Journal of the Nigerian Association of Mathematical Physics
Volume32,(November, 2015), pp 69 – 74
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Refinement of Hardy's Inequalities on Time Scales via Super-Quadratic and Sub-Quadratic Functions

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Abstract

This paper deals with a time scale on a new refined weighted Hardy in-equality for p >2 and provides the prove back to discussions. Some refinements of classical inequalities on time scales are obtained using properties of super-quadratic and sub-quadratic function.

Keywords: Hardy's inequalities, Time scales; super-quadratic functions, sub-quadratic functions

1.0 Introduction

1.1 Time Scale Calculus

The calculus of time scales was initiated by Stefan Hilger in [1] in order to create a theory that can unify discrete and continuous analysis. We first briefly introduce the time scales calculus and refer the interested reader elsewhere for more details [2,3,4,5,6]

A time scale (which is a special case of a measure chain) is an arbitrary non-empty closed subset of the real numbers throughout this paper, we will denote a time scale by the T. We will also, assume throughout that a time scale T has a topology that it inherits from the real numbers with the standard topology. The two most popular examples of time scales are the real number R and the discrete time scale Z. Let us start by defining the forward and backward jump operators.

Definition 1.2 Let T be a time scale for $t \in T$ we define the forward jump operator by

$$\sigma : T \rightarrow T$$
$$\sigma(t) = in\, f\ (s \in T : s > t),$$

while the backward jump operator $\rho : T \rightarrow T$ is defined by

$$\rho(t) = \sup\ (s \in T : s < t);$$

In this definition we put $inf\emptyset = supT(i{:}e{:};\ \sigma(t) = t$ if T has a maximum t) and $sup\,\emptyset = infT\ (i{:}e;\ \rho(t) = t\ if\ T, has\ a\ minimum\ t)$, where $\emptyset$ denotes the empty set. $if\ \sigma(t) > t$, we say that it is right-scattered, while $if \rho(t) < t$ we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated.

Also, $if\ t < supT\ and\ \sigma\ (t) = t$, then t is called right-dense, and $if\ t > infT\ and\ \rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. Finally, the graininess function $\mu : T \rightarrow [0;\infty]\ is\ defined\ by$

$$\mu(t) = \sigma(t) \rightarrow t$$

We also need below the set T^K which is derived from the time scale T as follows: if T has a left-scattered maximum m, then $T^K = T - (m)$. Otherwise, $T^K = T$. In summary,

$$T^{\,K} = \{^{T/(\rho(supT),supT]\ if\ sup\,T<\infty}_{T\qquad\qquad\qquad if\ supT=\infty}$$

Finally, if $f{:}\,T \rightarrow R$ is a function, then we define the function $f^{\sigma} : T \rightarrow R$ by

$$f^{\sigma}(t) = f(\sigma(t))\ for\ all\ t \in T;$$
$$i.e., f^{\,\sigma} = f\ o\ \sigma$$

Consider a function $f : T \rightarrow R$ and define the so-called delta (or Hilger) derivative of f at a point $t \infty T^k$.

Definition 1.3: Assume $f{:}\,T \rightarrow R$ is a function and let $t \in T^K$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any$\in > 0$, there is a neighborhood U of $t\ (i.e., U = (t - \delta, t + \delta) \cap T$ for more $\delta > 0)$ such that $\{f(\sigma(t)) - f(s)\}\ f^{\Delta}(t)[\sigma(t)s]\ \leq\in\ \sigma\ (t) - s\ for\ all\ s\ \in U.$

We call $f^{\Delta}(t)$the delta (or Hilger) $derivative\ of\ f\ at\ t.$

Moreover, we say that f is delta (or Hilger) differentiable on T^Kprovided $f^{\Delta}(t)$exists for all $t \in T^K$. The function $f^{\Delta}{:}\,T^K \rightarrow R$ is then called the (delta) derivative of $f\ on\ T^K$

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Theorem 1.4 Assume $f: T \to R$ is a function and $let\ t \in T^K$. Then we have the following.

(i) If *f* is differentiable at *t*, then *f* is continuous at *t*.

