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On the Combinatorics of Finite Dimensional Multi-Set and its Associated Probability Mass Function.

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Abstract

In this research paper, let $d \in \mathbb{N}$ be arbitrary but fixed, we consider the sets $X^{(d)}$ and $M(\lambda, X^{(d)})$ that is more general than the set $X = \{x_i : i = 1, 2, \ldots, k\}$ and thenconstructed the associated probability mass function (with proof) due to certain underlying multi-indexsuch that the index (running) variable$\bar{\imath}_d = i_1 i_2 \ldots i_d$or $i_1, i_2, \ldots, i_d$is not necessarily a point (i), but rather a vector ($\bar{\imath}_d$)=($i_1, i_2, \ldots, i_d$), where $i_r \epsilon [k_r]$, $k_r \epsilon \mathbb{N}$, $r \epsilon [d]$.

Keywords and Phrase: Permutations, Combinations, Multi-Set, Multi-Index, Probability.

1.0 Introduction

Enumerative combinatorics provides one of the basic fundamentals for discrete probability theory [1-9]. If E is subset of a finite set X ($E \subset X$) such that one chooses an element of X at random, the probability that the element chosen actually belong to E is given by $n(E) / n(X)$. Thus, the determination of the cardinalities of these sets underlies this probability model (the so-called uniform model). However, beyond this uniform model, enumerative combinatorics have proved to be efficient whereby permeating various forms of discrete probability theory andparametric evaluation which we intend to demonstrate. Let$X = \{x_i : i = 1,2, \ldots, k\}$ $and E \subset X$, then the usual probability (p_1) of successfully selecting (picking) an element of E is given by $p_1 = n(E) / n(X)$, and then define $p_2 = 1 - p_1$ for the failure. If this selection is repeated for n-number of times, then the probability that an element of Ewill be selected exactly $j$times in n-number of trials is given by the binomial function defined by $C_j^n p_1^j p_2^{n-j}$ for two possible outcomes and k-nomial function as $C_{j_1, j_2, \ldots, j_k}^n p_1^{j_1} p_2^{j_2} \ldots p_k^{j_k}$ for k-possible outcomes. In this research, we consider the sets $X^{(d)}$and $M(\lambda, X^{(d)})$that is more general than the set Xsuch that the index (running) variable ($\bar{\imath}_d$)=($i_1, i_2, \ldots, i_d$) is not necessarily a point, but rather a vector, where $i_r \epsilon [k_r]$, $k_r \epsilon \mathbb{N}$, $r \epsilon [d]$. To do this, let

$$X^{(d)} = \{ x_{\bar{\imath}_d} : i_r \epsilon [k_r], k_r \epsilon \mathbb{N}, r \epsilon [d] \}$$

and then define

$$M(\lambda, X^{(d)}) = \{ {x_{\bar{\imath}_d}}^{\alpha_{\bar{\imath}_d}} : \alpha_{\bar{\imath}_d} \epsilon \alpha^{(d)} \}$$

to be the Multiset induced by $X^{(d)}$ due to the function $\lambda: X^{(d)} \to \mathbb{N}$such that $\lambda(x_{\bar{\imath}_d}) = \alpha_{\bar{\imath}_d}$. Where $\alpha^{(d)}$ is a multi-index. We then give a classical combinatoria proof of the associated probability function.

1.1 Multiset and Multinomial

Definition1.1.1 [10-11]: A finite multiset $M(\lambda, X)$ ($or M$) on a set X is a function $\lambda: X \to \mathbb{N}$such that

$$\sum \lambda(x) < \infty$$

If $\lambda(x) = n \forall x \in X$, then M is called an n-multiset, hence we write $n(M) = n$. Suppose $X = \{x_i : i = 1,2, \ldots, k\}$ and $\lambda: X \to \mathbb{N}$ such that $\lambda(x_i) = \alpha_i$, we shall have $M = \{x_i^{\alpha_i} : i = 1,2, \ldots, k\}$, where α_i is called the multiplicity of $x_i (in M)$ and $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is called the (associated) multi-index (or weak composition), which is also a row matrix (vector). For simplicity we write $\alpha = (\alpha_{1i}, \alpha_{2i}, \ldots, \alpha_k)$. We quickly remark that the function $\lambda: X^{(d)} \to \mathbb{N}$ is the so-called "random variable" as often used by statisticians.

