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Fixed Point Theory for n-Th Order Ordinary Differential Equations

K. Rauf[1], O.T. Wahab[2], and J.O. Omolehin[3]

[1]Department of Mathematics, University of Ilorin, Ilorin, Nigeria.
[2]Department of Statistics and Mathematical Sciences, Kwara State University, Molete, Nigeria.
[3]Federal University Lokoja, Lokoja, Kogi State, Nigeria.

Abstract

In this article, we consider fixed point theorems with applications to nth order differential equation. In particular, we establish Banach fixed point to prove the famous Pickad theorem by transforming n-th order ordinary differential equation into system of first order ordinary differential equation and finally into vector ordinary differential equation of Euler's form. Some examples are considered. Our results extend and generalize several existing results in the literature.

Keywords: Pickad theorem, Fixed point, Lipschitz map, Nonlinear operator.
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1.0 Introduction

Problems concerning the existence of fixed points for Lipschitz map have been of considerable interest in the theory of nonlinear operator. The study of nonlinear operator had its beginning about the start of the twentieth century with investigations into the existence properties to certain initial value problems arising in ordinary differential equations. The earliest ways of dealing with such problems, which were largely planned in [1], involved the iteration of an integral operator to devise solutions to the problems. In 1922, these methods of Picard were given exact abstract formulation by Banach [2] and Caccіopoli [3] which is now generally referred to as Contraction Mapping Techniques. Since then, a number of authors have defined contractive type mappings on a complete metric space (X,d). Banach [2] defined a mapping which is a contraction for a positive number $c<1$. Also, Edelstein [4] considered a nonexpansive contractive type mappings. Alber and Guerre-Delabriere [5] introduced the weak contraction and showed that most of the results are still true for Banach space. Choudhury and Metiya [6] extend fixed point of weak contractions to cone metric spaces. Some works related to the concept of existence and uniqueness of solution, contraction mapping and ordinary differential equations could be sourced from [7-11].
In this article, we are concerned with a Banach fixed point techniques which is one of the most useful methods in the existence of fixed points theory. Furthermore, we shall use the Banach's theorem to prove the famous Picard's theorem which plays a vital role in the theory of ordinary differential equations.

2.0 Preliminary Results

Let us consider the general first order equation

$$y' = f(t,y)$$

Where f is defined for (t,y) on some set and continuous.

Suppose $f_1, f_2, \cdots, f_n$ are continuous-valued functions defined for $(t, y_1, y_2, \cdots, y_n)$ space. A wide class of (1) is the system.

$$\begin{aligned} y_1' &= f_1(t,y_1,y_2,\ldots,y_n) \\ y_2' &= f_2(t,y_1,y_2,\ldots,y_n) \\ &\vdots \\ y_n' &= f_n(t,y_1,y_2,\ldots,y_n) \end{aligned}$$

Corresponding author: Rauf Kamilu, E-mail: krauf@unilorin.edu.ng,, Tel.: +2348033965848, 8033578643(J.O.O)

This is a system of n ordinary differential equations of the first order, the derivatives $y_1^{\prime}, y_2^{\prime}, \ldots, y_n^{\prime}$ appear explicitly and they are analogue of (1).

2.1 $n-$th Order Equation

An equation of $n-th$ order

$$y^{(n)} = f(t, y, y^{\prime}, \ldots, y^{(n-1)})$$

may be treated as a system of the form (2).

Let $y = y_1,\ y^{\prime} = y_2, \cdots, y^{n-1} = y_n$.

Then (3) can be written as:

$$\begin{aligned} y_1^{\prime} &= y_2 \\ y_2^{\prime} &= y_3 \\ &\cdots \\ y_{n-1}^{\prime} &= y_n \\ y_n^{\prime} &= f(t, y, y_1, \ldots, y_n) \end{aligned}$$

which may be viewed as the type (2).

The clear difference between (1) and (2) is that a complex number y is now to deal with n such complex numbers $y_1, y_2, \cdots, y_n$.

Let $\mathbf{y}$ be a vector of the n complex numbers and we may write $\mathbf{y} = (y_1, y_2, \cdots, y_n)$. So, the complex number y_k is the $k-$th component of $\mathbf{y}$. The set of all such vectors is called the complex $n-$dimensional space C^n.

2.1.1 Systems as Vector Equations

Consider the first order system of equations

$$\begin{aligned} y_1^{\prime} &= f_1(t, y_1, y_2, \cdots, y_n) \\ y_2^{\prime} &= f_2(t, y_1, y_2, \cdots, y_n) \\ &\vdots \\ y_n^{\prime} &= f_n(t, y_1, y_2, \cdots, y_n) \end{aligned}$$

It is assumed that $f_1, f_2, \cdots, f_n$ are complex-valued functions defined for $(t, y_1, y_2, \ldots, y_n)$ on some set, where t is real and $y_1, y_2, \ldots, y_n$ are complex.

Clearly, $f_1,\ f_2, \ldots, f_n$ are functions of t and the vector $\mathbf{y}$, where $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in C^n.

