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Some Integral Inequalities of Hardy-type Operators

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Abstract

Hardy integral inequality has received attention of many researchers in recent time. The purpose of this paper is to obtain new integral inequalities of Hardy-type which complement some recent results. Furthermore, applications for measurable and convex functions are given. Improvements of some inequality are also obtained.

Keywords: Hard'y's inequality, Measurable functions, Weight functions & Hardy- type Operators.
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1.0 Introduction

In 1925, G. H. Hardy proved one of the most important classical one-dimensional integral inequalities. The classical Hardy integral inequality reads:

Theorem 1.1. *Let $f(x)$ be a non-negative p-integrable function defined on $(0,\infty)$, and $p > 1$. Then, f is integrable over the interval $(0,x)$ for each x and the following inequality:*

$$\int_0^\infty \left[\frac{1}{x}\left(\int_0^x f(y)\,dy\right)\right]^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p\,dx \tag{1}$$

holds, where $\left(\frac{p}{p-1}\right)^p$ is the best possible constant [1].

This inequality can be found in many standard books [2-6]. Inequality (1) has found much interest from a number of researchers and there are numerous new proofs, as well as, extensions, refinements and variants which are refered to as Hardy type inequalities.

In the paper [8], the author proved the following generalization which is an extension of [9].

Theorem 1.2. *Let $f(x)\in L^p(X)$, $g(x)\in L^q(X)$ and $fg\in L^p(X)$ be finite, non-negative measurable functions on $(0,\infty)$, $0 < t < a < b < \infty$ and $\frac{1}{p}+\frac{1}{q}+1=\frac{1}{r}$ with $1< p\le q<\infty$ such that $a<x<b$. Then, the following inequality holds:*

$$\left[\int_a^b \left(\frac{1}{x^q}(T(fg)^q)dx\right)\right]^{\frac{r}{q}} \le C\left[\left(\int_a^b t^{(p-1)}\,|f(t)|^p\,dt\right)\left(\int_a^b t^{(p-1)}\,|g(t)|^p\,dt\right)\right]^r \tag{2}$$

where,

$$C = \frac{(b-t)^{1-r}}{1-r}\left[\ln\left|\frac{(b-t)}{a}\right|^{\frac{1}{p^2}}\right.$$
$$\left. +\left[\frac{1}{p^2(1-r)}\right]\left(\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}(-1)^{k+1}(n-1)-(k-1)(p^2+1)\right)\ln\left[\frac{(b-t)}{a}\right]^R\right]$$

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and

$$R=\frac{1}{p^2}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}(n-k(p^2+1)) \quad \forall \ k(1)n$$

Adeagbo-Sheikh and Imoru in [10] also proved the following integral inequality of Hardy-type mainly by using Jensen's Inequality:

Theorem 1.3. *Let* g *be continuous and nondecreasing on* $[a,b]$, $0\le a\le b<\infty$, *with* $g(x)>0$ *for* $x>0$. *Let* $q\ge p\ge 1$ *and* $f(x)$ *be nonnegative and Lebesgue-Stieltjes integrable with respect to* $g(x)$ *on* $[a,b]$. *Suppose* δ *is a real number such that* $\frac{-p}{q}<\delta<0$, *then*

$$\left[\int_a^b g(x)^{\frac{\delta q}{p}}\left(\int_a^x f(t)dg(t)\right)^q dg(x)\right]^{\frac{1}{q}}\le C(a,b,p,q,\delta)\left[\int_a^b g(x)^{(p-1)(1+\delta)}f(x)^p dg(x)\right]^{\frac{1}{p}} \qquad (3)$$

where,

$$C(a,b,p,q,\delta)=(-\delta)^{\frac{q(1-p)}{p}}\left(\frac{p}{p+\delta q}\right)^{\frac{p}{q}}g(b)^{p+\delta q}(g(b)^{-\delta}-g(a)^{-\delta})^{\frac{q}{p}(p-1)}>0.$$

For other recent developments of the Hardy-type inequalities, see the papers [11-16]. In this article, we point out some other Hardy-type inequalities which will complement the above results (2) and (3).

