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Solution of Riccati Differential Equation with Variable Co-Efficient By Differential Transform Method

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Abstract

In this paper differential transform method (DTM) is implemented to solve some Riccati differential equations with variable coefficient. This technique does not require any discretization, linearization or small perturbations and therefore, it reduces significantly the numerical computation. The results derived by these methods are compared with the numerical results derived by Runge Kutta 4 (RK4) method.

Keywords: Riccati equation, differential transform method and classical Runge Kutta 4 method.

1.0 Introduction

The Riccati differential equation is named after the Italian nobleman Count Jacopo Francesco Riccati (1676 – 1754). The applications of this may be found not only in random processes, optimal control and diffusion problems [1] but also in stochastic realization theory, optimal control, fluid and solid mechanics, robust stabilization, network synthesis and financial mathematics [2] and [3]. This equation is perhaps one of the simplest nonlinear ordinary differential equation which plays a very important role in the solution of nonlinear integrable partial differential equations.

One of the characteristics of Riccatti equation is that it is the compatibility condition between two linear partial differential equations for an auxiliary function, the so called wave function [4]. Among the consequences of the existence of a lax pair is the fact that one can obtain for them a denumerable number of exact solutions, the so called soliton solutions.

The soliton solutions and their superposition's can be obtained recursively as solutions of the appropriate Boacklund transformation, a differential relation between two different solutions of the nonlinear equation, starting from a trivial, in general constraint, solution of the nonlinear partial differential equation. Moreover, it is the only first order nonlinear ordinary differential equation which possesses the Pamlev's property [5]. As shown by Lie [6], the Riccati equation, though a nonlinear equation possesses a superposition formula, as it is the case of all linear equations. The Riccati equation can also be linearised, i.e. it can be reduced to a linear Schrodinger spectral problem by the Cole-Hopt transformation [7]. This Riccati equation plays an important role in the solution of various nonlinear systems. The solution of these equations can be obtained numerically by the forward Euler method and Runge-Kutta method. An unconditionally stable scheme was presented by Dubols and Saidi [8].

An analytic solution of the nonlinear Riccati equation was obtained by El-Tawil et al. [9] using Adomian decomposition method. Recently, Tan and Abbasbandy [10] implemented the homotopy analysis method (HAM) to solve a quadratic Riccati equation. Abbasbandy [11] also solve one example of quadratic Riccati differential equation (with constraint coefficient) by He's variation iteration method by using Adomian's polynomials. Biazar and Eslami [12] also solved the quadratic Riccati differential equation (with constant coefficient) using the Differential Transform method. Batiha et al solved some of the Riccati equations by variational iteration method [13]. The concept of differential transform was not introduced by Zhou [14] in solving linear and nonlinear initial value problems in electrical circuit analysis.

The traditional Taylor series method takes a long time for computation of higher order derivatives. Instead, differential transform method (DTM) is an iterative procedure for obtaining analytic Taylor series solution of differential equations and is much easier. Here we have derived the solution of some Riccati equations (with variable coefficients) by DTM. For the first example we have got the exact analytical solution [13]. The results of the second example obtained by DTM are compared with those obtained by Runge Kutta 4 method and it is observed that there exists a sufficient amount of agreement between the two sets of results.

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2.0 Differential Transform Method

Recently a great deal of interest has been shown on the application of DTM for the solution of various problems e.g. boundary value problems, algebraic equations and partial differential equations [15]-[18]. This method provides the solution in a rapidly convergent series with components that are computed both elegantly and accurately. The main advantage of the method is that, it can be applied directly to various types of differential and integral equations, which are linear and nonlinear, homogenous and non-homogenous, with constant and with variable coefficients. Another important advantage is that it reduces the size of computation work and at the same time, maintains a high level of accuracy. To illustrate the method let us consider a function f(x) whose differential transform is defined as follows:

$$F(k) = \frac{1}{k!}\frac{d^k f(x)}{dx^k} \quad x = 0 \ldots \tag{1}$$

In (1), f(x) is the original function F(k) is the transformed function. The Taylor series expansion of the function f(x) about a point x = 0 is given as,

$$f(n) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=0} \ldots \tag{2}$$

Replacing $\frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=0}$ by F(k) we have,

$$f(n) = \sum_{k=0}^{\infty} x^k F(k) \ldots \tag{3}$$

which may be defined as the inverse differential transform from (1) and (2) it is easy to obtain the following mathematical operations:

(i) if $f(x) = g(x) \pm h(n), then\ F(k) = G(k) \pm H(k)$.

(ii) if $f(x) = g(x), then\ F(k) = cG(k)$, where c is a constant.

(iii) if $f(x) = \frac{d^n g(x)}{dx^n}, then\ F(k) = \frac{(k+n)!}{k!} G(k+n)$.

(iv) if $f(x) = g(x)h(x), then\ F(k) = \sum_{l=0}^{k} G(k)\, H(k-l)$

(v) if $f(x) = x^n, then\ F(k) = \delta(k-n)$, where δ is the Kronecker felta.

(vi) if $f(x) = \int_0^x g(t)dt, then\ F(k) = \frac{G(k-1)}{k}$, where $k \geq 1$.

(vii) if $f(x) = u(x)v(x)w(x).\ .\ .\ .\ .\ p(x), then\ F(k) = \sum_{r=0}^{k}\sum_{s=0}^{k-r} \ldots \sum_{m=0}^{k-r-s \ldots m} U(r)V(s)W(m) \ldots P(k-r-s-, \ldots m)$, where $F(k), G(k), H(k), U(k), V(k), W(k), P(k)$ are the differential transform of the functions $f(x), g(x), h(x), u(x), v(x), w(x), p(x)$ respectively.