Therefore, we may write

$$\begin{aligned} f_1(t, \mathbf{y}) &= f_1(t,\ y_1, y_2, \cdots, y_n) \\ f_2(t, \mathbf{y}) &= f_2(t,\ y_1, y_2, \cdots, y_n) \\ &\vdots \\ f_n(t, \mathbf{y}) &= f_n(t,\ y_1, y_2, \cdots, y_n) \end{aligned}$$

In (5), we have n functions $f_1,\ f_2, \cdots, f_n$ which may be regarded as a vector-valued function

$$\mathbf{f} = (f_1,\ f_2, \ldots, f_n)$$

which may be given by

$$\mathbf{f}(t, \mathbf{y}) = f_1(t, \mathbf{y}),\ f_2(t, \mathbf{y}), \ldots, f_n(t, \mathbf{y}).$$

Suppose

$$\mathbf{y}^{\prime} = (y_1^{\prime}, y_2^{\prime}, \ldots, y_n^{\prime})$$

then the system (5) may now be written as

$$\mathbf{y}' = \mathbf{f}(t,\mathbf{y})$$

Remark: The vector differential equation (6) now has the form (1).

Definition 2.1. A vector-valued function $\mathbf{f}$ is said to satisfy a Lipschitz condition on Ω if there is a number $K > 0$ such that

$$\left|\mathbf{f}(t,\mathbf{y}) - \mathbf{f}(t,\mathbf{z})\right| \le K\left|\mathbf{y} - \mathbf{z}\right|$$

for all $\mathbf{y},\mathbf{z} \in C^n$ and $(t,\mathbf{y}),(t,\mathbf{z}) \in \Omega$. The constant K is called the Lipschitz constant.

Proposition 2.1. Let $\mathbf{f}$ be a vector-valued function defined for $(t,\mathbf{y})$ on a set Ω given by

$$\Omega := \left\{(t,\mathbf{y}) : \left|t - t_0\right| \le a, \left|\mathbf{y} - \mathbf{y_0}\right| \le b,\ a, b > 0\right\}$$

If $\partial \mathbf{f} / \partial y_k \ (k = 1,2,\ldots,n)$ is continuous on Ω and there is a constant $K > 0$ such that

$$\left|\frac{\partial \mathbf{f}}{\partial y_k}\right| \le K$$

for $(t,\mathbf{y}) \in \Omega$, then $\mathbf{f}$ satisfies a Lipschitz condition on Ω.

Proof: See [12].

Proposition 2.2. Consider the vector differential equation

$$\mathbf{y}' = \mathbf{f}(t,\mathbf{y})$$

where the components $f_1, f_2, \ldots, f_n$ of $\mathbf{f}$ are of the form

$$\begin{aligned}
f_1(t,\mathbf{y}) &= a_{11}(t)y_1 + a_{12}(t)y_2 + \ldots + a_{1n}(t)y_n + b_1(t) \\
f_2(t,\mathbf{y}) &= a_{21}(t)y_1 + a_{22}(t)y_2 + \ldots + a_{2n}(t)y_n + b_2(t) \\
&\vdots \\
f_n(t,\mathbf{y}) &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \ldots + a_{nn}(t)y_n + b_n(t)
\end{aligned}$$

where $a_{11}(t),\ldots,a_{nn}(t), b_1(t),\ldots,b_n(t)$ are complex-valued functions defined for real t in some interval I. If all the a_{ij} are continuous on an interval $I : \left|t - t_0\right| \le a$, where $a > 0$, then the corresponding vector-valued function {\bf f} satisfies a Lipschitz condition on the strip

$$\Omega : \left|t - t_0\right| \le a, \left|y - y_0\right| \le b \ or \ \left|y\right| < \infty, \ \ a, b > 0$$

Proof: See [12].

Proposition 2.3. The vector differential equation (6) defined on Ω is equivalent to the integral equation

$$\mathbf{y} = \mathbf{y}_o + \int_{t_0}^{t} \mathbf{f}(\tau,\mathbf{y}(\tau))d\tau$$

$$\mathbf{y}_0 = (\alpha_1, \alpha_2, \ldots, \alpha_n), \mathbf{f}(\tau,\mathbf{y}(\tau)) = (f_1, f_2, \ldots, f_n) \text{ and}$$

$$f_k(\tau,\mathbf{y}(\tau)) = \sum_{j=1}^{n} a_{jk}(\tau)y_k(\tau) + b_k(\tau), k = 1,2,\ldots,n$$

We complete this section with a proposition which is sequel to our work.

Proposition 2.4. Let X be a metric space. Then X is said to be complete if every cauchy sequence in X has a limit x which is in X. A subset Y of a metric space X is complete if it is closed [13].

3.0 Problem Formulation

In this section, we discuss the Banach fixed point theorem which states sufficient conditions for the existence and uniqueness of a fixed point and also gives a constructive procedure for obtaining sharp results to the fixed point. We start with the following definitions:

Definition 3.1. Let X be a nonempty set and T be a mapping of X into itself. A point $x \in X$ is said to be a Fixed point of the mapping T if

$$Tx = x$$

i.e. the image Tx coincides with x.

Definition 3.2. Let $X = (X, d)$ be a metric space. A mapping $T : X \to X$ is called a Lipschitz map if there is a real number $c > 0$ such that for all $x, y \in X$

$$d(Tx, Ty) \le cd(x, y)$$

for all $x, y \in X$ and T is called a contraction on X if there is a positive real number $c < 1$ such that for all $x, y \in X$.