2.0 Main Results

The following lemma is of particular interest [8].

Lemma 2.1. *Let* $1<b<\infty$, $1<p$, $\frac{1}{p}+\frac{1}{q}=1$, *and let* $f(x)$ *be a non-negative measurable function such that* $0\le\int_a^b f^p(t)dt<\infty$. *Then the following inequality holds:*

$$\left(\int_x^b f(t)^q \mathrm{d}t\right)^{\frac{1}{q}}<\left(\sqrt[p^2]{\left|\ln\frac{b}{x}\right|}\right)^{(p-1)^2}\left(\int_x^b t^{p-1}f(t)^{\frac{p^2}{p-1}}\mathrm{d}t\right)^{\frac{1}{p}} \qquad (4)$$

Proof:

Let $I=(\int_x^b f(t)^q\,dt)^{1/q}$, then, $I=\left[\int_x^b t^{1/q}f(t)^q t^{-1/q}dt\right]^{1/q}$

by Holder's inequality, we have,

$$I\le\left(\int_x^b t^{\frac{p}{q}}f(t)^{pq}\mathrm{d}t\right)^{\frac{1}{pq}}\left(\int_x^b t^{-1}\mathrm{d}t\right)^{\frac{1}{q^2}}=\left(\sqrt[p^2]{\left|\ln\frac{b}{x}\right|}\right)^{(p-1)^2}\left(\int_x^b t^{p-1}f(t)^{\frac{p^2}{p-1}}\mathrm{d}t\right)^{\frac{1}{p}}$$

We need to show that there exists $x_0\in(a,b)$ such that for any $x\in(a,x_0)$, equality in (4) does not hold. If otherwise, there exist a decreasing sequence $(x_n)_{n\in\mathbf{N}}$ in (a,b), $x_n\searrow a$ such that for $n\in\mathbf{N}$ the inequality (4), written $x=x_n$, becomes an equality. Then, to every $n\in\mathbf{N}$ there exists corresponding real constants c_n and $d_n\ge 0$ not both zero, such that $c_n[t^{1/q}f(t)]^p=d_n[t^{-\frac{1}{q}}]^q$ almost everywhere in (x_n,b). There exists positive integer N such that for $n>N$, $f(t)\ne 0$ almost everywhere in (x,b). Hence, $c_n=c\ne 0$ and $d_n=d\ne 0$ for $n>N$, and also

$$\int_a^b f^p(t)\mathrm{d}t=\lim_{n\to\infty}\int_{x_n}^b f^p(t)\mathrm{d}t=\frac{c}{1-p}\left(b^{1-p}-x_n^{1-p}\right)=\infty$$

This contradicts the facts that $0<\int_a^b f^p(t)\mathrm{d}t<\infty$. The lemma is proved.

Theorem 2.1. *Let* $f(x)\in L^p(X)$, $g(x)\in L^q(X)$ *be finite non-negative measurable functions on* $(0,\infty)$,

$0<a<t<b<\infty$ *and* $\frac{1}{p}+\frac{1}{q}+1=\frac{1}{r}$

with $1<p\le q\le\infty$ *such that* $a<x<b$, *then the following inequality holds:*

$$\left[\int_a^b \frac{1}{x^q}\left(\int_x^b (fg)^q \mathrm{d}t\right)\mathrm{d}x\right]^{\frac{r}{q}} \le C\left(\int_a^b t^{p-1}(fg)^{\frac{p^2}{p-1}}\mathrm{d}t\right)^r \tag{5}$$

Where

$$C=\frac{(t-a)^{1-r}}{1-r}\left[\ln\left|\frac{b}{(t-a)}\right|^{\frac{2}{p-1}}+\frac{2}{(1-r)(p-1)}\left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(n-1)-(k-1)p\right)\ln\left|\frac{b}{t-a}\right|^{R}\right]$$

and

$$R=\frac{1}{p-1}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[(n+1)-(k-1)p\right] \qquad \forall k(1)n$$