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To see this, given any finite Multiset M, then there exist $\lambda: X^{(d)} \to \mathbb{N}$ such that $\lambda(x_i) = \alpha_i(\ i = 1,2,\ldots,k)$ and $\sum\lambda\ (x_i) = n$ since M is finite. If we let $\lambda\ (x_i) = X_i$, then $X_i$is a random variable that count the occurences of outcome $x_i$in X (i.e. $X_i = \alpha_i;\ i = 1,2,\ldots,k$).

1.2 One Dimensional Multinomial Distribution

If for each n independent trials, there are k possible outcomes $x_1, x_2, \ldots, x_k$, with the corresponding probabilities $p_1, p_2, \ldots, p_k$ ($\sum p_i = 1$), and if X_i (random variable) records the number of occurrances of x_i in these n trials, hence for every one-dimensional multi-index $(\alpha_{1i}, \alpha_{2i}, \ldots, \alpha_k)$of n, then

$$p(X_i = \alpha_i;\ i = 1,2,\ldots,k) = C^n_{\alpha_1,\alpha_2,\ \ldots,\alpha_k} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} \qquad (1.1)$$

is the underlying probability mass function (or simply pmf) which is well known. In the sequel we shall represent the above description on X that defines the pmf by $M(p,\alpha,X)$ ($orM(p,\lambda,X)$) and then give a formal proof for the pmf for completeness purpose. We now proceed to define certain concept and notations which will serve as a building block in this paper.

Definition1.2.1 [12-14] By Multi-index, we mean a k-tuple vector (a row matrix or a column matrix) α_i, where each (α_i: $[k] = \{1,2,\ldots,k\}$) is a non-negative interger. We define

The associated integer $|\alpha_i|$by

$$|\alpha| = \sum_{i=1}^{k} \alpha_i \qquad (1.2)$$

The associated monomial x^α by

$$x^\alpha = \prod_{i=1}^{k} x_i^{\alpha_i} \qquad (1.3)$$

The associated factorial $\alpha!$ by

$$\alpha! = \prod_{i=1}^{k} \alpha_i\,! \qquad (1.4)$$

Let $X = \{x_1, x_2, \ldots, x_k\}$ be a distinct finite set of points. If we associate to each element $x_i \in X$ with the numberα_i in αthen certainly there exist a non-empty set $M(\alpha, X)$ induced by a non-negative integer $\lambda: X \to \mathbb{N}$such that x_i has multiplicity $\alpha_i$in $M(\alpha,X) or\ (M(\lambda,X))$, which is define by

$$M(\lambda, X) = \{x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_k^{\alpha_k}\} \qquad (1.5)$$

is the multiset associated with X with respect to the non-negative integer function on X. Now consider the expansion of $(x_1 + x_2 + \cdots + x_k)^n$, observe that if $k = 2,3$ then we have the binomial, trinomial expansion respectively. For arbitrary but fixed positive integer k the expansion of $(\sum_{i=1}^{k} x_i)^n$is a multinomial expansion of X in one running (index) variable i, which can be referred to as one category or class of data. Observe that each $x_i$has certain number of repeaitition or multiplicity in the expansion of $(\sum_{i=1}^{k} x_i)^n$. There is no loss of generality if we assume that the multiplicity of x_i in the expansion of $(\sum_{i=1}^{k} x_i)^n$is $\alpha_i;\ i = 1,2,\ldots,k$ provided $\sum\alpha_i = n$. Thus, this will certainly induce a multiset representation due to the multinomial expansion; as such we have $\{x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_k^{\alpha_k}\}$as in (1.5). Furthermore, observe that each term (string) in this multinomial (k-nomial) expansion can be given in the general form.

$$C(\alpha_1, \alpha_2,\ \ \ldots, \alpha_k) x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k} \qquad (1.6)$$

Where $C(\alpha_1, \alpha_2,\ \ \ldots, \alpha_k)$is the associated k-nomial coefficient for each term. The following lemma gives the actual formular for $C(\alpha_1, \alpha_2,\ \ \ldots, \alpha_k)$ and corresponding probability mass function.