Remark. If $c = 1$, then (11) becomes $d(Tx, Ty) < d(x, y)$ which may not be replaced for (11). In this case, T is called nonexpansive [10].

Definition 3.3. Let X be a metric space. A mapping $T : X \to X$ is said to be weakly contractive on X if

$$d(Tx, Ty) \le d(x, y) - \varphi(d(x, y))$$

for all $x, y \in X$ and $\varphi[0, \infty) \to [0, \infty)$ is continuous and non-decreasing function such that $\varphi(t) = 0$ if and only if $t = 0$. Clearly, if $\varphi(t) = \kappa t$ where $0 < \kappa < 1$, then (12) reduces to (11).

Remark. In the light of the two definitions above, we remark that a linear map $T : X \to Y$ which is continuous is also bounded and vice versa [1].

Proposition 3.1: Let T be a contraction mapping, then for any positive integer n, T^n is also a contraction mapping.

Proof: Let T be a contraction mapping $T : X \to X$, (by Definition 3.2) there exists $c < 1$ for $x, y \in X$ such that $d(Tx, Ty) \le cd(x, y)$. Now,

$$\begin{aligned}
d(T^n x, T^n y) &= d(T(T^{n-1}x), T(T^{n-1}y)) \\
&\le cd(T^{n-1}x, T^{n-1}y) \\
&\le c^2 d(T^{n-2}x, T^{n-2}y) \\
&\vdots \\
&= c^n d(T^{n-n}x, T^{n-n}y) \\
&\le c^n d(x, y) \\
d(T^n x, T^n y) &\le c^n d(x, y)
\end{aligned}$$

Since $c < 1$, then $c^n < 1$ for all n. Therefore, T^n is a contraction.

Remark. If c is a constant of contraction T then c^n is a constant of contraction T^n.

Proposition 3.2: Every contraction mapping on a metric space (X, d) is a continuous mapping.

Proof: Let $T : X \to X$ be a contraction mapping of a metric space X, then there is a positive constant $c < 1$ such that $d(Tx, Ty) \le cd(x, y)$ for all $x, y \in X$

Let $\grave{o} > 0$ be given, we want to find $\delta > 0$ such that whenever $d(x, y) < \delta \Rightarrow d(Tx, Ty) < \grave{o}$

Choose $0 < \delta < \frac{\grave{o}}{c}$. Then, for $x, y \in X$

$$d(x, y) < \delta$$

$$\Rightarrow d(Tx, Ty) \le cd(x, y) < c.\frac{\grave{o}}{c} = \grave{o}$$

Hence the proof. See [14] for similar proof.

Theorem 3.1 (Banach Fixed Point Theorem)

Let X be a non-empty metric space. Suppose that X is complete and $T : X \to X$ is a contraction on X. Then, T has precisely one fixed point $x \in X$.

Proof: Let $x_0 \in X$ be arbitrarily fixed and define the iterative sequence $\{x_n\}$ by

$$x_0, x_1 = Tx_0, x_2 = T^2 x_0, \ldots, x_n = T^n x_0$$

We have constructed the sequence of various images of x_0 under repeated application of T.

Next, we show that $\{x_n\}$ is Cauchy.

By (10) and (11), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq cd(x_{n-1}x_n) \\ &\leq c^2 d(x_{n-2}, x_{n-1}) \\ &\vdots \\ &= c^n d(x_{n-n}, x_{n-n+1}) \\ &\leq c^n d(x_0, x_1) \end{aligned}$$

Let $m > n$ for $n, m \in N$, then by geometric progression and proposition (3.4)

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &\leq c^n d(x_0, x_1) + c^{n+1} d(x_0, x_1) + \ldots + c^{m-1} d(x_0, x_1) \\ &= c^n d(x_0, x_1)\left(1 + c + c^2 + \ldots + c^{m-n-1}\right) \\ &= c^n \left(\frac{1 - c^{m-n}}{1 - c}\right) d(x_0, x_1) \end{aligned}$$

Since $c < 1$, then $1 - c^{m-n} < 1$ for $m - n > 0$

So that,

$$d(x_n, x_m) \leq \frac{c^n}{1 - c} d(x_0, x_1)$$

On the right, $c < 1$ and $d(x_0, x_1)$ is fixed. So, as $n \to \infty$, $c^n \to 0$ which make the right hand side inequality as small as we please.

This proves that $\{x_n\}$ is Cauchy.

Since X is a complete metric space, then $\{x_n\}$ converges to a point (say x) in X, i.e

$$x_n \to x, \ as \ n \to \infty$$

Also, since T is a contraction, (by Proposition (3.5)) T is continuous.

Therefore,

$Tx_n \to Tx$ whenever (16) holds.

Next is to show that the limit x is the fixed point of the mapping T.

By (10),

$$\begin{aligned} d(Tx, x) &\leq d(x, x_n) + d(x_n, Tx) \\ &= d(x, x_n) + d(Tx_{n-1}, Tx) \\ &\leq d(x, x_n) + cd(x_{n-1}, x) \end{aligned}$$

By (16), $x_n \to x$ and $x_{n-1} \to x, \ as \ n \to \infty$

Thus,

$$d(Tx, x) = 0 \ \leftrightarrow \ Tx = x$$

And finally, we show that the limit x is the only fixed point of T.

