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Application of Euler Method (EM) for the Solution of Some First Order Differential Equations With Initial Value Problems (IVP's)

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Abstract

The attempt to solve problems in science and technology, gradually led to mathematical models. Mathematical models involve equations in which functions and derivatives play important roles. However the theoretical development of this branch of mathematics – Ordinary Differential Equations (ODE), has its origin rooted in a small number of mathematical problems. Therefore,Differential equations (DE) can be solved using many methods that are generally accepted in Mathematics. However, it is believed that one method should be more accurate, efficient, sufficient and unique than the other. Thus; solutions of First order Differential Equations (FOD's) with Initial Value Problems (IVP's) by Euler Method (EM) will be of central concern. However numerical computational algorithm, convergence rate, approximation errors and uniqueness will be seriously inspected and to asertain Euler Method modification requirement in order to be more stable and reliable over other methods for the FODE's with IVP's.

Keywords: Error estimate, Initial Value problem (IVP),(FODE),Euler Method (EM), Exact Solution (ES). Convergence rate, Analytical Solution, First Order Differential Equation Numerical Solution.

1.0 Introduction

According to some historians of Mathematics, the study differential *equations* began in 1675, when Gottfried Wilhelm von Leibnitz (1646-1716) wrote the equations

$$\int (x^2 + 2dx) = \frac{1}{3}x^3 + 2x \tag{1}$$

The search for general methods of integrating differential equations began when Isaac Newton (1642-1727) classified first order differential equations into three classes. These are

$$\frac{dy}{dx} = f(x) \tag{2}$$

$$\frac{dy}{dx} = f(x, y) \tag{3}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u \tag{4}$$

Equation (1) to (2) contain only ordinary derivatives of one or more dependent variables, with respect to a single independent variable and is known today as ordinary equations. Equation (3) involves the partial derivatives of one dependent variable and today is called partial differential equations. Newton will express the right side of the equation in powers of the dependent variable and assumed as a solution in an infinite series. The coefficient of the infinite series were then determined. Even though Newton noted that the constant coefficient could be chosen in an arbitrary manner and concluded that the equation possessed an infinite number of particular solution, it wasn't until the middle of the 18^{th} century that the full significance of this fact, i.e., the general solution of a first order equation depends upon an arbitrary constant, was realized.

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Only in special cases can a particular differential equation be integrable in a finite form, i.e., be finitely expressed in terms of known functions. In the general case one must depend upon solutions expressed in an infinite series in which the coefficients are determined by recurrence formula.. The study of differential equations continues to contribute to the solution of such practical problems in control theory, in orbital machine and in many other branches of science and technology and also to ask challenging questions in abstract subjects to pure mathematics working. In such apparently abstract subjects such as functional analysis and the theory of differential manifolds.

Parker and Sochacki theorem on Existence and Uniqueness states that if both $f(x, y)$ and $\frac{\partial y}{\partial x}$ are continuous in some region around the point (x_0, y_0) then there is a unique solution to the IVP [1]