Lemma 1.2.2

Let$M_{(1)}(p, \alpha\ , X)$ denote a description on a finite multiset with multiplicity $\alpha_i$and probability $p_i$for each $x_i\ \in X$, then the probability that $x_i\ \in X$is selected exactly α_i times $i = 1,\ldots,k$ in n-trials is;

$$P(X_i = \alpha_i; i = 1,\ldots,k) = \binom{n}{\alpha_1, \alpha_2, \ldots, \alpha_k} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$$

Proof

If α_i is the munber of times each α_i can be repeatedly be selected in n-trials such that $\sum_i \alpha_i = n$, then we must select the $x_i{}'s$, say $x_1$in $\binom{n}{\alpha_1}$ ways, x_2 in $\binom{n-\alpha_1}{\alpha_2}$ways, … , x_k in $\binom{n-\alpha_1-\cdots-\alpha_{k-1}}{\alpha_k}$ ways. Thus, altogether the number of ways of making these selections is given by

$$\binom{n}{\alpha_1}\binom{n-\alpha_1}{\alpha_2} \times \cdots \times \binom{n-\alpha_1-\cdots-\alpha_{k-1}}{\alpha_k} = \binom{n}{\alpha_1, \alpha_2, \ldots, \alpha_k} = \frac{n!}{\prod_{i=1}^{k} \alpha_i!}$$

Thus, consequently, we have

$$P(X_i = \alpha_i; i = 1, \dots, k) = \frac{n!}{\prod_{i=1}^{k} \alpha_i!} \prod_{i=1}^{k} p_i^{\alpha_i}$$

The proof of lemma1.2.2 can be found in most standard text as cited in $[1 - 9]$, we give the poof here for completness purpose.

Now, we extend our description above to two-dimensional multiset and its associated two-dimensional multi-index $\alpha^{(2)} = (\alpha_{ij})$, by considering the expansion $\left(\sum_{i=1}^{k}.\sum_{j=1}^{m} x_i\right)^n$ where the multiplicity α_{ij} $(i \in [k],\ j \in [m],\ k, m \in \mathbb{N})$ for each term $x_{ij} \in X^{(2)}(i \in [k],\ j \in [m],\ k, m \in \mathbb{N})$ induces a $k \times m$ array (vector) where each α_{ij} $(i \in [k],\ j \in [m],\ k, m \in \mathbb{N})$ is a non-negative integer. Hence for the vector

$$\alpha^{(2)} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_{k1} & \cdots & \alpha_{km} \end{pmatrix}, or \alpha^{(2)} = (\alpha_{ij}): i \in [k], \qquad j \in [m]$$

We define

The associated integer $|\alpha^{(2)}|$ by

$$\alpha^{(2)} = \sum_{i=1}^{k}.\sum_{j=1}^{m} \alpha_{ij}\ ; where \left|\alpha_i^{(2)}\right| = \sum_{j=1}^{m} \alpha_{ij}\ ;\ i \in [k] \qquad (1.7)$$

The associated monomial x^{α} by

$$x^{\alpha^{(2)}} = \prod_{i=1}^{k}.\prod_{j=1}^{m} x_{ij}^{\alpha_{ij}}\ ; \text{where } x_i^{\alpha^{(2)}} = \prod_{j=1}^{m} x_{ij}^{\alpha_{ij}} \qquad (1.8)$$

The associated factorial $\alpha!$ by

$$\alpha^{(2)}! = \prod_{i=1}^{k}.\prod_{j=1}^{m} \alpha_{ij}\ !;\ \text{where } \alpha_i^{(2)}! = \prod_{j=1}^{m} x_{ij}^{\alpha_{ij}}\ ! \qquad (1.9)$$