Suppose x and $\tilde{x}$ are two fixed points, then

$$\begin{aligned} d(x, \tilde{x}) &= d(Tx, T\tilde{x}) \\ &\leq cd(x, \tilde{x}) \end{aligned}$$

Thus,

$$d(x, \tilde{x}) = 0, \text{ if and only if } x = \tilde{x}$$

Hence, x is the only fixed point of T.

This completes the proof.

Corollary 3.1

Let X be a complete metric space and T is such that $T : X \to X$. Suppose T^n is a contraction on X. Then, T^n has only one fixed point.

Remark: Generally in application, the mapping T is a contraction not on the entire space X but merely on a subset of X. Since a closed subset of a complete space X is complete, T has a fixed point on the closed subset provided there is a restriction on the choice of x_0 so that the x_n lie in the closed subset.

This is justified by the following two theorems.

Theorem 3.2: Let $X = (X, d)$ be a complete metric space and let $T : X \to X$ be a contraction on a closed ball $\overline{B} = \{x{:}d(x,x_0) \le r\}\ \forall\ x_0, x \in \overline{B} \subset X$.

Moreover, assume that

$$d(x_0, Tx_0) < (1-c)r$$

Then, T has precisely one fixed point $x \in X$.

Proof: We need to show that all x_n's as well as x lie in $\overline{B}$

Set $n = 0$ in (15) and let m be replaced by n, then

$$d(x_0, x_n) \le \frac{1}{1-c} d(x_0, x_1) < r$$

Hence, all x_n's are in $\overline{B}$ and $x \in \overline{B}$ since $x_n \to x$ and $\overline{B}$ is closed.

The assertion of this theorem now follows from theorem (3.8) in [15].

We shall devote the rest of this paper to show how the arguments of Baire Category theorem can be adapted to show existence and uniqueness of solutions of vector differential equation (6) in [16].

4.0 Main Results

We begin with the following propositions which can be easily proved.

Proposition 4.1. Let $\mathbf{\Phi}$ be a vector-valued differentiable function satisfying $\mathbf{y}_o = \mathbf{\Phi}(t_o)$ for all $(t, \mathbf{\Phi}(t))$ in $\mathbf{\Omega}$. Suppose $\mathbf{\Phi}$ is a solution of (6), then

$$\mathbf{\Phi}(t) = \mathbf{\Phi}(t_0) + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{\Phi}(\tau)) d\tau$$

and the vector form is $\mathbf{\Phi}(t) = \left(\phi_1(t), \phi_2(t), \ldots, \phi_n(t)\right)$

Proposition 4.2. Let $\mathbf{\Phi}_o$ be fixed and defined by

$$\mathbf{\Phi}_o(t) = \mathbf{y}_o$$

then, by the iterative process in (13), we have

$$\mathbf{\Phi}_1(t) = \quad T\mathbf{\Phi}_o(t) = \mathbf{y}_o + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{\Phi}_o(\tau)) d\tau$$

$$\mathbf{\Phi}_2(t) = \quad T^2\mathbf{\Phi}_o(t) = \mathbf{y}_o + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{\Phi}_1(\tau)) d\tau$$

$$\vdots$$

In general, we have

$$\mathbf{\Phi}_m(t) = T^m\mathbf{\Phi}_o(t) = \mathbf{y}_o + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{\Phi}_{m-1}(\tau)) d\tau, \ (m = 0,1,2,\ldots)$$

As $m \to \infty$, the limit is given by (18) i.e $\mathbf{\Phi}_m(t) \to \mathbf{\Phi}(t)$

By (16), $T\mathbf{\Phi}_m(t) \to T\mathbf{\Phi}(t)$ so that

$$T\mathbf{\Phi}(t) = \mathbf{\Phi}(t)$$

Interpretation: In a picturesque, the mapping is like a machine (say S) which transforms the limit function $\mathbf{\Phi}$ into a new function $S\mathbf{\Phi}$ defined by

$$S\mathbf{\Phi}(t) = \mathbf{\Phi}(t_0) + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{\Phi}(\tau))d\tau$$

This means that a solution of the system (6) is the function which moves through the machine untouched, starting with $\mathbf{\Phi}_o(t) = \mathbf{y}_o$, S converts $\mathbf{\Phi}_o$ into $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_1$ into $\mathbf{\Phi}_2$ and, in general, we have $S\mathbf{\Phi}_m = \mathbf{\Phi}_{m+1}$. Consequently, we arrive at $\mathbf{\Phi}$ such that $S\mathbf{\Phi} = \mathbf{\Phi}$.

Next is to show that the sequence $\mathbf{\Phi}_m$ merit the nomenclature. Before that we give the following suitable remark.

Remark: Suppose $\mathbf{\Phi}_m$ as well as $\mathbf{\Phi}$ exist on the interval I containing t_o, then Baire's theorem asserts that the limit $\mathbf{\Phi}$ may not be attained on the neighborhood of $\mathbf{\Phi}_o$ unless on the successive neighborhoods of $\mathbf{\Phi}_o$.