(ii) If *f* is continuous at *t* and *t* is right-scattered, then *f* is differentiable at *t* with:

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

(iii) If t is right-dense, then *f* is differentiable at *t* iff the limit.

$$lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists as a finite number. in this case

$$f^{\Delta}(t) lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

(iv) If *f* is differentiable at *t*, then;

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Definition 1.5

A function $f: T \to R$ is called rd-continuous provided it is continuous at right-dense points in T and its left-sided limits exist (finite) at left-dense points in T. The set of rd-continuous functions $f: T \to R$ will be denoted here by:

$$C_{rd} = C_{rd}(T) = C_{rd}(T, R)$$

The set of functions $f : T \to R$ that are differentiable and whose derivative is rd-continuous is denoted by:

$$C^1_{rd} = C^1_{rd}(T) = C^1_{rd}(T, R)$$

Theorem 1.6 (Existence of Pre-Antiderivatives).Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that

$$F^{\Delta}(t) = f(t) \quad holds\ for\ all \quad t \in D$$

Definition 1.7 Assume $f : T \to R$ is a regulated function. Any function F as in theorem 1.60 ([5],Chap.1) is called a pre-antiderivative of *f*. We define the indefinite integral of a regulated function f by:

$$\int f(t)\Delta t = F(t) + C,$$

Where *C* is an arbitrary constant and F is a pre-antiderivative of *f*. We define the Cauchy integral by:

$$\int_r^s f(t)\Delta t = F(s) - F(r) for\ all\ r, s \in T.$$

A function $F : T \to R$ is called an antiderivative of $f : T \to R$ provided

$$F^{\Delta}(t) = f(t) holds\ for\ all\ t, \in T^K$$

Theorem 1.71: (Existence of Anti-derivatives). Every rd-continuous function has an antiderivative. In particular if $t_o \epsilon T$, then F defined by

$$F(t) := \int_{t_o}^{t} f(T)\Delta T\ for\ t \in T$$

is an antiderivative of *f*.

Theorem 1.72: $If\ f \in C_{rd}\ and\ t \in T^K, then$

$$\int_t^{\sigma(t)} f(T)\Delta T = \mu(t) f(t).$$

Theorem 1.73: $if\ f^{\Delta} \geq 0\ then\ f\ is\ non\ decreasing$

Theorem 1.74: $if\ a, b, c, \in T,\ \sigma \in R, and\ f, g \in C_{rd}, then$

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t;$

(ii) $\int_a^b (\sigma f)(t)\Delta t = \sigma \int_a^b f(t)\Delta t$

(iii) $\int_a^b f(t)\Delta t = -\int_a^b f(t)\Delta t;$

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$

(v) $\int_a^b f(\sigma(t)) g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)\, g(t)\, \Delta t;$

(vi) $\int_a^a f(t) g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g\,(\sigma(t))\Delta t;$

(vii) $\int_a^a f(t)\Delta t = 0$

2.0 Time Scale Inequalities of Superquadratic and Subquadratic Function

The concept of superquadratic functions in one variable, as a generalization of the class of convex functions, was recently introduced by Abramovich et al in [7,8]. Examples and properties of superquadratic functions can be found elsewhere [9,10,11].

Definition 2.1 A function $[\varphi:(0,\infty)\to R$ is superquadratic provided that for all $x > 0$ the exists a constant $C(x) \in R$ such that

$$\varphi:(y)-\varphi(x)-\varphi(y-x)\geq C(x)(y-x)$$

for all $y \geq 0$. We say that φ is subquadratic is $-\varphi$ is a superquadratic function.
For example, function $\varphi(x) = x^p$ is superquadratic for $p \geq 2$ and subquadratic for $p \in [0;\ 2]$.
The following Lemma shows that positive superquadratic functions are also functions.

Lemma 2.1: Let φ be a superquadratic function with $C(x)$ as in Definition above, then.
(i) $\varphi(0)\leq 0$
(ii) $if\ \varphi(0) = \varphi(0) = 0, then\ C\ (x) = \varphi(x) where\ \varphi\ is\ differentiable\ at\ x > 0.$
(iii) $if\ \varphi \geq 0, then\ \varphi\ is\ convex\ and\ \varphi(0) = \varphi(0) = 0$
Theorem 2.1 (Fubini's theorem on time scales).
If $f:\Omega\ x\ \Delta \to R\ is\ a\ \mu_\Delta\ x\ \lambda_\Delta$- integrable function and if we define the function $\varphi(y) = f_\Omega f(x,y)\Delta_x for\ a.e$
$y \in \Delta\ and\ \varphi(x) = f_\Delta f(x,y)\Delta_y for\ a\ e\ x \in \Omega, the\ \varphi\ is\ \lambda_\Delta$- integrable on $\Delta, \varphi\ is\ \varphi(x) is\ \mu_\Delta -$ integrable on Ω and

