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Approximate Method for Solving Factional Riccati Differential Equation

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Abstract

In this paper, Chebyshev Spectral method is presented for solving the non-linear Fractional Riccati Differential Equation (FRDE). The fractional derivative is described in the Caputo sense. The properties of the Chebyshev polynomials are used to reduce FRDE to the solution of non-linear system of algebraic equation using Newton iteration method. Numerical results are introduced to satisfy the accuracy and applicability of the proposed method.

Keywords: Fractional Riccati differential equation; caputo fractional derivative; Chebyshev Spectral method.

1.0 Introduction

It is well known that the fractional differential equation (FDEs) have been the focus of many studies due to their frequent appearance in various applications such as in fluid mechanics, solid mathematics, viscoelasticity, biology, physics and engineering applications, for more details see for example [1], [2]. As a result, considerable attention has been given to the efficient numerical solutions of FDEs of physical interest, because it is difficult to find exact solutions. Different numerical methods have been proposed in the literature for solving FDEs [3 – 16].

The Riccati differential equation is named after the Italian Nobleman count Jacopo Francesco Riccati (1676 – 1754). The book of Reid [17] contains the fundamental theories of Riccati equation, with applications to random processes, optimal control, and diffusion problems. Besides important engineering science applications that today are considered classical, such as stochastic realization theory, robust stabilization, and network synthesis, the newer applications include such areas as financial mathematics [18]. The solution of this equation can be reached using classical numerical methods such as the Forward Euler methods and Runge-Kutta method. An unconditionally stable scheme was presented by Dubois and Saidi [19]. Behnasawi et al. [20] presented the usage of Adomian decomposition method to solve the non-linear Riccati differential equation in an analytic form.

Tau and Abbasbandy [21] employed the analytic technique called Homotopy Analysis method to solve the quadratic Riccati equation.

The fractional Riccati differential equation is studied by many authors and using different numerical methods. In [22] this problem is solved using the homotopy analysis method, in [29] the same problem is solved by using variational iteration method and in [23] solved using the Adomian decomposition method.

We describe some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

Definition 1

The caputo fractional derivative operator D^α of order α is defined in the following form

$$D^\alpha f(n) = \frac{1}{r(m-\propto)}\int_0^x \frac{f^m(t)}{(x-t)^{\propto-m+1}}\,dt,\ \alpha > 0, x > 0$$

where $m-1 < \alpha \le m, m \in N.$

similar to integer – order differentiation, caputo fractional derivative operation is a linear operation

$D^\propto(\lambda f(n) + \mu g\ (x)) = \lambda D^\propto f(x) + \mu D^\propto g(x),$

where λ and μ are constants. For the caputo's derivative we have [2]

$$D^\alpha C = 0, \quad C \text{ is a constant} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$D^\propto x^n \quad \left\{ \begin{array}{ll} 0, & \text{for } n\in N_0 \text{ and } n < [\alpha]; \\ \frac{r(n+1)}{r(n+1-\alpha),} x^{n-\propto} & \text{for } n\in N_0 \text{ and } n \ge [\alpha]; \ldots\ldots\ldots (2) \end{array} \right\}$$

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We use the ceiling function $[\alpha]$ to denote the smallest integer greater than or equal to α, and $N_0 = \{0,1,2,....\}$. Recall that for $\alpha \in N_0$, the caputo differential operator coincides with the usual differential operator of integer order.
For more details on fractional derivatives, definitions and its properties, see [1, 2].
In this work, the Chebyshev collocation method is an efficient technique (see for example [24 –25]is used to study the numerical solution of the non-linear FRDE. An approximate formula of the fractional derivative is presented. We extend the application of the Chebyshev collocation method in order to derive analytical approximate solutions to non-linear fractional Riccati differential equation [26].

$$D^{\propto}u(t) = P(t) + Q(t)u(t) + R(t)u^2(t),\ t > 0,\ 0 < \propto \leq 1, \quad .... \quad (3)$$

subject to the initial condition

$$u(o) = u^o,$$

here P(t), Q(t) and R(t) are known real functions and u^o is a constant.
Our paper is organized as follows; section 2, derivation of the approximate formula for fractional derivations using Chebystev series expansion is givensection 3. The procedure solution using Chebyshev collocation method is given. Finally, in section 4 the paper ends with a brief conclusion.