Proposition 4.3: Let $\{\mathbf{\Phi}_m\}_{m=1}^{\infty}$ be sequence of vector-valued function defined on the interval $I : |t - t_o| \le a,$ and let β be smaller than $a, \frac{b}{M}$ where $M > 0$. Then, $\{\mathbf{\Phi}_m\}_{m=1}^{\infty}$ exists on the interval

$$I : |t - t_o| \le \beta < min\left\{a, \frac{b}{M}\right\} \$$$

for $(t, \mathbf{\Phi}_m)$ in Ω.

Proof: From (18)

$$\mathbf{\Phi}_m(t) = \mathbf{y}_o + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{\Phi}_{m-1}(\tau))d\tau, \ (m = 0,1,2,\ldots)$$

$$\begin{aligned} \Rightarrow \qquad |\mathbf{\Phi}_m(t) - \mathbf{y}_o| &= \left|\int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{\Phi}_{m-1}(\tau))d\tau\right| \\ &\le \left|\int_{t_0}^{t} |\mathbf{f}(\tau, \mathbf{\Phi}_{m-1}(\tau))| d\tau\right| \\ &\le M\left|\int_{t_0}^{t} d\tau\right| \\ &\le M\,|t - t_o| \end{aligned}$$

Since $I : |t - t_o| \le \frac{b}{M}$,

$$\rightarrow |\mathbf{\Phi}_m(t) - \mathbf{y}_o| \le b$$

This shows that $(t, \mathbf{\Phi}_m)$ are in Ω for $t \in I$.

Clearly $\mathbf{\Phi}_o$ exists on I for $m = 0$ and satisfies the inequality (20).

Now, for $m = 1$ in (18)

$$\mathbf{\Phi}_1(t) = \mathbf{y}_o + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{\Phi}_o(\tau))d\tau$$

$$|\mathbf{\Phi}_1(t) - \mathbf{y}_o| = \left|\int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{\Phi}_o(\tau))d\tau\right| \le \left|\int_{t_0}^{t} |\mathbf{f}(\tau, \mathbf{\Phi}_o(\tau))| d\tau\right| \le M\left|\int_{t_0}^{t} d\tau\right| \le M\,|t - t_o|$$

which implies that $\mathbf{\Phi}_1$ satisfies (20) and since $\mathbf{f}$ is continuous on Ω, then $\mathbf{f}(\tau, \mathbf{\Phi}_o(\tau))$ is continuous on I and so $\mathbf{\Phi}_1$ exists on I.

By induction, $\mathbf{\Phi}_m$ satisfy (20) for all m and $\mathbf{f}(\tau, \mathbf{\Phi}(\tau))$ as well as $\mathbf{\Phi}_m$ are continuous and exist on I.

We now show that $\mathbf{\Phi}_m$ converge on I to a solution of the system (6). This is given in our next theorem. See [12].

Theorem 4.1: Let f be a continuous vector-valued function defined on

$$\Omega := \left\{(t,\mathbf{y}) : |t-t_0| \le a, |\mathbf{y}-\mathbf{y}_0| \le b, (a,b>0)\right\}$$

and bounded on Ω, say

$$|\mathbf{f}(t,\mathbf{y})| \le M$$

Suppose $\mathbf{f}$ satisfies a Lipschitz condition on Ω with respect to its second argument.

Then, the iterative function sequence $\{\mathbf{\Phi}_m\}_{m=1}^{\infty}$ obtained in (18) converge on the interval $[t_0-\beta, t_0+\beta]$ where

$$\beta < \min\left\{a, \frac{b}{M}, \frac{1}{K}\right\}$$

to a solution $\mathbf{\Phi}$ of the system (6)

Proof: Let $C(I)$ be the metric space of all complex-valued continuous function on the interval $I = [t_0-a, t_0+a]$. For $t \in [t_0-a, t_0+a]$ and $\mathbf{\Phi}(\mathbf{t}), \mathbf{\Psi}(\mathbf{t}) \in C(I)$, the metric on $C(I)$ is defined by

$$d(\mathbf{\Phi}(\mathbf{t}), \mathbf{\Psi}(\mathbf{t})) = \sup_{t \in [t_0+a, t_0+a]} |\mathbf{\Phi}(\mathbf{t}) - \mathbf{\Psi}(\mathbf{t})|$$

$C(I)$ is complete [13].

Let $J = [t_0-\beta, t_0+\beta] \subset I$, then $C(J)$ is a closed subspace of $C(I)$ which is also complete by proposition 2.4.