$$\begin{cases} y' & = f(x, y) \\ y(x_0) & = y_0 (IVP) \end{cases} \quad (5)$$

Valid in some interval aroundx_0. In other words, if the slope field is sufficiently smooth at each point, then there is unique integral curve passing through any given point. How do we prove such a theorem? This method uses a sequence of approximate solutions and prove that these approximations converge at least in a small interval aroundx_0. Euler Method is quite simple to use in practice: one simply ‘’connect the dots’’ in the slope field. The disadvantage to this method is that it only gives an approximation ‘’at the dots’’. In other words, Euler Method only approximates the values of the solution at a finite list of points. It does not give us formula for an approximate function at every point. However, Euler Method has the advantage that its accuracy can be improved with only minor modifications. Eulerian method are far more efficient computationally than other methods suchasPicard method but it introduces an important technique that will be useful for the error analysis of Eulerian methods. An approximation method is useless without an estimate of the error.Parker and Sochacki (2000) showed that a large class of ODE’s could be converted to polynomial form using substitutions and using a system of equation. While this class of ODE’s is dense in the analytic functions, it does not include all analytic functions. They also showed one can approximate the solution by a polynomial system and the resulting error bound when using these approximations [2]. Parker and Sochacki also showed that if $x_0 \neq 0$, one computes the iteration as if $x_0 = 0$ and then the approximated solution to the ODE is $y^n(x + x_0)$. This algorithm is called the modified Picard method (MPM). While the MPM algorithm easily computes the approximations, since it only depends on calculating derivatives and integrals of the underlying polynomials, it has some limitations. They also showed how to handle the PDE including the initial conditions. However, the method requires the initial conditions in polynomial form. While in some PDE’s this is the case, many time one computes a Taylor polynomial that approximates the initial condition to high degree. This results in a substantial increase in computational time. For some problems, the initial condition is not explicitly known, but only a digitized form of the data. For example, in image processing, most of the data have already been digitized and we have to interpolate the data using polynomials in other to apply the Modified Euler Method (MEM). If this is done, the resulting polynomial may not effectively approximate the derivatives of the original function. The polynomial approximation might contain large number of oscillations that do not represent the underlying data accurately. Finally, we would also like to handle boundary conditions in a simple manner, but keep the extensibility of the Modified Euler Method (MEM), which does not allow for a boundary condition. Picard’s method, sometimes called the method of successive approximations, gives a means of proving the existence of solutions to DE. Emile Picard, a French Mathematician, developed the method in the early 20^{th} century. It has proven to be so powerful that it has replaced the Cauchy- Lipchitz method that was previously employed for such endeavours.

Picard developed his method while he was a Professor at the University of Paris. It arose out of a study involving the Picard-Lindel of existence theorem that had been formulated at the end of the 19^{th} century. Picard’s method is utilized in similar situations as those that employ the Taylor series method. It is a method that converts the differential Equation into an equation involving integrals. Some DE’s are difficult to solve, but Picard’s method provides a numerical process by which solution can be approximated. The method consists of constructing a sequence of functions that will approach the desired solution upon successive iteration. It is similar to the Taylor series method in that successive iterations also approach the desired solution to a DE. Picard’s method allows us to find a series solution about some fixed point. The number of terms or iterations that is required to reach the desired solution depends on how far from the chosen point the solution must apply. The closer the chosen point to the known point, the fewer terms that are needed. It can be shown that the series is convergent and provides a solution to the differential equation of interest although the number of terms will depend upon how rapidly the series converges as well[3]. The details of Picard’s method involve starting with an initial value problem and expressing it as an integral equation. This is done by integrating both sides with respect to one variable from a defined starting point to a defined termination point, $x_0 to x_1$. The initial value given is substituted into the resulting integral equation. This yields the simple fraction evaluated at the initial value summed with the remaining integral, after a simple substitution and appropriate arrangements of the limits on the remaining integral, the result can be used to generate successive approximations of a solution to the initial equation. The number of iteration steps is determined by two factors; how quickly the series converges and how far away from the point of interest is the value given in the initial problem [4]. The term ‘’Picard iteration’’ occurs in two places in undergraduate mathematics. In numerical analysis it is used when discussing fixed point iteration for finding a numerical approximation to the equation$x = g(x)$. In differential equations, Picard iteration is a constructive procedure for establishing the existence of a solution to a DE $y^{'} = f(x, y)$ that passes through the point(x_0, y_0)[5]. Picard iteration is a widely used procedure for solving the nonlinear equation governing flow in variably saturated porous media. The method is simple to code and computationally cheap, but has been known to fail or converge slowly [6]. Picard showed that an entire function can omit not more than one finite value without being reduced to a constant function and if there exist at least two values, each of which is taken on only a finite number of times, the function is a polynomial [7]. Otherwise the function takes on every value, other than the exceptional one, an infinite number of times. His beautiful proof of what is known as Picard's Big [8]. Picard iteration is a special kind of fixed point iteration. We call x a fixed point of a function if $\mathrm{x} = \mathrm{f}(x)$.Suppose a sequence is defined by:$x_{n+1} = f(x_n), x_1 = \ [someguessatthefixedpoint]$. Often you will find that x_n converges to a fixed point of f. The process of taking the successive terms of such a sequence is called *iteration*. We are going to apply this iterative idea to differential equations and we come up with the Picard method. Basically, we are going to apply fixed point iteration to a whole differential equation [9]. The goal here is to use Picard method to find a solution to the given FODE with IVP of the form in (1) ODE frequently occurs as mathematical models in many branches of science, engineering and economy. Unfortunately it is seldom that these equations have solutions that can be expressed in closed form, so it is common to seek approximate solutions by means of numerical methods [10]; nowadays this can usually be achieved very inexpensively to high accuracy and with a reliable bound on the error between the analytical solution and its numerical approximation. In this section we shall be concerned with the construction and the analysis of numerical methods for FODE of the form in (1). For the real – valued function y of the real variable$x$, where $y^{'} = \frac{dy}{dx}$ In other to select a particular integral from the infinite family of solution curves that constitute the general solution to (1), the FODE will be considered in tandem with an initial condition: given two real number We seek a solution to (1) for$x > x_0 \ni y(x_0) = y_0$. The FODE (1) together with the IVP is called FODE with IVP. In general, even if $f(x, y)$is a continuous function, there is no guarantee that the IVP in (1) possesses a unique solution. Fortunately, under a further mild condition on the function$f$, the existence and uniqueness of a solution to (1) can be ensure: the result is encapsulated in the next theorem[11].