2.0 Derivation of the Approximate Method

The well known Chebyshev polynomials are defined on the interval [-1, 1] and can be determined with the aid of the following recurrence formular [25].

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x),\ T_o(x) = 1,\ T_1(x) = x,\ n = 1,2,.... \quad (4)$$

The analytic form of the Chebyshev polynomials
$T_n(x)$ of degree n is given by

$$T_n(x) = n \quad \sum_{i=0}^{\frac{n}{2}} (-1)^i 2^{n-2i-c} \frac{(n-i-1)!}{(i)!(n-2i)!} X^{n-2i} \quad .... \quad (5)$$

where $[\frac{n}{2}]$ denotes the integer part of $\frac{n}{2}$. The orthogonality condition is

$$\int_{-1}^{1} \frac{Ti(x)Tj(n)}{\sqrt{1-x^2}}\,dx = \begin{cases} \pi, for\ i = j = 0 \\ \frac{\pi}{2}, for\ i = j \neq 0 \\ 0, for\ 1 \neq j \end{cases}$$

In order to use these polynomials on the interval [O,L] we define the so called shifted Chebyshev polynomials by introducing the change of variable $x = \frac{2}{L}t - 1$. So, the shifted Chebyshev polynomials are defined as

$$T_n^*(t) = T_n\left(\frac{2}{L}t - 1\right) = T_{2n}\left(\sqrt{\frac{t}{L}}\right).$$

The analytic form of the shifted Chebyshev polynomials $T_n^*(t)$ of degree n is given by

$$T_n^*(t) = n \sum_{k=0}^{n} (-1)^{n-k} \frac{2^{2k}\ (n+k-1)!}{L^k\ (2k)!(n-k)!} t^k,\ n = 1,2. .... \quad (6)$$

The function u(t), which belongs to the space of square integrable function in [O,L], may be expressed in terms of shifted Chebyshev polynomials as

$$u(t) = \sum_{i=0}^{\infty} c_i\ T_i^*(t) \ .... \quad (7)$$

where the coefficients c_i are given by

$$c_0 = \frac{1}{\pi}\int_O^L \frac{u(t)T_0^*(t)}{\sqrt{Lt-t^2}}\,dt c_i = \frac{2}{\pi}\int_O^L \frac{u(t)T_0^*(t)}{\sqrt{Lt-t^2}}dt,\ i = 1,2,...\ .... \quad (8)$$

In practice, only the first (m+1) terms of shifted Chebyshev polynomials are considered.
Then we have

$$U_m(t) = \sum_{i=0}^{m} c_i\ T_i^*(t) \ ....\,..\,...................... \quad (9)$$

Kahder [24] introduced a new approximate formula of the fractional derivative and used it to solve numerically the fractional diffusion equation. The main approximate formular of the fraction derivative of $U_m(t)$ is given in the following theorem.

Theorem 1

let u(t) be approximated by Chebyshev polynomials as in (9) and also suppose $\propto > 0$,
then $D^{\propto}(U_{m(t)}) = \sum_{i=[\propto]}^{m} \sum_{k=[\propto]}^{i} c_i \quad w_{i,k}^{(\propto)} t^{k-\propto}$ .... (10)
where $w_{i,k}^{(\propto)}$ is given by

$$w_{i,k}^{(\propto)} = (-1)^{i-k} \frac{2^{2k} i\ (i+k-1)r\ (k+1)}{L^k (i-k)!(2k)r\ (k+1-\propto)} \ .... \quad (11)$$