2.0 Main Results

Lemma 2.1

Let $M_{(2)}(p, \alpha^{(2)}, X^{(2)})$denote a description on a finite multiset with multiplicity $\alpha_{ij}$and probability $p_{ij}$for each $\alpha_{ij} \in X^{(2)}$, then the probability that $\alpha_{ij} \in X^{(2)}$is selected exactly α_{ij}times $(i \in [k],\ j \in [m],\ k, m \in \mathbb{N})$ in n-trials is;

$$\mathrm{P}\left(X_{ij} = \alpha_{ij} :\ i \in [k], j \in [m]\right) = \frac{\mathrm{n}!}{\prod_{i=1}^{k}.\prod_{j=1}^{m} \alpha_{ij}} \prod_{i=1}^{k}.\prod_{j=1}^{m} p_{ij}^{\alpha_{ij}}$$

Proof

Somehow, our choice of selection is a little beat coupled, since for each fixed $i \in [k]$ we have to make selection from the set$\{x_{ij} : j \in [m]\}$. Let $|\alpha_i|$ be the number of times x_{ij}could be repeatedly be selected for every fixed $i \in [k]$ and let α_{ij} be the number of times x_{ij}could be repeatedly be selected given that the ith row has been selected for each $j \in [m]$. Thus, for each $i \in [k]$, the number of ways of selecting the $1^{st}, 2^{nd}, \dots, m^{th}$ element in $X^{(2)}$constituting $(\alpha_{i1}, \alpha_{i2}, \ \dots, \alpha_{im})$ is given by

$$\binom{\left|\alpha^{(2)}\right| - \sum_{r=1}^{i-1} |\alpha_r^{(2)}|}{\left|\alpha_i^{(2)}\right|} \binom{\left|\alpha_i^{(2)}\right|}{\alpha_{i1}} \binom{\left|\alpha_i^{(2)}\right| - \alpha_{i1}}{\alpha_{i2}} \cdots \binom{\left|\alpha_i^{(2)}\right| - \sum_{j=1}^{m-1} \alpha_{ij}}{\alpha_{im}} \forall i \in [k]$$

$$= \binom{\left|\alpha^{(2)}\right| - \sum_{r=1}^{i-1} |\alpha_r^{(2)}|}{\left|\alpha_i^{(2)}\right|} \prod_{j=1}^{m} \binom{\left|\alpha_i^{(2)}\right| - \sum_{r=1}^{j-1} \alpha_{ir}}{\alpha_{im}} \forall i \in [k]$$

Thus, as i ranges over the set $[k]$, we have altogether

$$\prod_{i=1}^{k} \binom{\left|\alpha^{(2)}\right| - \sum_{r=1}^{i-1} |\alpha_r^{(2)}|}{\left|\alpha_i^{(2)}\right|} \prod_{j=1}^{m} \binom{\left|\alpha_i^{(2)}\right| - \sum_{r=1}^{j-1} \alpha_{ir}}{\alpha_{im}}$$

$$= \frac{\left(\sum_{i=1}^{k}.\sum_{j=1}^{m} \alpha_{ij}\right)!}{\prod_{i=1}^{k}.\prod_{j=1}^{m} \alpha_{ij}\ !} = \frac{\mathrm{n}!}{\prod_{i=1}^{k}.\prod_{j=1}^{m} \alpha_{ij}!}$$

Thus, consequently, we have

$$\mathrm{P}\big(X_{ij} = \alpha_{ij} : i \in [k], j \in [m]\big) = \frac{\mathrm{n}!}{\prod_{i=1}^{k} . \prod_{j=1}^{m} \alpha_{ij}} \prod_{i=1}^{k} . \prod_{j=1}^{m} p_{ij}^{\alpha_{ij}}$$