2.0 Material and Method

2.1 Euler Method (EM)

This is the most simple but crude method to solve differential equation of the form in (5). Considering the FODE with the IVP in (5), then the solution to (5) is equivalently given as finding solution to the integral equation:

$$\begin{cases} \mathrm{y}^{'} = \mathrm{f}(\mathrm{x}, \mathrm{y}) \\ \mathrm{y}(\mathrm{x}_0) = \mathrm{y}_0(\mathrm{IVP}) \end{cases} \quad (6a)$$

$$y_{n+1} = y_n + hf(x_n, y_n), n = 0,1,2, \dots (6b)$$

Proof

To show that equation (6b) is the equivalent solution to any first order DE of the form in equation (5) by Euler Method (EM) also suffices that:

Let $y^{'} = f(x, y)$ be the FODE with IVP $\mathrm{y}(x_0) = y_0 iefrom (6a)$

$$\Rightarrow \frac{dy}{dx} = f(x, y)$$

$$dy = f(x, y)dx$$

Let $x_1 = x_0 + h$, where h is small. Then by Taylor's series

$y_1 = y(x_0 + h) = y_0 + h\left(\frac{dy}{dx}\right)_{x_0} + \frac{h^2}{2}\left(\frac{dy}{dx}\right)_{c_1}$, where c_1 lies between $x_0 and x$

$$y_0 + hf(x_0, y_0) + \frac{h^2}{2}y^{''}(c_1)$$

If the step size h is chosen small enough, then the second-Order term may be neglected and hence y_1 is given by:

$$\Rightarrow y_1 = y_0 + hf(x_0, y_0)$$

$$\Rightarrow y_2 = y_1 + hf(x_1, y_1)$$

$$\Rightarrow y_3 = y_2 + hf(x_2, y_2)$$

And so on

In general,

$$y_{n+1} = y_n + hf(x_n, y_n), n = 0,1,2, \dots (6c)$$

where $\mathrm{x_k} = \mathrm{x_0} + \mathrm{kh}$ (6d)

Thus: equation (6a) gives the$(\mathrm{n} + 1)$th iteration , hence the Proof of Euler Method (EM).

This method is very slow. To get a reasonable accuracy with Euler's method, the value of h should be taken as small.It may be noted that the Euler's method is a single-step explicit method. According to Atkinson *et al*. [2] '' Euler method is a first-order numerical procedure proposed by Leonhard Euler for solving ODE's with IVP's''. It is the most basic explicit method for numerical integration of ODE's and is considered as the simplified Runge-Kutta method.

This is one of the oldest and simplest methods of solving IVP's numerically. This method is used for solving ODE's thereby roughly estimating the coordinates of the next point in the solution and with this knowledge, the original estimate is re-predicted or corrected which leads to the Modified Euler Method also known as the Heun's Method. The Euler method is a first order Method, which means that the local error per step is proportional to the square of the step size and the global error (error at a given time) is proportional to the step size. The Euler method is often not accurate enough and is only more accurate if the step size$h$ is smaller. It also suffers from stability problems. For these reasons, the Euler method is not often used in practice. It serves as the basis to construct more complicated methods. Although Euler method integrates a first order ODE, any ODE of order n can be represented as a first order ODE. Thus, to treat the equation,

$$y^n(x) = f\left(x, y'(x)\right), \dots, y^{n-1}(x) \qquad (6e)$$

We introduce auxiliary, variables

$$g_1(x) = y(x), g_2(x) = y'(x), \dots g_n(x) = y^{n-1}(x) \qquad (6h)$$

This is a first order system in the variable and can be handled by Euler's method or in fact by any other scheme for first order systems.

