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On The Solution of Riccati Equation Using Laplace Transform Decomposition Algorithm (LTDA)

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Abstract

In this paper, a laplace transform decomposition algorithm (LTDA) is proposed to solve Riccati equation. Comparisons are made among the Adomian Decomposition Method (ADM) and the proposed method. It is shown that the proposed method is equivalent to the adomian decomposition method for solving the Riccati equation.

Keywords: Laplace transforms decomposition algorithm, Adomian decomposition method, Riccati equation and Homotopy Perturbation Method.

1.0 Introduction

Sometimes using several numerical methods to solve a problem may give similar results. It is noticeable that applying different numerical methods to solve a problem may provide just the same results. For example, using the ADM and the successive approximation method for linear integral equations [1], the ADM and the power series method for differential equations [2] and the ADM and the Jacobi iterative method for system of linear equation [3], give will get just the same results.

Since the beginning of the 1980s, the ADM has been applied to a wide class of functional equations [4,5]. The ADM has been demonstrated to provide accurate and computable solutions for a wide class of linear or non-linear equations involving differential operators by representing non-linear terms in Adomian's polynomials [6,7]. This procedure requires neither linearization nor perturbation.

The LTDA is an approach based on the ADM, which is to be considered an effective method in solving many problems for it provides, in general, a rapidly convergent series solution. Since the use of the laplace transform replaces differentiation with simple algebraic operations on the transform, the algebraic equation is then solvable by decomposition. The LTDA approximates the exact solution with a high degree of accuracy using only few terms of the iterative scheme [8].

Many authors have applied this method to solve Bratu's equation [9], the Duffing equation [10], the Klein-Gordon equation [11, 12], some nonlinear coupled partial differential equation [13] and integro-differential equation [14]. It is shown that this algorithm solves Riccati equation too.

Until recently, the application of the Homotopy Perturbation Method in nonlinear problems has been developed by scientists and engineers because this method continuously deforms a simple problem that is easy to solve into the difficult problem under study. Most perturbation methods assume an existing small parameter but most nonlinear problems have no small parameter whatsoever. Many new methods have been proposed to eliminate the small parameter [15,16]. The homotopy theory becomes a powerful mathematical tool when it is successfully coupled with perturbation theory [17,18].

The Riccati equations are one of the most important classes of nonlinear differential equations and play a significant role in many fields of applied science [19]. The importance of the Riccati equation usually arises in optimal control problems. Several authors have proposed different methods to solve the Riccati equation. For example, Bulut and Evans [20] applied the decomposition method for solving the Riccati equation. El-Tawal et al [21] applied the Multistage Adomian's decomposition method for solving the Riccati differential equation and compared the result with the standard ADM. Abbasbandy [22] applied the HPM to solve the Riccati equation and compared the obtained results for this equation. Also, he [23] used the iterated homotopy perturbation method to solve this equation.

In this paper, we consider the Riccati equation

$$\frac{dy(t)}{dt} = 2y(t) - y^2(t) + 1 : y(0) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

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and apply the LTDA and ADM for solving it.
We shall show that both methods converge rapidly.

ADM for the Riccati equation. consider the Riccati equation (1). Denoting $\frac{d}{dt}$ by G, we have G^{-1} as an integral from 0 to t. operating with G^{-1} and using the initial condition, we obtain

$$y(t) = \int_0^t dt + 2\int_0^t y(t)dt - \int_0^t y^2(t)dt \ldots\ldots \quad (2)$$

To apply the ADM to (2), let

$y(t) = \sum_{n=0}^{\infty} y_n(t)$ and $y^2(t) = \sum_{n=0}^{\infty} A_n(t)$, where the $A_n(t)'s$ are the Adomian polynomials depending on $y_0(t), y_1(t), y_2(t) \ldots\ldots y_n(t)$. Upon substitution, (2) can be written as

$$\sum_{n=0}^{\infty} y_i(t) = t + 2\int_0^t \sum_{i=0}^{\infty} y_i(t)\,dt - \int_0^t \sum_{i=0}^{\infty} A_i(t)\,dt \ldots\ldots \quad (3)$$

Based on the recursion scheme of the ADM, we define

$$y_0(t) = t \ldots\ldots \quad (4)$$

$$y_{n+1} = 2\int_0^t y_n(t)dt - \int_0^t A_n(t)dt \text{ for } n \geq 0 \ldots\ldots \quad (5)$$

Note that the Adomian polynomials $A_n(t)'s$ for the quadratic nonlinearity are written as [24,25]

$$A_0(t) = y_0^2(t)$$
$$A_1(t) = 2y_0(t)y_1(t)$$
$$A_2(t) = y_1^2(t) + 2y_0(t)y_2(t)$$
$$A_3(t) = 2y_1(t)y_2(t) + 2y_0(t)\,y_3(t), \ldots\ldots \quad (6)$$

