Journal of the Nigerian Association of Mathematical Physics
Volume 31, (July, *2015*), pp 399 – 402

Construction of Arithmetic Probability Distribution on A Defined Interval [a,b].

[1]*O.C. Okoli* , [2]*N.A. Nsiegbe and* [3]*D.F. Nwosu*

[1,2,3]**Department of Mathematics, P.M.B 02 Uli, Chukwuemeka Odumegwu Ojukwu University, Anambra State, Nigeria.**
[3]**Department of Mathematics/Statistics, Federal Polytechnic Nekede, Owerri.**

Abstract

Considering that events occurring at unit interval may decrease or increase arithmetically over time, we seek to define a probability distribution function on [a,b] with respect to the arithmetic distribution property of the values. It is an improvement on the well-known discrete uniform distribution.

Keywords and Phrase: Probability mass function, Arithmetic progression, Expectation.

1.0 Introduction

By constructing a probability density on [a,b] called arithmetic probability density function. We observed that this probability density function will only give the expected probability value if the distribution of the random variable maintains a particular order of the progression, so that in this work we will only consider the case of increasing order of the progression, consequently that of decreasing order of progression fellows immediately. In order to explain the input of this order of progression to the probability so defined, we shall introduce the following terminologies with their corresponding definitions.

Definition 1.1 [1]

Let $\{x_i : i = 1,2,\dots,n\}$ be arithmetically distributed on the interval [a,b], then the collection is said to be;

a. arithmetically non-decreasing if $x_{i+1} \geq x_i \;\forall\; i \in N_n. i.e. d \geq 0.$
b. strictly arithmetically increasing if $x_{i+1} > x_i \;\forall\; i \in N_n. i.e. d > 0.$
c. arithmetically non-increasing if $x_{i+1} \leq x_i \;\forall\; i \in N_n. i.e. d \leq 0.$
d. strictly arithmetically decreasing if $x_{i+1} < x_i \;\forall\; i \in N_n. i.e. d < 0.$
 where d is the common difference.

In most standard Text, we are acquainted with the discrete uniform distribution function and the host of other distribution functions [2]. We strongly believe that there is need to define a density function on [a,b] which will be a generalisation of the existing uniform density function on [a,b], so that this can serve as an ideal representation of probability density function on [a,b] for this case of arithmetic distribution when compare with the discrete uniform distribution function already known.

Definition 1.2 [3-4]

If S is a sample space with probability measure and X is a real-valued function defined over the elements of S, then X is called a random variable.

2.0 Main Results

Definition 2.1

Let X be a discrete random variable in [a,b] that can take on the values $x_i: i = 1,2,\dots,n$ and $x_{i+1} \leq x_i$ such that the probability density is given by

$$f(x) = P(X \leq x) = \frac{2b(n-1)-2(x-1)(b-a)}{(n^2-n)(a+b)}; \quad x = 1,2,\dots,n$$

is said to have arithmetic density distribution on [a,b]. And the cumulative distribution function F is given by

$$F(x) = P(X \leq x) = \begin{cases} 0 & if\ x < 1, \\ \frac{2x(bn-a)-(x^2+x)(b-a)}{(n^2-n)(a+b)} & if\ 1 \leq x < n, \\ 1 & if\ x \geq n. \end{cases}$$

Corresponding author: O.C. Okoli, E-mail: odicomatics@yahoo.com, Tel.: +234-8036941434

In the sequel, we shall state the results so obtained as theories and then give a formal proof for each of the theorems.