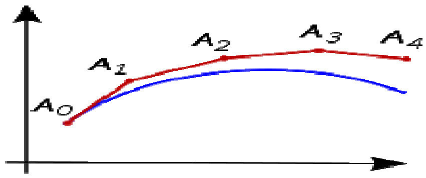

Figure 1 Illustration of the Euler method. The unknown curve is in blue, and its polygonal approximation is in red.

The idea is that while the curve is initially unknown, its starting point, which we denote by A_0, is known (see the picture on top right). Then, from the differential equation, the slope to the curve at A_0 can be computed, and so, the tangent line.

Taking a small step along that tangent line up to a point A_1. along this small step, the slope does not change too much so A_1 will be close to the curve. If by pretending that A_1 is still on the curve, the same reasoning as for the point A_0 above can be used. After several steps, a polygonal curve $A_0 A_1 A_2 A_3 \dots$ is computed. In general, this curve does not diverge too far from the original unknown curve, and the error between the two curves can be made small if the step size is small enough and the interval of computation is finite.

Euler's Method is used to roughly estimate the coordinates of the next point in the solution, and with this knowledge, the original estimate is re-predicted or corrected.

3.0 Error Analysis

$$E^{global}_{method(i)} = |y(x_i) - Y(x_i)|, i = 1,2, \dots, \qquad (7)$$

$$re-writtenas: E^{FGE}_{method(i)} = |y(x_i) - Y(x_i)|, i = 1,2, \dots, \qquad (8)$$

$$andthelocalerroras: E^{local}_{method(i+1)} = |y(x_{i+1}) - y(x_i)|, i = 1,2, \dots, \qquad (9)$$

$where : y(x_i) = SolutionbyDiscreteVariableMethod(DVM)and$

$Y(x_i) = ExactSolution(SolutionbyAnalyticalMethod\ (AM)\ )$

Problem 1

Find the values of y(0.1) and y(0.2) from the following differential equation

$$\frac{dy}{dx} = x^2 + y$$

with initial condition

$y(0) = 0. Alsofindthevaluesofy(0.1)andy(0.2)$

Solution 1

let $h = 0.05, x_0 = 0;\ y_0 = 0$ by the ivp then;

by $x_k = x_{k-1} + h$ where $k = 1,2,3, \dots,$

when $k = 1$

$\Rightarrow x_k = x_1 = x_{1-1} + h = x_0 + h$

$\Rightarrow x_1 = x_0 + h$ where $x_0 = 0$ and $h = 0.05$

ie $x_1 = 0 + 0.05 = 0.05$

$\therefore$ by equation (6a)when $n = 0$

$y_{n+1} = y_n + hf(x_n, y_n), n = 0,1,2, \dots$

ie $y_{0+1} = y_0 + hf(x_0, y_0), n = 0$

ie $y_1 = y_0 + hf(x_0, y_0), n = 0$

ie $y_1 = y(0.05) = y_0 + hf(x_0, y_0), n = 0$

ie $y_1 = y(0.05) = y_0 + h(x_0^2 + y_0), n = 0$

$y_1 = 0$ (10)

hence

Again by (6b)$x_k = x_{k-1} + h$ where $k = 1,2,3,\ldots,$

when $k = 2$

$\Rightarrow x_k = x_2 = x_{2-1} + h = x_1 + h$

$\Rightarrow x_2 = x_1 + h$ where $x_1 = 0.05$ and $h = 0.05$

ie $x_2 = 0.05 + 0.05 = 0.1$

$\therefore$ by equation (6a)when $n = 1$

$y_{n+1} = y_n + hf(x_k, y_n), n = 0,1,2,\ldots$ and $k = 1,2,3,\ldots.$

ie $y_{1+1} = y_1 + hf(x_1, y_1), n = 1, k = 2$

ie $y_2 = y_1 + hf(x_2, y_1)$

ie $y_2 = y(0.1) = y_1 + hf(x_2, y_1)$

ie $y(0.1) = y_1 + h(x_1^2 + y_1)$, where $x_2 = 0.1, y_1 = 0$ and $h = 0.05$

hence; $y_2 = 0.0005$ (11)