It is important to remark that in lemma2.1, given the set $X^{(2)} = \{x_{ij} : i \in [k], j \in [m]\}$and $\lambda: X^{(2)} \to \mathbb{N}$**,** then that there exist a two-dimensional finite multiset $M_{(2)}\big(\alpha^{(2)}, X^{(2)}\big) = \{x_{ij}^{\alpha_{ij}} : i \in [k], j \in [m]\ \}$ on $X^{(2)}$ with it corresponding multi-index $\alpha^{(2)} = \big(\alpha_{ij}\big)$ $(i \in [k], j \in [m])$. By lemma 2.1, we wish to generalise the result for an arbitrary d-dimensional finite multiset $M_{(d)}\big(\alpha^{(d)}, X^{(d)}\big)$ with the corresponding d-dimensional multi-index $\alpha^{(d)} = \alpha_{i_1,i_2,\dots,i_d}$, by considering the expansion $\left(\sum_{i_1=1}^{k_1} \dots \sum_{i_d=1}^{k_d} x_{i_1,i_2,\dots,i_d}\right)^n$ where the multiplicity $\alpha_{i_1,i_2,\dots,i_d}$ $(i_r \epsilon [k_r\ ], k_r \epsilon \mathbb{N}, r \epsilon [\ d\ ])$ for each term $x_{i_1,i_2,\dots,i_d} \in X^{(d)}(i_r \epsilon [k_r\ ], k_r \epsilon \mathbb{N}, r \epsilon [\ d\ ])$ induces a $k_1 \times k_2 \times \dots \times k_d$ vector (array) where each $\alpha_{i_1,i_2,\dots,i_d}$ $(i_r \epsilon [k_r\ ], k_r \epsilon \mathbb{N}, r \epsilon [\ d\ ])$ is a non-negative integer. Similarly for the vector $\alpha^{(d)}$. We define

The associated integer $|\alpha^{(2)}\ |$ by

$$\alpha^{(d)} = \sum_{i_1=1}^{k_1} \dots \sum_{i_d=1}^{k_d} \alpha_{i_1,i_2,\dots,i_d} \qquad (1.10)$$

$$where\ \ \left|\alpha_{i_1,i_2,\dots,i_u}^{(d)}\right| = \sum_{i_{u+1}=1}^{k_{u+1}} \dots \sum_{i_d=1}^{k_d} \alpha_{i_1,i_2,\dots,i_u i_{u+1},\dots,i_d}\ ; 1 \le u < d$$

The associated monomial x^{α} by

$$x^{\alpha^{(d)}} = \prod_{i_1=1}^{k_1} \dots \prod_{j_d=1}^{k_d} x_{i_1,i_2,\dots,i_d}^{\alpha_{i_1,i_2,\dots,i_d}} \qquad (1.11)$$

$$\text{where } x_{i_1,i_2,\dots,i_u}^{\alpha^{(2)}} = \prod_{i_{u+1}=1}^{k_{u+1}} \dots \prod_{i_d=1}^{k_d} x_{i_1,i_2,\dots,i_u i_{u+1},\dots,i_d}^{\alpha_{i_1,i_2,\dots,i_u i_{u+1},\dots,i_d}}\ ; 1 \le u < d$$

The associated factorial $\alpha^{(d)}!$ by

$$\alpha^{(d)}! = \prod_{i_1=1}^{k_1} \dots \prod_{j_d=1}^{k_d} \alpha_{i_1,i_2,\dots,i_d}! = \left|\alpha_{\bar{\imath}_d}\right|! \qquad (1.12)$$

Where

$$\alpha_{i_1,i_2,\dots,i_u}^{(d)}! = \prod_{i_{u+1}=1}^{k_{u+1}} \dots \prod_{j_d=1}^{k_d} \alpha_{i_1,i_2,\dots,i_u i_{u+1},\dots,i_d}!\ and\ \left|\alpha_{\bar{\imath}_u}\right|! = \left|\alpha_{i_{u-1}1}\right|! \left|\alpha_{i_{u-1}2}\right|! \cdots \left|\alpha_{i_{u-1}k_u}\right|!$$

Lemma 2.2

For arbitrary but fixed $d \in \mathbb{N}$ let $r \epsilon [\ d\ ]\ and\ \ k_r \epsilon \mathbb{N}$, , let $i_r \epsilon [k_r\ ]$, then the following holds;

$$\prod_{r=1}^{d} . \left(\prod_{i_r=1}^{k_r} \binom{\left|\alpha_{\bar{\imath}_{r-1}}\right| - \sum_{j_r=1}^{i-1} |\alpha_{\bar{\imath}_{r-1} j_r}|}{\left|\alpha_{\bar{\imath}_{r-1} i_r}\right|} \right) = \frac{n!}{\alpha^{(d)}!}$$

Where $\left|\alpha_{\bar{\imath}_{r-1}}\right| - \sum_{j_r=1}^{i-1} |\alpha_{\bar{\imath}_{r-1} j_r}| = 0\ \forall\ r \epsilon [\ d\ ]$