Define the mapping $T : C(J) \to C(J)$ and $T\mathbf{\Phi}(t) = \mathbf{\Phi}(t)$ for $\mathbf{\Phi} \in C(J)$

Consider a ball B in $C(J)$ with radius b centered at $\mathbf{y_o}$ given by

$$B = \left\{\mathbf{\Phi} \in C(J) : |\mathbf{\Phi}(t) - \mathbf{y}_o| \le b\right\}$$

We show that $B \supset T(B)$, for suppose

$$T\phi_m(t) \to T\phi(t)$$

$$\begin{aligned}
T\mathbf{\Phi}(t) &= \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{\Phi}(\tau))d\tau \\
\Rightarrow d(T\mathbf{\Phi}(t), \mathbf{y}_0) &= \sup |T\mathbf{\Phi}(t) - \mathbf{y}_0| \\
&= \sup \left|\int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{\Phi}(\tau))d\tau\right| \\
&\le \sup \left|\int_{t_0}^{t} |\mathbf{f}(\tau, \mathbf{\Phi}(\tau))| d\tau\right| \\
&\le M \sup |t-t_0| \\
&\le M\beta < b
\end{aligned}$$

which implies for $\mathbf{\Phi} \in T(B) \Rightarrow \mathbf{\Phi} \in B$, and thus, T maps $C(J)$ into itself.

Next is to show that T is a contraction on $C(J)$.

By the Lipschitzian assumptions (7) and for $\mathbf{\Phi}(t), \mathbf{\Psi}(t) \in C(J))$.

We have

$$\begin{aligned}
d(T\mathbf{\Phi}, T\mathbf{\Psi}) &= \sup |T\mathbf{\Phi}(t) - T\mathbf{\Psi}(t)| \\
&= \sup \left|\mathbf{y}_0 + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{\Phi}(\tau))d\tau - \left(\mathbf{y}_0 + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{\Psi}(\tau))d\tau\right)\right| \\
&\le \sup \left|\int_{t_0}^{t} |\mathbf{f}(\tau, \mathbf{\Phi}(\tau)) - \mathbf{f}(\tau, \mathbf{\Psi}(\tau))| d\tau\right| \\
&\le \sup K |\mathbf{\Phi}(\tau) - \mathbf{\Psi}(\tau)| \left|\int_{t_0}^{t} d\tau\right| \\
&\le K \sup |\mathbf{\Phi}(\tau) - \mathbf{\Psi}(\tau)| \sup |t-t_0| \\
&\le K\beta d(\mathbf{\Phi}, \mathbf{\Psi})
\end{aligned}$$

From (21), choose $c = K\beta < 1$, so that T is a contraction on $C(J)$.
The conclusion of the theorem follows from Theorem 3.8.
Observe that the existence result proved above is local. Moreso, I depends on M, K and the initial condition.

Remark: Let $\mathbf{f}$ be a continuous vector-valued function and global on the strip

$$\Omega' := \{(t, \mathbf{y}) : |t - t_o| \leq a, |\mathbf{y}| < \infty\}$$

Then the iterative sequence $\{\mathbf{\Phi}_m(t)\}_{m=1}^{\infty}$ exist on $|t - t_0| \leq a$ and converge to a solution of the system (6).

We now discuss the existence and uniqueness of solution of an n-th order differential equation given by (3). We consider the following theorem:

Theorem 4.2: Let $\mathbf{f}$ be a complex valued continuous function in (4) defined on

$$\Omega : |t - t_0| \leq a,\ |\mathbf{y} - \mathbf{y}_0| \leq b \quad (a, b > 0)$$

such that

$$|F(t, \mathbf{y})| \leq N$$

for all $(t, \mathbf{y})$ in Ω. Suppose there exists a constant $L > 0$ such that

$$|F(t, \mathbf{y}) - F(t, \mathbf{z})| \leq L|\mathbf{y} - \mathbf{z}|$$

for all $(t, \mathbf{y})$ and $(t, \mathbf{z})$ in Ω. Then, there is only and only one solution of ϕ of (3) on the interval

$$I : |t - t_0| \leq \beta < min\left\{a, \frac{b}{M}, \frac{1}{K}\right\}$$

Which satisfies

$$\phi(t_0) = \alpha_1, \phi'(t_0) = \alpha_2, \ldots, \phi^{n-1}(t_0) = \alpha_n,$$

$$(\mathbf{y} = (\alpha_1, \alpha_2, \ldots, \alpha_n))$$

Proof: Consider the system $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$ with component of f_k given by (4). Then

$$\begin{aligned} |\mathbf{f}(t, \mathbf{y})| &= |y_2| + |y_3| + \ldots + |y_n| + |F(t, \mathbf{y})| \\ &\leq |\mathbf{y}| + |F(t, \mathbf{y})| \\ &\leq |\mathbf{y}_0| + b + N = M \end{aligned}$$

where $M = max\{|\mathbf{y}_0| + b + N, b > 0\}$.

Also,

$$\begin{aligned} |\mathbf{f}(t, \mathbf{y}) - \mathbf{f}(t, \mathbf{z})| &= |y_2 - z_2| + \ldots + |y_n - z_n| + |F(t, y)| - |F(t, z)| \\ &\leq |\mathbf{y} - \mathbf{z}| + L|\mathbf{y} - \mathbf{z}| \\ &= (1 + L)|\mathbf{y} - \mathbf{z}| \end{aligned}$$

Thus satisfies the Lipschitz conditions with Lipschitz constant $K = 1 + L$.
The conclusion of the theorem follows from theorem (4.4)

Corollary 4.1: Let $a_1, a_2, \ldots, a_n, b$ be continuous complex-valued function on the interval I containing a point t_0. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are any n constants, there exists one and only one solution ϕ of the equation

$$y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y = b(t)$$

on I satisfying

$$\phi(t_0) = \alpha_1, \phi'(t_0) = \alpha_2, \ldots, \phi^{n-1}(t_0) = \alpha_n$$

Proof: The proof follows readily from the proof of theorem 4.5.