Proof:

$$\left[\int_a^b \frac{1}{x^q}\left(\int_x^b (fg)^q \mathrm{d}t\right)\mathrm{d}t\right]^{\frac{r}{q}}$$

$$\le\left[\int_a^b \frac{1}{x^q}\left(\int_x^b |f|^q \mathrm{d}t\right)\left(\int_x^b |g|^q \mathrm{d}t\right)\mathrm{d}x\right]^{\frac{r}{q}}$$

$$\le\left[\int_a^b \frac{1}{x^q}\left(\ln\left|\frac{b}{x}\right|\right)^{\frac{2}{p-1}}\left(\int_x^b t^{p-1}(fg)^{\frac{p^2}{p-1}}\mathrm{d}t\right)^{\frac{1}{p}}\mathrm{d}x\right]^{\frac{r}{q}}$$

$$=\left[\int_a^b x^{-q}\left(\ln\left|\frac{b}{x}\right|\right)^{\frac{2}{p-1}}\left(\int_a^t t^{p-1}(fg)^{\frac{p^2}{p-1}}\mathrm{d}t\right)^{\frac{1}{p}}\mathrm{d}x\right]^{\frac{r}{q}}$$

$$\le\int_a^b x^{-r}\left(\ln\left|\frac{b}{x}\right|\right)^{\frac{2}{p-1}}\mathrm{d}x\left(\int_a^b t^{p-1}(fg)^{\frac{p^2}{p-1}}\mathrm{d}t\right)^r$$

$$=C\left(\int_a^b t^{p-1}(fg)^{\frac{p^2}{p-1}}\mathrm{d}t\right)^r$$

where C is as stated in the statement of the theorem and this proves the theorem. The next results are on convex functions as it applies to Hardy-type inequalities.

Lemma 2.2. *A local minimum of a function f is a global minimum if and only if f is strictly convex.*

Proof

The necessary part follows from the fact that if a point x is a local optimum of a convex function f. Then, $f(z)\ge f(x)$ for any z in some neighborhood U of x. For any y, $z=\lambda x+(1-\lambda)y$ belongs to U and $\lambda<1$ sufficiently close to 1 implies that x is a global optimum. For the sufficient part, we let f be a strictly convex function with convex domain. Suppose f has a local minimum at a and b such that $a\ne b$ and assuming $f(a)\le f(b)$. By strict convexity and for any $\lambda\in(0,1)$, we have,

$$f(\lambda a+(1-\lambda)b)<\lambda f(a)+(1-\lambda)f(b)\le\lambda f(b)+(1-\lambda)f(b)=f(b)$$

Since any neighborhood of b contains points of the form $\lambda a+(1-\lambda)b$ with $\lambda\in[0,1]$, thus the neighborhood of b contains points x for which $f(x)<f(b)$.

Hence, f does not have a local minimum at b, a contradiction. It must be that $a = b$, this shows that f has at most one local minimum.

Lemma 2.3. *Let* $0 < b < \infty$ *and* $-\infty \le a < c \le \infty$. *If* φ *is a positive convex function on (a,c), then*

$$\int_0^b \varphi\left[\frac{1}{x^q}\int_0^x h(t)\mathrm{d}t\right]\mathrm{d}x$$

$$\le \frac{1}{1-q}\int_0^b \varphi\big(h(t)\big)\big(b^{1-q} - t^{1-q}\big)\mathrm{d}t$$

Proof

$$\int_0^b \varphi\left[\frac{1}{x^q}\int_0^x h(t)\mathrm{d}t\right]\mathrm{d}x \le \int_0^b \frac{1}{x^q}\left(\int_0^x \varphi(h(t))\mathrm{d}t\right)\mathrm{d}x$$

$$= \int_0^b \varphi\big(h(t)\big)\left(\int_t^b \frac{1}{x^q}\mathrm{d}x\right)\mathrm{d}t$$

$$= \int_0^b \varphi\big(h(t)\big)\left(\frac{b^{1-q} - t^{1-q}}{1-q}\right)\mathrm{d}t$$

$$= \frac{1}{1-q}\int_0^b \varphi\big(h(t)\big)\big(b^{1-q} - t^{1-q}\big)\mathrm{d}t$$

Hence the proof.