similarly; by (6b) $x_k = x_{k-1} + h$ where $k = 1,2,3,\ldots,$

when $k = 3$

$\Rightarrow x_k = x_3 = x_{3-1} + h = x_2 + h$

$\Rightarrow x_3 = x_2 + h$ where $x_2 = 0.1$ and $h = 0.05$

ie $x_3 = 0.1 + 0.05 = 0.15$

$\therefore$ by equation (6a)when $n = 2$

$y_{n+1} = y_n + hf(x_k, y_n), n = 0,1,2,\ldots, k = 1,2,3,\ldots.,$

ie $y_{2+1} = y_2 + hf(x_3, y_2), n = 2, k = 3$

ie $y_3 = y_2 + hf(x_3, y_2)$

ie $y_3 = y(0.15) = y_2 + hf(x_3, y_2)$

ie $y(0.15) = y_2 + h(x_3^2 + y_2)$, where $x_3 = 0.15, y_2 = 0.0005$ and $h = 0.05$

hence; $y_3 = 0.0017$ (12)

similarly; by (6b) $x_k = x_{k-1} + h$ where $k = 1,2,3,\ldots,$

when $k = 4$

$\Rightarrow x_k = x_4 = x_{4-1} + h = x_3 + h$

$\Rightarrow x_4 = x_3 + h$ where $x_3 = 0.2$ and $h = 0.05$

ie $x_4 = 0.15 + 0.05 = 0.2$

$\therefore$ by equation (6a)when $n = 3$

$y_{n+1} = y_n + hf(x_k, y_n), n = 0,1,2,\ldots, k = 1,2,3,\ldots.,$

ie $y_{3+1} = y_3 + hf(x_4, y_3), n = 3, k = 4$

ie $y_4 = y_3 + hf(x_4, y_3)$

ie $y_4 = y(0.2) = y_3 + hf(x_4, y_3)$

ie $y(0.15) = y_2 + h(x_4^2 + y_3)$, where $x_4 = 0.2, y_3 = 0.00165$ and $h = 0.05$

hence; $y_4 = 0.0037$ (13)

Table 1 : Result generated From Euler Method (EM) and Exact Solution (ES) for the step size $h = 0.05 \leq x_n \leq 0.2$

n	x_n	y_n	ExactSolution (ES)	Associated Error (AE)
1	0.05	0.0000	0.0000	0.0000
2	0.1	0.0005	0.0003	0.0002
3	0.15	0.0017	0.0012	0.0005
4	0.2	0.0037	0.0028	0.0009

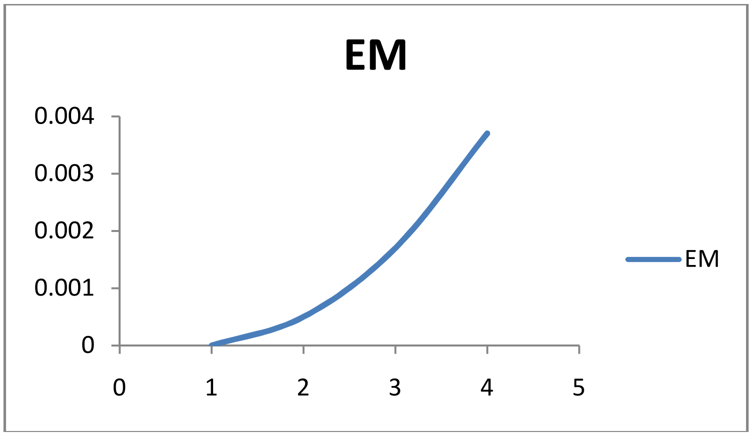

Figure 2: Graphical illustration of Solution by Euler Method(EM)