3.0 Solution of the Riccati Equation with Variable Coefficients Using DTM.

The Riccati equation with variable coefficients is given as:

$$\frac{dy}{dx} = p(x) + q(x)y + r(x)y^2 \ldots \tag{4}$$

$$y(0) = g(x) \ldots \tag{5}$$

where $p(x), q(x), r(x)$ and $g(x)$ are functions of x. applying differential transform (DT) to equation (4) we have,

$DT\left[\frac{dy}{dx} = p(x) + q(x)y + r(x)y^2\right]$, i.e.

$$(k+1)T(k+1) = P(k) + \sum_{i=0}^{k} Q(L)T(k-l) + \sum_{s=0}^{k}\sum_{m=0}^{K-S} R(s)Y(m)T(k-s-m) \tag{6}$$

where $P(x)Q(x)R(x)\ and\ T(x)$ are the differential transforms of $p(x), q(x), r(x) and\ y(x)$ respectively. To illustrate the ability and reliability of the differential transform method for the Riccati differential equations with variable coefficients we solve a few examples.

Example 1. The above general equation (4) and (5) when $p(x) = x^5 + 1$,

$q(x) = -2x^4y(x) + x^3y^2(x) \ldots$ (7)

with initial condition as $y(0) = 0$. Applying Differential Transform (DT) to (7) we have

$$DT\left[\frac{dy}{dx} = 1 + x^5 - 2x^4y(x) + x^3y^2(x)\right]$$

$$DT\left[\frac{dy}{dx}\right] = DT[1] + DT[x^5] - 2DT[x^4y(x)] + DT[x^3y^2(x)] \ldots \tag{8}$$

$$(k+1)T(k+1) = \delta(k) + \delta(k-5) - 2\sum_{l=0}^{k} \delta(l-4)T(k-1) + \sum_{s=0}^{k}\sum_{m=0}^{k-s} \delta(s-3)$$

$$T(m)T(k-s-m) \ldots \tag{9}$$

The inverse differential transform of $y(x)$ is

$y(x) = \sum_{k=0}^{\infty} T(k)x^k$.......................... ... (10)
$T(0) = 0$.................................... (11)
for k= 0 in the above equation and using equation (10), we have
$T(1) = \delta(0) = 1$.......................... ..... (12)
for $k = 1$, we have
$T(2) = 0$.................... .... (13)
for $k = 2$, we have

$$3T(3) = -2\sum_{l=0}^{2}\delta(l-4)T(2-l) + \sum_{s=0}^{2}\sum_{m=0}^{2-s}\delta(5-3)T(m)T(2-s-m).$$

$T(3) = 0$.................... .... (14)
Similarly for $k = 3, 4, 5, 6, 7, 8$ *and* 9 we have
$T(4) = T(5) = T(6) = .\ .\ .\ .\ . = 0$...................... (15)
From equation (10) we have the exact solution of the equation (7) given as
$y(x) = x$.................................... (16)
Example 2. Let us now solve the above general (4) and (5) when
$p(x) = 3, q(x) = 3x^2, r(x) = -1$ *and* $g(x) = 1$. Therefore the Riccati Equation takes the form
$\frac{dy}{dx} = 3 + 3x^2y - xy^2$.............................. (17)
With initial condition as $p(0) = 1$. Applying Differential Transform (DT) to (17) we have
$DT\left[\frac{dy}{dx} = 3 + 3x^2y - xy^2\right] = DT\left[\frac{dy}{dx}\right] = DT[3] + 3DT[x^2y] - DT[xy^2]$……… (18)

$$(k+1)T(k+1) = 3\delta(k) + 3\sum_{l=0}^{k}\delta(l-2)T(k-l) - \sum_{s=0}^{k}\sum_{m=0}^{k-s}\delta(s-l)T$$

$(m)T(k-s-m)$..................................... (19)
The inverse differential transform of $y(x)$ is
$y(x) = \sum_{k=0}^{\infty} T(k)x^k$.................................. (20)
using the initial condition $y(0) = 1$ we have
$T(0) = 1$.................................... (21)
for $k = 0$ in the above equation and using equation (20), we have
$T(1) = 3$.................................... (22)
for $k = 1$, we have
$2T(2) = \frac{1}{2}$,.................................... (23)
for $k = 2$, we have

$$3T(3) = 3\sum_{l=0}^{2}\delta(l-2)T(2-l) - \sum_{s=0}^{2}\sum_{m=0}^{2-s}\delta(s-1)T(m)T(2-s-m).$$

$= T(3) = -1$.................................... (24)
Similarly for $k = 3, 4, 5, 6, 7, 8$ *and* 9 we have

$$T(4) = \frac{1}{4}, T(5) = \frac{7}{10}, T(6) = \frac{3}{8}, T(7) = \frac{53}{140}$$

$T(8) = \frac{9}{20}, T(9) = \frac{133}{2520}, T(10) = \frac{2099}{5600}$.................... (25)
From equation (20) we have

$$y(x) = 1 + 3x - \frac{1}{2}x^2 - x^3 + \frac{1}{4}x^4 + \frac{7}{10}x^5 + \frac{3}{8}x^6 - \frac{53}{140}x^7 - \frac{9}{20}x^8 + \frac{133}{2520}x^9 + \frac{2099}{5600}x^{10} + \ldots \quad (26)$$

4.0 Results and Conclusion

The straight forward applicability, computational effectiveness and the accuracy of the results obtained by DTM is evident from example 1, where the exact analytical solution is obtained by this method. It should be noted that the differential transform gives more satisfactory results for small terms which are evident.

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