Lemma 2.4. *Let* $h(x,t)$ *be non-negative for* $x, t \ge 0$, λ *non decreasing and* $-\infty \le a \le b \le \infty$. *then*

$$\int_a^x h(x,t)^{1/pq}\mathrm{d}\lambda(t) \le \left[\int_a^x \mathrm{d}\lambda(t)\right]^{1-\frac{1}{p}}\left[\int_a^x h(x,t)^{1/q}\mathrm{d}\lambda(t)\right]^{\frac{1}{p}}$$

Proof

Let Φ be continuous and convex, If Φ has a continuous inverse which is neccessarily concave, then by Jensen's inequality we have

$$\phi^{-1}\left[\frac{\int_a^x h(x,t)\mathrm{d}\lambda(t)}{\int_a^x \mathrm{d}\lambda(t)}\right] \ge \frac{\int_a^x \phi^{-1}[h(x,t)]\mathrm{d}\lambda(t)}{\int_a^x \mathrm{d}\lambda(t)}$$

Taking $\phi(u) = u^p$, $p \ge 1$, we obtain

$$\left[\frac{\int_a^x h(x,t)\mathrm{d}\lambda(t)}{\int_a^x \mathrm{d}\lambda(t)}\right]^{\frac{1}{p}} \ge \frac{\int_a^x h(x,t)^{\frac{1}{p}}\mathrm{d}\lambda(t)}{\int_a^x \mathrm{d}\lambda(t)}$$

for $1 \le p \le q$, we have

$$\left[\frac{\int_a^x h(x,t)^{\frac{1}{q}}\mathrm{d}\lambda(t)}{\int_a^x \mathrm{d}\lambda(t)}\right]^{\frac{1}{p}} \ge \frac{\int_a^x h(x,t)^{\frac{1}{pq}}\mathrm{d}\lambda(t)}{\int_a^x \mathrm{d}\lambda(t)}$$

which implies that

$$\int_a^x h(x,t)^{\frac{1}{pq}}\mathrm{d}\lambda(t) \le \left[\int_a^x \mathrm{d}\lambda(t)\right]^{1-\frac{1}{p}}\left[\int_a^x h(x,t)^{\frac{1}{q}}\mathrm{d}\lambda(t)\right]^{\frac{1}{p}}$$

This complete the proof.

Theorem 2.3. *If $0<b\leq\infty$ and $-\infty\leq a<c\leq\infty$, let f, g be defined on (0,b) such that $a< f(x), g(x)<c$, then*

$$\int_0^b exp\left[\frac{1}{x^q}\int_0^x \ln(fg)\mathrm{d}t\right]\mathrm{d}x\leq\frac{e}{1-2q}\int_0^b t(fg)(b^{1-2q}-t^{1-2q})\mathrm{d}t.(8)$$

Proof:

$$\int_0^b exp\left[\frac{1}{x^q}\int_0^x \ln(fg)\mathrm{d}t\right]\mathrm{d}x=\int_0^b exp\left(\frac{1}{x^q}\int_0^x(\ln t(fg)-\ln t)\mathrm{d}t\right)\mathrm{d}x$$

$$=\int_0^b\left[exp\left(\frac{1}{x^q}\int_0^x \ln t(fg)\mathrm{d}t\right)\times exp\left(\frac{-1}{x^q}\int_0^x \ln t\mathrm{d}t\right)\right]\mathrm{d}x$$

Since $f(x)=e^x$ is a convex function, applying Jensen's inequality to the above gives

$$\int_0^b exp\left[\frac{1}{x^q}\int_0^x \ln(fg)\mathrm{d}t\right]\mathrm{d}x\leq\int_0^b\frac{1}{x^q}\left[\int_0^x t(fg)\mathrm{d}t\times\frac{1}{x^{q-1}}exp(-\ln x+1)\right]\mathrm{d}x$$

$$=e\int_0^b\frac{1}{x^{2q}}\left(\int_0^x t(fg)\mathrm{d}t\right)\mathrm{d}x$$

$$=e\int_0^b t(fg)\left(\int_t^b\frac{1}{x^{2q}}\mathrm{d}x\right)\mathrm{d}t$$

$$=\frac{e}{1-2q}\int_0^b t(fg)(b^{1-2q}-t^{1-2q})\mathrm{d}t$$

and the result follows.