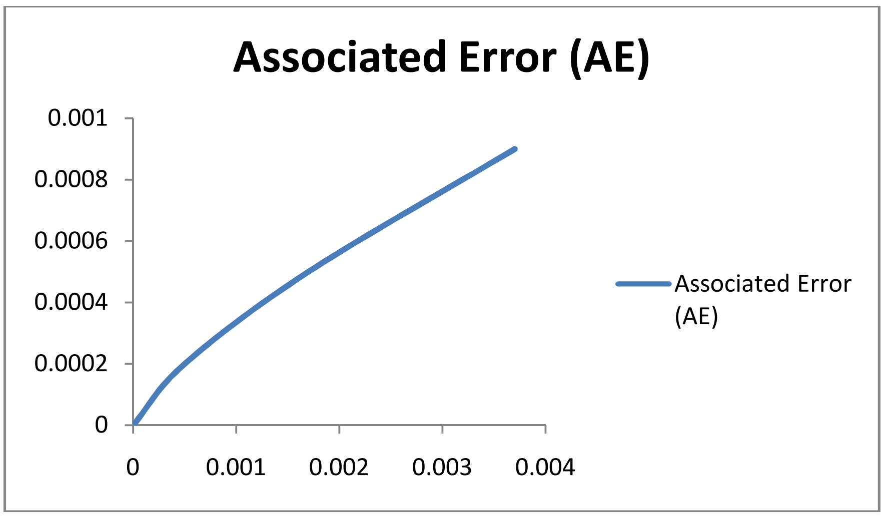

Figure 3: Graphical illustration of the Associated Error (AE)

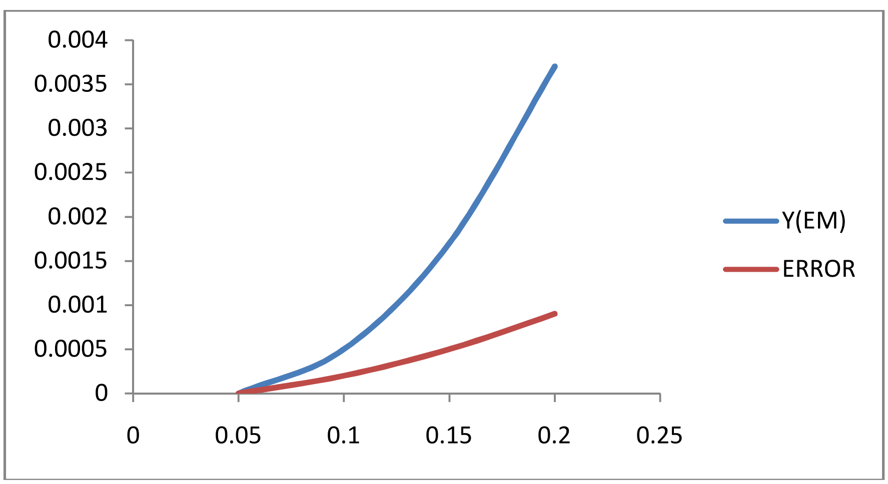

Figure 4: Graphical illustration of Solution byEuler Method (EM) relative to the Associated Error (AE)

4.0 Analytical Solution of The Problem

The equation considered in this scope can also be solved through the analytical method using the method of integrating factor as follows:
By the equation described in problems1, 2 and 3:

Problem 2

Find the values of $y(0.1)$ and $y(0.2)$ from the given differential equation below:

$$\frac{dy}{dx} = x^2 + y$$

with initial condition

$y(0) = 0.$ Also find the values of $y(0.1)$ and $y(0.2)$

Solution 2

Given that $\frac{dy}{dx} = x^2 + y2$

Using the method of integrating factor the solution to the given problem 2 is given below:

ie: $\frac{dy}{dx} = x^2 + y \equiv y' - y = x^2$ (14)

$\Rightarrow \frac{dy}{dx} - y = x^2 \equiv y' - y = x^2$, where $y' = \frac{dy}{dx}$, $p(x) = (-1)$

$q(x) = x^2$ and integrating factor $(I.F) = e^{\int p(x)dx}$

by the $\mathbf{I.F} = e^{\int p(x)dx}$

$\therefore$ $\mathbf{I.F} = e^{-x}$ (15)

now; multiplying equation (14) by equation (15)

ie: (14) becomes; $y'e^{-x} - ye^{-x} = x^2e^{-x}$

$\Rightarrow y'e^{-x} - ye^{-x} = \frac{d(ye^{-x})}{dx} = x^2e^{-x}$

ie: $\frac{d(ye^{-x})}{dx}dx = x^2e^{-x}dx$

ie: $d(ye^{-x}) = (x^2e^{-x})dx$

integratung both sides:

$$\int d(ye^{-x}) = \int (x^2e^{-x})dx$$

$$\Rightarrow ye^{-x} = \int (x^2e^{-x})dx \quad (16)$$

applying method of integration by part to the R. H. S

$$\int udv = uv - \int vdu, \quad (17)$$

where u = function to be differentiated and

v = function to be integrated

$$\text{ie: } \int (x^2e^{-x})dx = RHS \quad (18)$$

where $dv = e^{-x}dx$ and $v = \int dv = v$

ie: $v = \int e^{-x}dx = [e^{-x}] \div \frac{d(-x)}{dx} = \frac{e^{-x}}{-1} = -e^{-x}$

hence; $v = -e^{-x}$

again by $u = x^2 \Rightarrow \frac{du}{dx} = \frac{d(x^2)}{dx} \Rightarrow du = (x^2)dx$

