Then, there exists a finite open interval I about a, and an $\varepsilon > 0$ such that
$| t | < \varepsilon \Rightarrow a + tu \in I \subseteq A$. ///

Let $\varnothing \neq A \subseteq \mathbb{R}^n$, $n \geq 2$, $\theta \neq u \in \mathbb{R}^n$, $f : A \rightarrow \mathbb{R}$, and $a \in A$ an interior point of A. By the preceding THEOREM, there exists a finite open interval I and an $\varepsilon > 0$ such that

$|t| < \varepsilon \Rightarrow a + tu \in I \subseteq A$

Hence,

$-\varepsilon < t < \varepsilon \Rightarrow a + tu \in I \subseteq A.$

Hence, in particular,

$t \in N_\varepsilon^{'}(0) \Rightarrow a + tu \in I \subseteq A.$

Therefore, the function

$$f^{Du} : N_\varepsilon^{'}(0) \rightarrow \mathbb{R}$$

$$t \mapsto \frac{1}{t}(f(a+tu) - f(a))$$

with domain the deleted ε-neighbourhood of 0 in $\mathbb{R}$, $N_\varepsilon^{'}(0)$, is well-defined. Of course, 0 is a point of accumulation of $N_\varepsilon^{'}(0)$. If $\lim_{t\to 0} f^{Du}(t)$ exists it is called the *directional derivative of f at a in the direction of u*, and denoted $D_u f(a)$.

Let $\varnothing \neq A \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, $a \in A$ an interior point of A, and $f : A \rightarrow \mathbb{R}$. The directional derivatives of f at a, $De_1 f(a)$, $De_2 f(a)$, …, $De_n f(a)$, in the direction of the fundamental vectors

$e_1, e_2, \ldots, e_n$, respectively, are called in the literature the *partial derivatives of f at a*. $D_1 f(a) = De_1 f(a)$ is called the *first partial derivative* of f at a, $D_2 f(a) = De_2 f(a)$ is called the *second partial derivative of f at a*, …., $D_n f(a) = De_n f(a)$ is called the *nth partial derivative of f at a*.

III *Equivalence of the Route and the Puritanical Definition* Let us re-examine the definition of $D_k f(a) = De_k f(a)$ for $k \in \{1, 2, ..., n\}$. So let $\varnothing \neq A \subseteq \mathbb{R}^n$, $a = (a_1, a_2, \ldots, a_n) \in A$ interior to A and $f : A \rightarrow \mathbb{R}$. By what was shown in II, there exists a finite open interval $I = I_1 \text{ x } I_2 \text{ x } \ldots \text{ x } I_n$ of $\mathbb{R}^n$ and an $\varepsilon > 0$ such that

$|t| < \varepsilon \Rightarrow a + te_k \in I \subseteq A, k = 1, 2, \ldots, n.$

That is,

$|t| < \varepsilon \Rightarrow (a_1, a_2, \ldots, a_n) + t(0, 0, \ldots, 0, 1, 0, \ldots, 0) \in I_1 \text{ x } I_2 \text{ x } \ldots \text{ x } I_n \subseteq A.$

That is,

$|t| < \varepsilon \Rightarrow (a_1, a_2, \ldots, a_{k-1}, a_k + t, a_{k+1}, \ldots, a_n) \in I_1 \text{ x } I_2 \text{ x } \ldots \text{ x } I_n \subseteq A, k = 1, 2, \ldots, n.$

Hence,

$|t| < \varepsilon \Rightarrow a_k + t \in I_k, k = 1, 2, \ldots, n.$ …(∇)

and so,

$t \in N_\varepsilon(0) \Rightarrow a_k + t \in I_k \ (k = 1, 2, \ldots, n.)$ ....(Δ)

By definition,

$$D_k f(a) = De_k f(a) = \lim_{t\to 0} f^{Dek}(t)$$

where

$$f^{D_{e_k}} : N_\varepsilon^{'}(0) \rightarrow \mathbb{R}$$

$$t \mapsto \frac{1}{t}(f(a+te_k) - f(a))$$

So,

$$D_k f(a) = \lim_{t\to 0} f^{Dek}(t) = \lim_{t\to 0} \frac{f(a+te_k) - f(a)}{t}$$

$$= \lim_{t\to 0} \frac{f\big((a_1, a_2, \ldots, a_n) + t(0, 0, \ldots, 0, 1, 0, \ldots, 0)\big) - f\big((a_1, a_2, \ldots, a_n)\big)}{t}$$

$= \lim_{t\to 0} \dfrac{f\left((a_1, a_2, ..., a_{k-1}, a_k + t, a_{k+1}, ..., a_n)\right) - f\left((a_1, a_2, ..., a_n)\right)}{t}$

which clearly,

$= \lim_{t\to 0} \dfrac{F(a_k + t) - F(a_k)}{t} \quad = \quad \lim_{h\to 0} \dfrac{F(a_k + h) - F(a_k)}{h}$

where

$F \; : \quad N_\varepsilon(0) \;\to\; \mathbb{R}$

$x \;\mapsto\; f((a_1, a_2, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_n))$.

By(∇), clearly,

$N_\varepsilon(0) \subseteq J_{a_k I_k}$ ….(ρ)

Define

$F^* : \; J_{a_k I_k} \;\to\; \mathbb{R}$,

$x \;\mapsto\; f((a_1, a_2, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_n))$

Clearly, $F = F^*| N_\varepsilon(0)$.

By (ρ) and A Converse of Section 2, therefore

$D_k f(a) = \lim_{h\to 0} \dfrac{F(a_k + h) - F(a_k)}{h}$

$= \lim_{h\to 0} \dfrac{F^*(a_k + h) - F^*(a_k)}{h}$

That is,

$D_k f(a) = \lim_{h\to 0} \dfrac{F^*(a_k + h) - F^*(a_k)}{h}$ …(ρρ)

Now

$F^{**}J_{a_k I_k} \; : \; J_{a_k I_k} - \{0\} \to \mathbb{R}$,

$h \quad \mapsto \dfrac{F^*(a_k + h) - F^*(a_k)}{h}$

So, by the Characterization of differentiability

$\lim_{h\to 0} \; F^{**}J_{a_k I_k}(h) = F^{*\prime}(a_k)$

$= \lim_{x\to a_k} \dfrac{F^*(x) - F^*(a_k)}{x - a_k}$

$= \lim_{x\to a_k} \dfrac{f(a_1, a_2, ..., a_{k-1}, x, a_{k+1}, ..., a_n) - f(a_1, ..., a_n)}{x - a_k}$

$= \lim_{x\to a_k} f^{pk^* a_k}(x)$, which is the puritanical definition

And we have finished furnishing the second clarification advertised in the abstract.

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