Practical Example 1.
Let us consider the bending of an elastic plate's equation

$$y'''' - 2\lambda^2 y'' + 4\lambda^2 y = 0,\ \lambda \neq 0$$

with the initial conditions

$$y(0) = 0,\ y'(0) = 0,\ y''(0) = 0,\ y'''(0) = 2.$$

Solution
Let

$$y = y_1, y' = y_1' = y_2, y'' = y_2' = y_3, y''' = y_3' = y_4$$

then $y'''' = y_4' = 2\lambda^2 y_3 - 4\lambda^4 y_1$

and

$$\begin{aligned} y_1' &= y_2 \equiv f_1(t, y, y_1, \ldots, y_n) \\ y_2' &= y_3 \equiv f_2(t, y, y_1, \ldots, y_n) \\ y_3' &= y_4 \equiv f_3(t, y, y_1, \ldots, y_n) \\ y_4' &= 2\lambda^2 y_3 - 4\lambda^4 y_1 \equiv f_4(t, y, y_1, \ldots, y_n) \end{aligned}$$

Hence,

$$\mathbf{f}(t, \mathbf{y}) = (y_2, y_3, y_4, 2\lambda^2 y_3 - 4\lambda^4 y_1)$$

$$\frac{\partial f_1}{\partial y_2} = 1,\ \frac{\partial f_1}{\partial y_1} = \frac{\partial f_1}{\partial y_3} = \frac{\partial f_1}{\partial y_4} = 0$$

$$\frac{\partial f_1}{\partial y_3} = 1,\ \frac{\partial f_1}{\partial y_1} = \frac{\partial f_1}{\partial y_2} = \frac{\partial f_1}{\partial y_4} = 0$$

$$\frac{\partial f_1}{\partial y_4} = 1,\ \frac{\partial f_1}{\partial y_1} = \frac{\partial f_1}{\partial y_2} = \frac{\partial f_1}{\partial y_3} = 0$$

$$\frac{\partial f_1}{\partial y_1} = -4\lambda^4,\ \frac{\partial f_1}{\partial y_3} = 2\lambda^2,\ \frac{\partial f_1}{\partial y_2} = \frac{\partial f_1}{\partial y_4} = 0$$

Therefore,

$$\left|\frac{\partial \mathbf{f}}{\partial y_1}\right| = 4\lambda^4, \quad \left|\frac{\partial \mathbf{f}}{\partial y_2}\right| = 1, \quad \left|\frac{\partial \mathbf{f}}{\partial y_3}\right| = 1 + 2\lambda^2, \quad \left|\frac{\partial \mathbf{f}}{\partial y_4}\right| = 1$$

Thus, $\mathbf{f}$ satisfies the Lipschitz condition with Lipschitz constant $L = 4\lambda^4 > 0$, for $\lambda \neq 0.$
Let T be a mapping defined by

$$T\mathbf{y} = \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{\Phi}(\tau))\, d\tau$$

$$\Rightarrow d(T\mathbf{y}, T\mathbf{z}) = |T\mathbf{y}(t) - T\mathbf{z}(t)|$$

$$= \left|\int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{y}(\tau))\, d\tau - \int_{t_0}^{t} \mathbf{f}(\tau, \mathbf{z}(\tau))\, d\tau\right|$$

$$= \left|\int_{t_0}^{t} \left(\mathbf{f}(\tau, \mathbf{y}(\tau)) - \mathbf{f}(\tau, \mathbf{z}(\tau))\right) d\tau\right|$$

$$\leq \left|\int_{t_0}^{t} \left|(y_2, y_3, y_4, 2\lambda^2 y_3 - 4\lambda^4 y_1) - (z_2, z_3, z_4, 2\lambda^2 z_3 - 4\lambda^4 z_1)\right| d\tau\right|$$

$$\leq \left|\int_{t_0}^{t} \left|(y_2 - z_2, y_3 - z_3, y_4 - z_4, 2\lambda^2 y_3 - 2\lambda^2 z_3 - 4\lambda^4 y_1 + 4\lambda^4 z_1)\right| d\tau\right|$$

$$\leq |t|\left(|y_2 - z_2| + |y_3 - z_3| + |y_4 - z_4| + |2\lambda^2 y_3 - 2\lambda^2 z_3| + |4\lambda^4 y_1 - 4\lambda^4 z_1|\right)$$

$$\le |t|\left(|\mathbf{y}-\mathbf{z}| + 2\lambda^2 |y_3 - z_3| + 4\lambda^4 |y_1 - z_1|\right)$$
$$\le |t|\left(|\mathbf{y}-\mathbf{z}| + 4\lambda^4 \left(|y_3 - z_3| + |y_1 - z_1|\right)\right)$$
$$\le |t|\left(|\mathbf{y}-\mathbf{z}| + 4\lambda^4 |\mathbf{y}-\mathbf{z}|\right)$$
$$\le |t|\left(1+4\lambda^4\right)|\mathbf{y}-\mathbf{z}|$$
$$\le |t|K|\mathbf{y}-\mathbf{z}|$$

where $K = 1+4\lambda^4 \equiv 1+L$ and $c = |t|K < 1$.