Proof: since the caputo's fractional differentiation is a linear operation we have

$$D^{\propto}(U_{m(t)}) = \sum_{i=0}^{m} C_i D^{\propto}\ (T_i^*(t)). .... \quad (12)$$

Employing equations (1) and (2) into equation (6), we have

$$D^{\propto}T_i^*(t) = 0, \quad i = 0, 1, \ldots, [\propto] - 1, \propto > 0 \ldots\ldots\ldots\ldots \quad (13)$$

Therefore, for i = [∝], . . . , m, and by using

Equations (1) and (2) m Equation (6), we get

$$D^{\propto}T_i^*(t) = i, \sum_{k=[\propto]}^{i} (-1)^{i-k} \frac{2^{2k}\ (i+k-1)!\ r\ (k+1)}{L^k\ (i-k)!(2k)!r\ (k+1-\propto)} t^{k-\propto} \ldots\ldots\ldots\ldots \quad (14)$$

A combination of Equations (13), (14) and (11) leads to the desired result and completes the proof of the theorem.

Theorem 2

The caputo fractional derivative of order ∝ for the shifted Chebysher polynomials can be expressed in terms of the shifted Chebysher polynomials themselves in the following form.

$$D^{\propto}(T_i^*(t)) = \sum_{k=[\propto]}^{i} \sum_{j=0}^{k-[\propto]} O_{i,j,k}\, T_j^*(t) \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

where

$$0_{i,j,k} = \frac{(-1)^{i-k}\, 2i\, (i+k-i)!\, r\left(k-\propto+\frac{1}{2}\right)L^{k-\propto}}{hj\, r\left(k+\frac{1}{2}\right)(i-k)!\ r\ (k-\propto-j+1)\, r\ (k+j-\propto+1)}, j = 0, 1, \ \ldots.$$

Proof using the properties of the shifted Chebyshev polynomials [25], then $t^{k-\propto}$ in (14) can be expanded in the following form [28]

$$t^{k-\propto} = \sum_{j=0}^{k-[\propto]} C_{j,k}\, T_j^*(t) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

Where $C_{j,k}$ can be obtained using (8) such that u(t) = $t^{k-\propto}$, then we can claim the following

$$C_{j,k} = \frac{2}{hj\pi}\int_O^L \frac{t^{k-\propto}T_{j(t)}^*}{\sqrt{Lt-t^2}} \,\mathrm{dt}, \text{ ho} = 2, \text{ hj} = 1, \text{ j} = 1, 2, \ldots.$$

But at j = 0 we have, $c_{ko=\frac{1}{\pi}}$ $\quad \int_O^L \frac{t^{k-\propto}T_{0(t)}^*}{\sqrt{Lt-t^2}}\,\mathrm{dt} = \frac{L^{k-\propto}}{\sqrt{\pi}} \frac{r\ (k-\propto+\frac{1}{2})}{r\ (\ k-\propto+1)},$

also, for any j, using the formula (6), we can claim

$$C_{kj} = \frac{j}{\sqrt{\pi}}\sum_{r=0}^{j} (-1)^{j-r} \frac{(j+r-1)!\ 2^{2r+1}\ \ r\left(k+r-\propto+\frac{1}{2}\right)L^{k-\propto}}{(j-r)!\ (2r)!\ r\ (k+r-\propto+1)},$$

j = 1,2,3,. . . .

Employing equations (14) and (16) gives

$$D^{\propto}(T_i^*(t)) = \sum_{k=[\propto]}^{i} \sum_{j=0}^{k-[\propto]} O_{i,j,k}\, T_j^*(t),\ i = [\propto], [\propto] + 1, \ldots.$$

where

$$0_{i,j,k} = \left\{ \begin{array}{l} \frac{i\ (-1)^{i-k}(1+k-1)!\ 2^{2k}\ k!r\ (k-\propto+\frac{1}{2})L^{k-\propto}}{(i-k)!(2k)!\sqrt{\pi}\,(r(k+1-\propto))^2}\ j = 0; \\ \frac{(-1)^{i-k}\ ij(l+k-1)!\ 2^{2k+1}\ k!}{\sqrt{\pi}\ r(k+1-\propto)(i-k)!\ (2k)!} \mathrm{x}\sum_{r=0}^{i} \quad \frac{(-1)^{j-r}(j+r-1)!\ 2^{2k}r\left(k+r-\propto+\frac{1}{2}\right) L-\propto}{(j-r)!\ (2r)!\ r\ (k+r-\propto+1)} \end{array} \right\}$$

j = 1,2,3,. . . .