Proof

$$\prod_{r=1}^{d} . \left(\prod_{i_r=1}^{k_r} \binom{\left|\alpha_{\bar{\imath}_{r-1}}\right| - \sum_{j_r=1}^{i-1} |\alpha_{\bar{\imath}_{r-1} j_r}|}{\left|\alpha_{\bar{\imath}_{r-1} i_r}\right|} \right)$$

$$= \frac{n!}{|\alpha_1|!\,|\alpha_2|! \cdots \left|\alpha_{k_1}\right|! \left(n - |\alpha_1| - |\alpha_2| - \dots - \left|\alpha_{k_1}\right|\right)!}$$

$$\times$$

$$= \frac{\left|\alpha_{i_1}\right|!}{\left|\alpha_{i_1 1}\right|! \left|\alpha_{i_1 2}\right|! \cdots \left|\alpha_{i_1 k_2}\right|! \left(n - \left|\alpha_{i_1 1}\right| - \left|\alpha_{i_1 2}\right| - \dots - \left|\alpha_{i_1 k_2}\right|\right)!}$$

$$\times$$

$$\vdots$$
$$\times$$
$$\frac{|\alpha_{\bar{\imath}_{d-2}}|!}{|\alpha_{\bar{\imath}_{d-2}1}|!\,|\alpha_{\bar{\imath}_{d-2}2}|!\cdots|\alpha_{\bar{\imath}_{d-2}k_{d-1}}|!\,\left(|\alpha_{\bar{\imath}_{d-2}}|-|\alpha_{\bar{\imath}_{d-2}1}|-|\alpha_{\bar{\imath}_{d-2}2}|-\cdots-|\alpha_{\bar{\imath}_{d-2}k_{d-1}}|\right)!}$$
$$\times$$
$$\frac{|\alpha_{\bar{\imath}_{d-1}}|!}{|\alpha_{\bar{\imath}_{d-1}1}|!\,|\alpha_{\bar{\imath}_{d-1}2}|!\cdots|\alpha_{\bar{\imath}_{d-1}k_d}|!\,\left(|\alpha_{\bar{\imath}_{d-1}}|-|\alpha_{\bar{\imath}_{d-1}1}|-|\alpha_{\bar{\imath}_{d-1}2}|-\cdots-|\alpha_{\bar{\imath}_{d-1}k_d}|\right)!}$$
$$=\frac{n!}{|\alpha_{\bar{\imath}_{d-1}1}|!\,|\alpha_{\bar{\imath}_{d-1}2}|!\cdots|\alpha_{\bar{\imath}_{d-1}k_d}|!}$$

Hence, the result follows immediately from the definition above, this complete the prove.

Theorem 2.3

Let $M_{(d)}(p\,,\alpha^{(d)},\,X^{(d)})$ denote a description on d-dimensional finite multiset $M_{(d)}(\,\alpha^{(d)},\,X^{(d)})$ with corresponding multiplicity $\alpha_{i_1,i_2,\dots,i_d}$and probability $p_{i_1,i_2,\dots,i_d}$ for each $x_{i_1,i_2,\dots,i_d}\,\epsilon\,X^{(d)}$ then the probability that $x_{i_1,i_2,\dots,i_d}\,\epsilon\,X^{(d)}$is selected exactly $\alpha_{i_1,i_2,\dots,i_d}$ times ($i_r\,\epsilon\,[k_r\,], k_r\,\epsilon\,\mathbb{N}, r\,\epsilon\,[\,d\,]$) in n-trials is;

$$\mathrm{P}\left(X_{i_1,i_2,\dots,i_d}=\alpha_{i_1,i_2,\dots,i_d}:i_r\,\epsilon\,[k_r\,],k_r\,\epsilon\,\mathbb{N},r\,\epsilon\,[\,d\,]\right)=$$
$$\frac{\mathrm{n}!}{\prod_{i_1=1}^{k_1}\cdots\prod_{j_d=1}^{k_d}\alpha_{i_1,i_2,\dots,i_d}}\prod_{i_1=1}^{k_1}\cdots\prod_{i_d=1}^{k_d}p_{i_1,i_2,\dots,i_d}^{\alpha_{i_1,i_2,\dots,i_d}}$$