$$\int_\Omega \Delta_x \int_\Delta f(x,y)\Delta_y = \int_\Delta \Delta_y \int_\Omega f(x,y)\Delta_x$$

holds.
Bibi et al [12] established the Fubini's theorem on the time scales while, Baric et al [13] obtained a refined Jensen's inequality on time scales for super quadratic functions.
We state the refinnement of Jensen's inequality in [14], which is useful in proofs of our results.
Lemma 2.2 Let $(\Omega\ \mu)$ be a probability measure space [15]. The inequality

$$\varphi\{\int_\Omega f(s)d\mu(s)\} \leq \int_\Omega \varphi\big(f(s)\big)d\mu\ (s) - \int_\Omega \varphi(f(s) - \int_\Omega \big(f(s)\big)d\mu\ (s))\ d\mu\ (s)$$

hold for all probability measures μ and all nonnegative μ-integrable functions *f* if and only if φ is superquadratic (2.2) holds in the reversed direction if and only if φ is subquadratic.
Presenting, Jensen's inequality on time scales for superquadratic functions.
Theorem 2.2: Let $a, b, \epsilon\, T$. Suppose $f:[a,b]_T^K \to [0,\infty)$ is rd-continuous and $\varphi:[0,\infty)\to R$ is continuous and superquadratic. Then:

$$\varphi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) \leq \frac{1}{b-a}\int_a^b \left[\varphi\big(f(s)\big) - \varphi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)\right]\Delta s$$

Proof of Theorem 2.2 $Let\ \varphi:[0,\infty) \to R$ be a superdratic function and $let\ x_o \in [0,\infty)$. According to (2.1), there is a constant $C(x_0)$ such that;

$$\varphi(y) \leq \varphi(x_o) + C(x_o)(y-x_o) + \varphi(y-x_o)$$

since *f* is rd-continuous.

$$x_o = \frac{\int_a^b f(t)\Delta t}{b-a}$$

is well define. The function $\varphi o f$ is also rd-continuous, so we may apply (4) with $y = f(s)$ and (2.4) to obtain

$$\varphi\big(f(s)\big) \geq \varphi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right) + C(x_0)\left(f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right) + \varphi\left(f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right)$$

Integrating above equation from a to b, we get

$$\int_a^b \left[\varphi\big(f(s)\big) - \varphi\left(f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right)\right]\Delta s - (b-a)\varphi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)$$

$$= \int_a^b \varphi\big(f(s)\big)\Delta s \int_a^b \varphi\left(f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right)\Delta S - \int_a^b \varphi\left(\frac{\int_a^b f(t)\Delta t}{b-a}\right)\Delta S$$

$$\geq C(X_o)\int_a^b \left[f(s) - \frac{\int_a^b f(t)\Delta t}{b-a}\right]\Delta S$$

$$= C(x_o)\int_a^b f(s)\Delta S - (b-a)\ .\, x_o$$

$$= 0$$

Theorem 2.3 Suppose $u \epsilon Crd(a\ b), R)$ is a nonnegative function such that the delta integral $\int_t^b \frac{u(x)}{(x-a)(\sigma(x)-a}\ \Delta x$ exists as a finite number, and the function v is defined by

$$v(t) = (t-a)\int_t^b \frac{u(x)}{(x-a)(\sigma(x)-a)}\ \Delta x, t\ \in (a,b)$$

$If\ \Phi :\ (c;\ d)\ \rightarrow\ R$ is continuous and convex, where $c,\ d\ \epsilon\ R$, the the inequality

$$\int_a^b u(x)\Phi(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t)\frac{\Delta x}{x-a} \leq \int_a^b v(x)\Phi\left(f(x)\right)\frac{\Delta x}{x-a}$$

holds for all delta integrable functions $f\ \in C_{rd}([a;\ b,R)$ such that $f(x) \in (c;\ d)$.
Proof. $Let\ f :\ [a,b)\ \rightarrow\ R$ is rd-continuous function with values in (c; d). Applying Jensen's inequality [14] and Fubini's Theorem [2.1] on time scales, we obtain.