Theorem 2.4. *Let g be a continuous and nondecreasing on $[a,b]$, $0\leq a\leq b\leq\infty$, with $g(x)>0$ for $x>0$ and $a\leq t<b$. Let $1\leq p\leq q$ and $f(x)$ be nonnegative and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a,b]$. Suppose r is a real number such that $0>r>-\infty$ then,*

$$\left[\int_a^b g(x)^{\frac{rq}{p}}\left(\int_0^x f(t)\mathrm{d}g(t)\right)^q\mathrm{d}g(x)\right]^{\frac{1}{q}}\leq C(a,b,p,q,r)\left[\int_a^b g(x)^{\frac{p-1}{r}}f(x)^p\mathrm{d}g(x)\right]^{\frac{1}{p}}.(9)$$

where

$$C(a,b,p,q,r)=\left(\frac{r}{r-1}\right)^{\frac{p-1}{p}}\left(\frac{p}{p+rq}\right)^{\frac{1}{q}}\left(g(b)^{\frac{r-1}{r}}-g(a)^{\frac{r-1}{r}}\right)^{\frac{p-1}{p}}\left(g(b)^{\frac{p+rq}{p}}-g(a)^{\frac{p+rq}{p}}\right)^{\frac{1}{q}}$$

Proof

In the inequality (7), we let $h(x,t)=g(x)^{rq}g(t)^{pq/r}f(t)^{pq}$ and $\mathrm{d}\lambda(t)=g(t)^{-1/r}\mathrm{d}g(t)$

Then, the left hand side of (2.5) becomes

$$\int_a^x g(x)^{\frac{r}{p}}g(t)^{\frac{1}{r}}f(t)g(t)^{\frac{-1}{r}}\mathrm{d}g(t)\quad=\int_a^x g(x)^{\frac{r}{p}}f(t)\mathrm{d}g(t)$$

$$=g(x)^{\frac{r}{p}}\int_a^x f(t)\mathrm{d}g(t)$$

and the right hand side reduces to

$$\left[\int_a^x g(t)^{\frac{-1}{r}}\mathrm{d}g(t)\right]\frac{p-1}{p}\left[\int_a^x g(x)^r g(t)^{\frac{p}{r}} f(t)^p g(t)^{\frac{-1}{r}}\mathrm{d}g(t)\right]^{\frac{1}{p}}$$

$$=\left[\int_a^x g(t)^{\frac{-1}{r}}\mathrm{d}g(t)\right]\frac{p-1}{p}\left[\int_a^x g(x)^r g(t)^{\frac{p-1}{r}} f(t)^p\mathrm{d}g(t)\right]^{\frac{1}{p}}$$

$$=\left[\frac{r}{r-1}g(t)^{\frac{r-1}{r}}\Big|_a^x\right]^{\frac{p-1}{p}} g(x)^{\frac{r}{p}}\left[\int_a^x g(t)^{\frac{p-1}{r}} f(t)^p\mathrm{d}g(t)\right]^{\frac{1}{p}}$$

$$=\left(\frac{r}{r-1}\right)^{\frac{p-1}{p}}\left[g(x)^{\frac{r-1}{r}}-g(a)^{\frac{r-1}{r}}\right]^{\frac{p-1}{p}} g(x)^{\frac{r}{p}}\left[\int_a^x g(t)^{\frac{p-1}{r}} f(t)^p\mathrm{d}g(t)\right]^{\frac{1}{p}}$$