Hence, T is a contraction.

Next is to show that $\mathbf{y}_m \to \mathbf{y}, \quad m = 1,2,3,\ldots$

Let $\mathbf{y}_m \equiv \mathbf{y}^m$ and $\mathbf{y}^0 = (0,0,0,2)$ be fixed, then,

$$\mathbf{y}^1 = (0,0,0,2) + \int_0^t \mathbf{f}(\tau, y_1^0 y_2^0 y_3^0 y_4^0) d\tau$$
$$= (0,0,0,2) + (0,0,2t,0) = (0,0,2t,2)$$
$$\mathbf{y}^2 = (0,0,0,2) + \int_0^t (y_2^1, y_3^1, y_4^1, 2\lambda^2 y_3^1 - 4\lambda^4 y_1^1) d\tau$$
$$= (0,0,0,2) + (0,t^2,2t,2\lambda^4 t^2) = (0,t^2,2t,2+2\lambda^4 t^2)$$
$$\mathbf{y}^3 = (0,0,0,2) + \int_0^t (y_2^2, y_3^2, y_4^2, 2\lambda^2 y_3^2 - 4\lambda^4 y_1^2) d\tau$$
$$= (0,0,0,2) + (\frac{t^3}{3}, t^2, 2t + \frac{2}{3}\lambda^2 t^3, 2\lambda^2 t^2) = (\frac{t^3}{3}, t^2, 2t + \frac{2}{3}\lambda^2 t^3, 2+2\lambda^2 t^2)$$
$$\mathbf{y}^4 = (0,0,0,2) + \int_0^t (y_2^3, y_3^3, y_4^3, 2\lambda^2 y_3^3 - 4\lambda^4 y_1^3) d\tau$$
$$= (\frac{t^3}{3}, t^2 + \frac{1}{6}\lambda^2 t^4, 2t + \frac{2}{3}\lambda^2 t^3, 2+2\lambda^2 t^2)$$
$$\mathbf{y}^5 = (0,0,0,2) + \int_0^t (y_2^4, y_3^4, y_4^4, 2\lambda^2 y_3^4 - 4\lambda^4 y_1^4) d\tau$$
$$= (0,0,0,2) + (\frac{t^3}{3} + \frac{1}{30}\lambda^2 t^5, t^2 + \frac{1}{6}\lambda^2 t^4, 2t + \frac{2}{3}\lambda^2 t^3, 2\lambda^2 t^2)$$
$$= (\frac{t^3}{3}, t^2 + \frac{1}{6}\lambda^2 t^4, 2t + \frac{2}{3}\lambda^2 t^3, 2+2\lambda^2 t^2)$$

Since $\mathbf{y}_4$ and $\mathbf{y}_5$ are sufficiently close to each other, then there is a cluster value (say $\mathbf{y}$), and therefore, $\mathbf{y}_m \to \mathbf{y}$ as $m \to \infty$.

Remark: This example shows that the local result is the only one we can hope for.

Practical Example 2.

Given a second order equation

$$y'' - 2\sqrt{|y|} = 0 (23)$$

with the conditions $y(0) = 0,\ y'(0) = 1$.

Solution: Let $y = y_1$ and $y' = y_2$. So,

$$y_1' = y_2 \ \equiv f_1$$
$$y_2' = 2\sqrt{|y_1|} \equiv f_2$$
$$\mathbf{f}(t,\mathbf{y}) = (y_2, 2\sqrt{|y_1|})$$

$$\frac{\partial f_1}{\partial y_2}=1, \qquad \frac{\partial f_1}{\partial y_1}=0$$

$$\frac{\partial f_2}{\partial y_1}=\frac{1}{|y|^{\frac{1}{2}}}, \quad \frac{\partial f_2}{\partial y_2}=0$$

$$\to\left|\frac{\partial \mathbf{f}}{\partial y_2}\right|=1, \qquad \left|\frac{\partial \mathbf{f}}{\partial y_1}\right|=\frac{1}{|y|^{\frac{1}{2}}}$$

$\mathbf{f}$ fails to satisfy the Lipschitz conditions at $\mathbf{y}=(0,0)$, and hence, the uniqueness fails.

Claim: Observe $\mathbf{f}$ is continuous but not Lipschitzian, however, it is possible to prove that the problem has a solution around the neighborhood of t_0 [17], though it's solution is not unique.

5.0 Conclusion

In conclusion, if we suppose $\mathbf{f}$ to be a continuous vector-valued function defined on

$$\hat{\Omega}:=\left\{(t,\mathbf{y}):|t|<\infty,\ |\mathbf{y}|<\infty\right\}$$

and satisfies Lipschitz conditions on each strip

$$(t,\mathbf{y}):|t|\le a,\ |\mathbf{y}|<\infty$$

where a is any positive number. Then, the vector differential equation (6) has a solution which exists for all real t. That is, the iterative sequence $\left\{\mathbf{\Phi}_m(t)\right\}_{m=1}^{\infty}$ converge to a solution which exist for all real t.

6.0 References

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