The solution components $y_n(t)$ from (5) can be calculated as $y_1(t) = 2\int_0^t y_0(t)dt - \int_0^t A_0(t)dt = 2\int_0^t tdt - \int_0^t y_0^2(t)dt$

$$y_1(t) = 2\frac{t^2}{2} - \int_0^t t^2\,dt = t^2 - \frac{t^3}{3} \quad (7)$$

$$y_2(t) = 2\int_0^t y_1(t)\,dt - \int_0^t A_1(t)dt$$
$$= 2\int_0^t \left(t^2 - \frac{t^3}{3}\right)dt - \int_0^t 2y_0(t)y_1(t)dt$$
$$= 2\frac{t^3}{3} - 2\frac{t^4}{12} - \int_0^t 2.t\left(t^2 - \frac{t^3}{3}\right)dt$$
$$= 2\frac{t^3}{3} - \frac{t^4}{6} - \int_0^t 2\,t^3 dt + \int_0^t \frac{2t^4}{3}\,dt$$
$$= 2\frac{t^3}{3} - \frac{t^4}{6} - \frac{2t^4}{4} + \frac{2t^5}{15} = \frac{2t^3}{3} - \frac{4t^4 - 12t^4}{24} + \frac{2t^5}{15}$$
$$y_2(t) = \frac{2t^3}{3} - \frac{16t^4}{24} + \frac{2t^5}{15}$$

$$y_2(t) = \frac{2+3}{3} - \frac{2t^4}{3} + \frac{2t^5}{15} \ldots\ldots \quad (8)$$

$$y_3(t) = 2\int_0^t y_2(t)dt - \int_0^t A_2(t)dt$$
$$y_3(t) = 2\int_0^t \left(\frac{2t^3}{3} - \frac{2t^4}{3} + \frac{2t^5}{15}\right)\;dt - \int_0^t (y_1^2(t) + 2y_0(t)y_2(t))dt$$
$$y_3(t) = 2\int_0^t \left(\frac{2t^3}{3} - \frac{2t^4}{3} + \frac{2t^5}{15}\right)\;dt - \int_0^t [(t^2 - \frac{t^3}{3})^2 + 2t\,(\frac{2t^3}{3} - \frac{2t^4}{3} + \frac{2t^5}{15})]$$

$$y_3(t) = \frac{t^4}{3} + \frac{4t^5}{15} + \frac{4t^6}{90} - \frac{t^5}{5} + \frac{4t^5}{15} + \frac{4t^6}{18} + \frac{t^7}{63} + \frac{4t^7}{105}$$

$$y_3(t) = \frac{t^4}{3} - \frac{t^5}{5} - \frac{288t^6}{1620} + \frac{357t^5}{6615}$$

Similarly, the solution components $y_3(t)$ are calculated for $n = 4,5, \ldots$ but are not listed for brevity. It is clear that better approximations can be obtained by evaluating more components of the decomposition series solution $y(t)$. We note here that the convergence question of this technique has been formally proved and justified in [26,27,28].

LTDA for the Riccati Equation; here the LTDA is applied to find the solution of (1). The method consists of applying the laplace transformation (denoted throughout in this paper by L) to both sides of (1), where

$$L[y^1(t)] = L[1] + 2L[y(t)] - L[y^2(t)] \ldots\ldots \quad (9)$$

Applying the formulas of the respective laplace transforms, we obtain

$$SL\,[y(t)] - y(0) = L[1] + 2L[y(t)] - L[y^2(t)] \ldots\ldots \quad (10)$$

Using the initial condition $y(0) = 0$, we have

$L\,[y(t)] = \frac{1}{5}\,L\,[1] + \frac{2}{5}\,L\;[y(t)] - \frac{1}{5}\,L\;[y^2(t)]$ . . . . (11)

The laplace transform decomposition technique consists of representing $y(t)$ by $\sum_{n=0}^{\infty} y_n(t)$ and $y^2(t)$ by $\sum_{n=0}^{\infty} A_n(t)$, where the $A_n(t)'s$ are the same as in equation (6) above.

Therefore, equation (11) becomes

$L[\sum_{n=0}^{\infty} y_n(t)] = \frac{1}{5}\,L\,[1] + \frac{2}{5}\,L\;[\sum_{n=0}^{\infty} y_n(t)] - \frac{1}{5}\,L\,[\sum_{n=0}^{\infty} A_n(t)]$ . . . . . (12)

Matching both sides of (12), the following iterative algorithm is obtained.

$L[y_0(t)] = \frac{1}{5}\,L\,[1] = \frac{1}{5^2}$ . . . . (13)

and

$L[y_{n+1}(t)] = \frac{2}{5}\,L\,[y_n(t)] - \frac{1}{5}\,L\,[A_n(t)]\; for\; n \geq 0$ . . . . (14)

Applying the inverse laplace transform to equation (13), we obtained

$y_0(t) = 0$. We now find the first few iterative of the recursive scheme. For n = 0, equation (14), becomes

$L[y_1(t)] = \frac{2}{5}\,L\,[y_0(t)] - \frac{1}{5}\,L\,[A_0(t)]$ . . . . (15)

Substituting $y_0(t)$ and $A_0(t)$ into 15 and applying the inverse laplace transform, we have

$y_1(t) = t^2 - {t^3}/{3}$ . . . . (16)

for n = 1, equation (14) converts to

$L[y_2(t)] = \frac{2}{5}\,L\,[y_1(t)] - \frac{1}{5}\,L\,[A_1(t)]$ .. . . (17)

Putting (16) into (17) and applying the inverse laplace transform yields

$y_2(t) = \frac{2t^3}{3} - \frac{2t^4}{3} + \frac{2t^5}{15}$ . . . . (18)

The other terms $y_3(t),\, y_4(t)$ are obtained recursively in a similar manner by using equation (14) and applying the inverse laplace transform. Notice that components of $y(t)$ obtained from LTDA method are just the same as those of the ADM. Therefore, these two methods are equivalent for solving the Riccati equation and so the convergence of the LTDA method is the same as the ADM, which was mentioned in previous sub-section.

2.0 Conclusion

In this work, we applied the powerful and efficient ADM and LTDA for solving the Riccati equation. This study showed that solution components $y_0(t),\, y_1(t),\, y_2(t)$ and $y_3(t)$ obtained by LTDA are just the same terms obtained by the ADM. Equality for other higher terms are likely to takes place too.

3.0 References

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