Proof

For simplicity we write α to mean $\alpha^{(d)}$. If we choose any $i_1,i_2,\dots,i_u$ component of $\alpha=(\alpha_{\bar{\imath}_d})$ such that $u<d$ and $u+1=d$, then the number of ways of selecting the $1^{st},2^{nd},\dots,k_d^{th}$elements in $X^{(d)}$. constituting $(\alpha_{\bar{\imath}_u1},\alpha_{\bar{\imath}_u2},\ \dots,\alpha_{\bar{\imath}_uk_d})$ is given by

$$\binom{|\alpha|-\sum_{j_u=1}^{i_u-1}|\alpha_{\bar{\imath}_{u-1}j_u}|}{|\alpha_{\bar{\imath}_u}|}\prod_{i_{u+1}=1}^{k_d}\binom{|\alpha_{\bar{\imath}_u}|-\sum_{j_{u+1}=1}^{i_{u+1}-1}|\alpha_{\bar{\imath}_uj_{u+1}}|}{|\alpha_{\bar{\imath}_ui_{u+1}}|}$$

as $i_1,i_2,\dots,i_u$ ranges over the array $\alpha=(\alpha_{i_1,i_2,\dots,i_u})$ inductively we have

$$\prod_{i_1=1}^{k_1}\binom{|\alpha|-\sum_{j_1=1}^{i_1-1}|\alpha_{j_1}|}{|\alpha_{i_1}|}\prod_{i_2=1}^{k_2}\binom{|\alpha_{i_1}|-\sum_{j_2=1}^{i_2-1}|\alpha_{i_1j_2}|}{|\alpha_{i_1i_2}|}$$
$$\cdots$$
$$\prod_{i_u=1}^{k_u}\binom{|\alpha_{\bar{\imath}_{u-1}}|-\sum_{j_u=1}^{i_u-1}|\alpha_{\bar{\imath}_{u-1}j_u}|}{|\alpha_{\bar{\imath}_{u-1}i_u}|}\prod_{i_{u+1}=1}^{k_{u+1}}\binom{|\alpha_{\bar{\imath}_u}|-\sum_{j_{u+1}=1}^{i_{u+1}-1}|\alpha_{\bar{\imath}_uj_{u+1}}|}{|\alpha_{\bar{\imath}_ui_{u+1}}|}$$
$$\cdots$$
$$\prod_{i_{d-1}=1}^{k_{d-1}}\binom{|\alpha_{\bar{\imath}_{d-2}}|-\sum_{j_{d-1}=1}^{i_{d-1}-1}|\alpha_{\bar{\imath}_{d-2}j_{d-1}}|}{|\alpha_{\bar{\imath}_{d-2}i_{d-1}}|}\prod_{i_d=1}^{k_d}\binom{|\alpha_{\bar{\imath}_{d-1}}|-\sum_{j_d=1}^{i_d-1}|\alpha_{\bar{\imath}_{d-1}j_d}|}{|\alpha_{\bar{\imath}_{d-1}i_d}|}$$
$$=\prod_{r=1}^{d}\cdot\left(\prod_{i_r=1}^{k_r}\binom{|\alpha_{\bar{\imath}_{r-1}}|-\sum_{j_r=1}^{i-1}|\alpha_{\bar{\imath}_{r-1}j_r}|}{|\alpha_{\bar{\imath}_{r-1}i_r}|}\right)$$

Using lemma 2.2 and definitions above, consequently, we have

$$\mathrm{P}(X_{\bar{\imath}_d}=\alpha_{\bar{\imath}_d})=\frac{\mathrm{n}!}{\prod_{\bar{\imath}_d}\alpha_{\bar{\imath}_d}!}\prod_{\bar{\imath}_d}p_{\bar{\imath}_d}^{\alpha_{\bar{\imath}_d}}=\binom{n}{(\alpha_{\bar{\imath}_d})}\prod_{\bar{\imath}_d}p_{\bar{\imath}_d}^{\alpha_{\bar{\imath}_d}}$$

3.0 Conclusion

The results we obtained in this paper are new; to the best of our knowledge, we are unaware of any such demonstration of our results in the manner we did in literature.

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