$\Rightarrow du = 2(x^{2-1}) = 2xdx$

$\therefore$ $v = -e^{-x}, u = x^2, dv = e^{-x}$ and $du = 2xdx$ (19)

substituting equation (19) into (17)to give the point process integral solution of (16)

ie: $\int udv = uv - \int vdu$

$$\Rightarrow \int (x^2e^{-x})dx = -x^2e^{-x} - \int (2x)(-e^{-x})dx$$

$$= -x^2e^{-x} + \int 2xe^{-x}dx \quad (20)$$

$$\left.\begin{aligned} &\text{again; } u = 2x, dv = e^{-x}dx \\ &\text{ie: } \frac{du}{dx} = \frac{d(2x)}{dx} \Rightarrow du = (2x)dx \\ &\Rightarrow du = 1.(2x^{1-1})dx. \\ &= 2dx \end{aligned}\right\} \quad (21)$$

$$\therefore\ v = -e^{-x}, u = x^2, dv = e^{-x}\text{and } du = 2xdx \quad (22)$$

using equation (17)

ie: $\int udv = uv - \int vdu,$

$\Rightarrow$ equation (20) becomes:

$$= -x^2e^{-x} + \left(uv - \int vdu\right) \quad (23)$$

$$\text{where } v = -e^{-x}, u = 2x, dv = e^{-x}\text{and } du = 2dx \quad (24)$$

$\therefore$ by substituting equation (24) into equation (23)we obtain equation (25):

$$\Rightarrow \int udv = \int 2xe^{-x}dx = -2xe^{-x} - \int (2dx)(-e^{-x})$$

$$\Rightarrow \int (x^2e^{-x})dx = -x^2e^{-x} + \left(-2xe^{-x} - \int (2dx)(-e^{-x})\right)$$

$$\Rightarrow \int (x^2e^{-x})dx = -x^2e^{-x} - xe^{-x} - (-)2\int (e^{-x})\,dx$$

$$= -(x^2e^{-x} + 2xe^{-x} + 2e^{-x}) + C$$

$$\text{thus; } \int (x^2e^{-x})dx = -(x^2e^{-x} + 2xe^{-x} + 2e^{-x}) + C$$

$\Rightarrow$ by equation (40): $ye^{-x} = \int (x^2e^{-x})dx = -(x^2e^{-x} + 2xe^{-x} + 2e^{-x}) + C$ using integration by part.

ie: $ye^{-x} = -(x^2e^{-x} + 2xe^{-x} + 2e^{-x}) + C$

ie: $\frac{ye^{-x}}{e^{-x}} = \frac{-(x^2e^{-x} + 2xe^{-x} + 2e^{-x}) + C}{e^{-x}}$, ie dividing both side by (e^{-x})

$$\Rightarrow y = \frac{-(x^2e^{-x} + 2xe^{-x} + 2e^{-x}) + C}{e^{-x}} = -\left(\frac{x^2e^{-x}}{e^{-x}} + \frac{2xe^{-x}}{e^{-x}} + \frac{2e^{-x}}{e^{-x}}\right) + \frac{C}{e^{-x}}$$

$$\therefore y(x) = Ce^{x} - (x^2 + 2x + 2) \quad (25)$$

Equation (25) gives the equivalence analytical solution ofproblem2

But by the given IVP; ie $y(0) = 0, \Rightarrow y = 0$ when $x = 0$

now substituting the IVP into equation (25)

to obtain the value of the constant term of integration (C)

$\therefore y(0) = -((0)^2 + 2(0) + 2) + Ce^0 = -2 + C \times 1 = 0$

ie: $C - 2 = 0, \Rightarrow C + 2 - 2 = 0 + 2, \Rightarrow C = 2$

$$\text{hence } C = 2 \quad (26)$$

$\therefore$ equation (25) becomes :

$y(x) = -(x^2 + 2x + 2) + 2e^x$

thus:

$$y(x) = 2e^x - (x^2 + 2x + 2) \quad (27)$$

REMARK

Equation (27) gives the general non-numerical solution of problem 2 for any given value of x.