Hence, inequality (7) becomes

$$g(x)^{\frac{r}{p}}\left(\int_a^x f(t)\mathrm{d}g(t)\right)\le\left(\frac{r}{r-1}\right)^{\frac{p-1}{p}}\left[g(x)^{\frac{r-1}{r}}-g(a)^{\frac{r-1}{r}}\right]^{\frac{p-1}{p}} g(x)^{\frac{r}{p}}$$

$$\times\left[\int_a^x g(t)^{\frac{p-1}{r}} f(t)^p\mathrm{d}g(t)\right]^{\frac{1}{p}}$$

for $q\ge p$, we have

$$g(x)^{\frac{rq}{p}}\left(\int_a^x f(t)\mathrm{d}g(t)\right)^q\le\left(\frac{r}{r-1}\right)^{\frac{q(p-1)}{p}}\left[g(x)^{\frac{r-1}{r}}-g(a)^{\frac{r-1}{r}}\right]^{\frac{q(p-1)}{p}} g(x)^{\frac{rq}{p}}$$

$$\times\left[\int_a^x g(t)^{\frac{p-1}{r}} f(t)^p\mathrm{d}g(t)\right]^{\frac{q}{p}}$$

Integrating both sides with respect to *g(x)* and then raising both sides to power p/q yields

$$\left[\int_a^b g(x)^{\frac{rq}{p}}\left(\int_a^x f(t)\mathrm{d}g(t)\right)^q\mathrm{d}g(x)\right]^{\frac{p}{q}}$$

$$\le\left[\left(\frac{r}{r-1}\right)^{\frac{q(p-1)}{p}}\int_a^b g(x)^{\frac{rq}{p}}\left(g(x)^{\frac{r-1}{r}}-g(a)^{\frac{r-1}{r}}\right)^{\frac{q(p-1)}{p}}\left(\int_a^x g(t)^{\frac{p-1}{r}} f(t)^p\mathrm{d}g(t)\right)^{\frac{q}{p}}\mathrm{d}g(x)\right]^{\frac{p}{q}}$$

Applying Minkowski integral inequality to the right hand side implies

$$\le\left(\frac{r}{r-1}\right)^{p-1}\int_a^b g(t)^{\frac{p-1}{r}} f(t)^p\left[\int_t^b\left(g(x)^{\frac{r-1}{r}}-g(a)^{\frac{r-1}{r}}\right)^{\frac{q(p-1)}{p}} g(x)^{\frac{rq}{p}}\mathrm{d}g(x)\right]^{\frac{p}{q}}\mathrm{d}g(t)$$

$$\le\left(\frac{r}{r-1}\right)^{p-1}\left(g(b)^{\frac{r-1}{r}}-g(a)^{\frac{r-1}{r}}\right)^{p-1}\int_a^b g(t)^{\frac{p-1}{r}} f(t)^p\left[\int_t^b g(x)^{\frac{rq}{p}}\mathrm{d}g(x)\right]^{\frac{p}{q}}\mathrm{d}g(t)$$

Since $r<0$

$$=\left(\frac{r}{r-1}\right)^{p-1}\left(\frac{p}{p+rq}\right)^{\frac{p}{q}}\left(g(b)^{\frac{r-1}{r}}-g(a)^{\frac{r-1}{r}}\right)^{p-1}\int_a^b g(x)^{\frac{p-1}{r}} f(x)^p\left(g(b)^{\frac{p+rq}{p}}-g(t)^{\frac{p+rq}{p}}\right)^{\frac{p}{q}}\mathrm{d}g(x)$$

$$\leq C(a,b,p,q,r)\int_a^b g(x)^{\frac{p-1}{r}} f(x)^p \mathrm{d}g(x)$$

Hence, we have

$$\left[\int_a^b g(x)^{\frac{rq}{p}}\left(\int_0^x f(t)\mathrm{d}g(t)\right)^q \mathrm{d}g(x)\right]^{\frac{1}{q}} \leq C(a,b,p,q,r)\left[\int_a^b g(x)^{\frac{p-1}{r}} f(x)^p \mathrm{d}g(x)\right]^{\frac{1}{p}}$$

Which complete the proof of the Theorem.

3.0 Conclusion

This work obtained an improvement on Adeagbo-Sheikh and Imoru results. Applications for measurable and convex functions are also given.

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