$$\int_a^b u(x)\Phi(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t)\ \frac{\Delta x}{x-a} \leq \int_a^b u(x)(\int_a^{\sigma(x)} \Phi(f(t))\Delta t)\frac{\Delta x}{(x-a)(\sigma(x)-a)}$$
$$= \int_a^b \Phi\big(f(t)\big)\int_t^b \frac{u\ (x)}{(x-a)(\sigma(x)-a)}\ \Delta x\Delta t = \int_a^b v(t)\Phi\big(f(t)\big)\frac{\Delta t}{t-a}$$

and the proof is complete.
Corollary 2.4 Let the assumptions of Theorem 2.3 be satisfied
(a) If $\emptyset$ is superquadratic and a < b < ∞, then

$$\int_a^b \phi\ (\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t)\frac{\Delta x}{x-a}$$
$$+\int_a^b\int_r^b \emptyset(f(t) - \frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t)\ \frac{\Delta x}{(x-a)(\sigma(x)-a)}\ \Delta t$$
$$\leq \int_a^b \left(1-\frac{x-a}{b-a}\right)\ \emptyset\ \left(f(x)\right)\frac{\Delta x}{x-a}$$

and

$$\int_a^\infty \emptyset(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t)\ \frac{\Delta x}{x-a}$$
$$= \int_a^\infty\int_t^\infty \emptyset\ (f(t) - \frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t)\frac{\Delta x}{(x-a)(\sigma(x)-a)}\ \Delta t$$
$$\leq \int_a^\infty \emptyset(\int(x)) - \frac{\Delta x}{x-a}$$

(b) The inequalities (2.7) and (2.8) hold in the reversed direction if $\emptyset$ is sub-quadratic.
Example 2.5. By taking T = R and a = 0 in Corollary 2.4, inequalities (2.7) and (2.8) read:

$$\int_0^b \emptyset(\frac{1}{x}\int_0^x f(t)dt)\frac{dx}{x} + \int_0^b\int_t^b \emptyset\ (f(t) - \frac{1}{x}\int_0^x f(t)dt)\frac{dx}{x^2}\ dt$$
$$\leq \int_0^b \left(1-\frac{x}{b}\right)\emptyset\ \left(f(x)\right)\frac{dx}{x},$$

and

$$\int_0^\infty \emptyset\ (\frac{1}{x}\int_0^x f(t)dt)\frac{dx}{x} + \int_0^\infty\int_t^\infty \emptyset(f(t) - \frac{1}{x}\int_0^x f(t)dt)\ \frac{dx}{x^2}dt$$
$$\leq \int_0^\infty \emptyset\big(f(x)\big)\frac{dx}{x}$$

Respectively.
Remark 2.6. The inequalities (2.9) and (3.0) coincide with Proposition 2.1 from [17], written for case u(x) = 1.
By using the well-known fact that the function $\emptyset(u) = u^p$ is superquadratic for $p\ \geq 2$ and subquadratic for $1\ <\ p\ \leq\ 2$ we obtain the following:
Example 2.7: Assume that $\emptyset(u)\ =\ u^p$ and $\int_a^b \int^p(x)\frac{\Delta x}{x-a}\ <\ \infty$. Then inequalities (2.6) and (2.7) read:

$$\int_a^b (\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} f(t)\Delta t)\ p\,\frac{\Delta x}{x-a}$$

$$= \int_a^b \int_t^b f(t) - \frac{1}{\sigma(x)-a} \int_a^{\sigma(x)} f(t)\Delta t \, p \frac{\Delta x}{(x-a)(\sigma(x)-a)} \Delta t$$
$$\leq \int_a^b \left(1 - \frac{x-a}{b-a}\right) f/p(x) \frac{\Delta x}{x-a}$$
$$for \; a < b < \infty \; and$$
$$\int_a^\infty \left(\frac{1}{\sigma(x)-a} \int_a^{\sigma(x)} f(t)\Delta t\right) p \frac{\Delta x}{x-a}$$
$$= \int_a^\infty \int_t^\infty f(t) - \frac{1}{\sigma(x)-a} \int_a^{\sigma(x)} f(t)\Delta t \, P \frac{\Delta x}{(x-a)(\sigma(x)-a)} \Delta t$$
$$\leq \int_a^\infty \left(1 - \frac{x-a}{b-a}\right) fp\,(x) \frac{\Delta x}{x-a}$$

respectively. Moreover, if $1 < p \leq 2$; the the inequalities (3.1) and (3.2) hold in the reversed direction.

Remark 2.9. The natural "breaking point" (the point where the inequality reverses) in Hardy type inequalities is usually p = 1. However, here we see that for our refined Hardy type inequality the natural breaking point is p = 2 and even more remarkable for p = 2 we have a new identity even in the case with time scales.

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