5.0 Numerical Computation of Exact Solution Of Problem (1 and 2)

Below are the analytical computation of the equivelence unknown solution given by equation

ie: $y(x) = 2e^x - (x^2 + 2x + 2)$(28)

so when $x = 0.05$

ie: $y(0.05) = 2e^{(0.05)} - ((0.05)^2 + 2(0.05) + 2)$

$$hence\ \ y(0.05) \cong 0 \quad (29)$$

when $x = 0.1$

$ie\ by: y(x) = 2e^x - (x^2 + 2x + 2)$
$\Rightarrow\ y(0.05) = 2e^{(0.1)} - ((0.1)^2 + 2(0.1) + 2)$
$hence\ \ y(0.1)\ \cong 0.0003$ (30)
$when\ x = 0.15$
$ie\ by: y(x) = 2e^x - (x^2 + 2x + 2)$
$\Rightarrow\ y(0.15) = 2e^{(0.15)} - ((0.15)^2 + 2(0.15) + 2)$
$hence\ \ y(0.15)\ \cong 0.0012$ (31)
$when\ x = 0.2$
$ie\ by: y(x) = 2e^x - (x^2 + 2x + 2)$
$\Rightarrow\ y(0.15) = 2e^{(0.2)} - ((0.2)^2 + 2(0.2) + 2)$
$\text{hence}\ \ y(0.2)\ \cong 0.0028$ (32)

Table 2: Result generated From Exact Solution (ES) for the step size h =0.05 and 0.05 x_n

n	x_n	Exact Solution (ES)
1	0.05	0.0000
2	0.1	0.0003
3	0.15	0.0012
4	0.2	0.0028

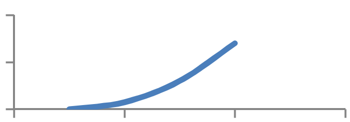

Figure 5: Graphical illustration of the Exact Solution (ES).

6.0 Results and Discussion

In Equations (6a-6d)and (27) show the derived general form of the Methods (Euler and Analytical Method (AM)) respectively. Similarly, Equations(9a)-(9c)gives Expression for the Local, Global and Final global Errors respectively. Also; equations (10) – (13)the approximate numerical solution to four decimal place of problem 1using the proved equations in (8a)- (8d)and (27) for EM and AM to obtain numerical results in equations (29)-(32) for AM through which ES is analytically computed and equations (29) to (32)gives numerically computed inexact or approximate solution by EM iteration scheme for the solution of problem 1 and 2. In addition, graphical illustrations for the general solutions and associated error were shown and displayed in Figures (1) to(2) for Euler and Analytical Method (AM) given by ES respectively. Tables 1 – 4 shows the numerical results together with their associated errors where necessary of the solutions obtained from solutions for the problems 1 and 2., using Euleras well asAM respectively. Table 1 show the numerical solution obtained from EM for the successive iterations. Similarly, numerical solution from AM and the AE were also displayed. Table 3 shows the numerical solution obtained from Euler Method (EM) for the successive iterations. More so, numerical solution from the Analytical Method and the associated error were also displayed.

Furthermore, Analytical Method (AM) was also applied in solving Problem (1) and solution was obtained for the two given points of x $(ie; x = 0.1\ and\ x = 0.2)$as required. Equations (33) to (34)gives the non numerical equivalence Exact solution (ES) to Problem 1

$y(x) = Ce^x\ - (x^2 + 2x + 2)$ (33)
$y(x) = 2e^x - (x^2 + 2x + 2)$ (34)

More so, the resulting numerical solution was obtained. See equations (35) to (38)for the ranges of values of (ie; x: $(0.05 \leq x_n \leq 0.2)$ respectively. Below are the numerical equations obtained for the Exact Solution. (ES).

$y(0.05)\ \cong 0$ (35)
$y(0.1)\ \cong 0.0003$ (36)
$y(0.15)\ \cong 0.0012$ (37)
$y(0.2)\ \cong 0.0028$ (38)

Table 3 displays the result from Exact Solution (ES) of problem 2