After some lengthy manipulation $0_{i,j,k}$ can put in the following form

$$0_{i,j,k} = \frac{(-1)^{i-k}\ 2i(i+k-1)!\ r\left(k-\propto+\frac{1}{2}\right)L^{k-\propto}}{hjr\left(k+\frac{1}{2}\right)(i-k)!\ r(k-\propto-j+1)r\ (k+j-\propto+1)}, j = 0,1,\ldots\ldots\ldots\ldots\ldots \quad (17)$$

and this completes the proof of the theorem.

Implementation of Chebyshev Spectral Method for Solving Fractional Riccati Differential Equation

In this section, we introduce a numerical algorithm using Chebyshev collocation spectral method for solving the Fractional Riccati Differential Equation of the form (3).

Example 1

In this example, we consider the Fractional Riccati Differential Equation of the form [26]

$$D^{\alpha}u(t) = u^{(2)}(t) - 1 = 0,\ t > 0,\ 0 < \propto \leq 1, \quad \ldots\ldots\ldots\ldots\ldots \quad (18)$$

The parameter ∝ refers to the fractional order of the time derivative. We also assume an initial condition.

$$U(o) = U_0 = 0$$

for∝ = 1, Equation (18) is the Standard Riccati Differential Equation . . . . (19)

$$\frac{du(t)}{dt} + u^1(t) - 1 = 0$$

The exact solution to this equation is

$$u(t) = \frac{e^{2t}-1}{e^{2t}+1}.$$

The procedure of the implementation is given by the following steps:

1. Approximate the function u(t) using the formula (9) and its caputo fractional derivative $D^{\alpha}u(t)$ using the presented formula (12) with m = 5, then the FRDE (18) is transformed to the following approximated form.

$\sum_{i=1}^{5}\sum_{k=1}^{i} c_i w_i^{\propto}, k\, t^{k-\propto} + (\sum_{i=0}^{5} C_i\, T_i^*(t))^2 - 1 = 0$,. . . . . (20)

Where $w_i^{(\propto)}, k$ is define in (11) $\sum_{i=1}^{5}\sum_{k=1}^{i} c_i w_i^{\propto}, kt^{k-\propto} + (\sum_{i=0}^{5} C_i\, T_i^*(t))^2 - 1 = 0$

2. The initial condition (19) is given by the following form

   $\sum_{i=0}^{5} C_i\, T_i^*(0) \quad = 0$,.. . . . . . (21)

   TheEquations (20) – (21) represent a system of non-linear algebraic equations which contains six equations for the unknowns C_i, i = 0,1,. . . . 5.
3. Solve the previous system using the Newton iteration method to obtain the unknowns C_i, i = 0,1,. . . . 5.

   Therefore, the approximate solution will take the form

   u(t) $C_o T_o^*(t) + C_1 T_1^*(t) + C_2 T_2^*(t) + C_3 T_3^*(t) + C_4 T_4^*(t) + C_5 T_5^*(t)$.

3.0 Conclusion

In this article, we used Chebyshev collocation method for solving the Fractional RiccatiDifferential Equation. Special attention is given to the study of the proposed problem. The properties of the Chebyshev polynomials are used to reduce the Fractional Riccati Differential Equation to a non-linear system of algebraic equations which is solved by Newton iteration method. The convergence analysis of the approximate formula is given. From the obtained numerical results we can conclude that this method gives results with an excellent agreement with the exact solution. All numerical results are obtained using maple program.

4.0 References

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