Theorem 2.1

Let $\{x_i : i = 1,2,\dots,n\}$ be arithmetically distributed in the interval [a,b] and $x_{i+1} \le x_i \; \forall \; i \in N_n$., then the probability density function f is given by

$$f(x) = P(X \le x) = \frac{2b(n-1)-2(x-1)(b-a)}{(n^2-n)(a+b)}; \quad x = 1,2,\dots,n.$$

Proof:

Suppose $\{t_x : x = 1,2,\dots,n\}$ is arithmetically distributed in [a,b] and $t_{x+1} \le t_x \; \forall \; x \in N_n$. Then it suffice to define

$$t_x = b - \frac{(b-a)(x-1)}{n-1} \quad \forall \; x = 1,2,\dots,n; \; t_x \in [a,b]$$

So that

$$f(x) = \frac{b-(x-1)h}{\sum_{x=1}^{n}[b-(x-1)h]}, \text{ where } h = \left(\frac{b-a}{n-1}\right).$$

By this we shall have

$$f(x) = \frac{2b-2(x-1)h}{2n(b+h)-n(n+1)h}$$
$$= \frac{2b(n-1)-2(x-1)(b-a)}{(n^2-n)(a+b)}.$$

Theorem 2.2

Let $\{x_i : i = 1,2,\dots,n\}$ be arithmetically distributed in the interval [a,b] and $x_{i+1} \le x_i \; \forall \; i \in N_n$., then the cumulative distribution function F is given by

$$F(x) = P(X \le x) = \begin{cases} 0 & if \; x < 1, \\ \frac{2x(bn-a)-(x^2+x)(b-a)}{(n^2-n)(a+b)} & if \; 1 \le x < n, \\ 1 & if \; x \ge n. \end{cases}$$

Proof:

Suppose $\{t_x : x = 1,2,\dots,n\}$ is arithmetically distributed in [a,b], then we shall have

$$F(x) = \sum_{y=1}^{x} f(y) = \sum_{y=1}^{x}\left(\frac{2[b-(y-1)h]}{2n(b+h)-n(n+1)h}\right)$$
$$= \frac{2x(b+h)-x(x+1)h}{2n(b+h)-n(n+1)h}$$
$$= \frac{2x(bn-a)-(x^2+x)(b-a)}{(n^2-n)(a+b)}$$

Thus,

$$F(x) = P(X \le x) = \begin{cases} 0 & if \; x < 1, \\ \frac{2x(bn-a)-(x^2+x)(b-a)}{(n^2-n)(a+b)} & if \; 1 \le x < n, \\ 1 & if \; x \ge n. \end{cases}$$

Having proved theorem 2.1, we shall show that f is actually a probability density function on [a,b] by verifying the conditions governing probability density functions.

i. $0 \le b-(x-1)h \le \sum_{x=1}^{n}[b-(x-1)h] \quad \forall \; 1 \le x \le n$

$$\Rightarrow 0 \le \frac{b-(x-1)h}{\sum_{x=1}^{n}[b-(x-1)h]} \le 1 \quad \forall \; 1 \le x \le n$$
$$\Rightarrow 0 \le f(x) \le 1 \quad \forall \; 1 \le x \le n$$

ii. $\sum_{x=1}^{n} f(x) = \sum_{x=1}^{n}\left(\frac{2b-2(x-1)h}{2n(b+h)-n(n+1)h}\right)$

$$= \frac{2n(b+h)-n(n+1)h}{2n(b+h)-n(n+1)h} = 1$$

Theorem 2.3

If X is a discrete random variable that is arithmetically non-increasing on [a,b] such that

$$f(x) = P(X \le x) = \frac{2b(n-1)-2(x-1)(b-a)}{(n^2-n)(a+b)}; \quad x = 1,2,\dots,n$$

then,

i. $E(x) = \frac{(n+1)(b+2a)}{3(a+b)}$

ii. $Var(x) = \frac{(n+1)[(n-2)(a+b)^2+2(n+1)ab]}{18(a+b)^2}$

iii. Coefficient of variation $= \left\{\frac{(n-2)(a+b)^2+2(n+1)ab}{2(n+1)(b+2a)}\right\}^{1/2}$

Proof:

i. $E(x) = \sum_{x=1}^{n} xf(x) = \sum_{x=1}^{n} \frac{2b(n-1)x-2(x^2-x)(b-a)}{(n^2-n)(a+b)}$

$= k[2b(n-1)\sum_{x=1}^{n} x - 2(b-a)(\sum_{x=1}^{n} x^2 - \sum_{x=1}^{n} x)],$
where k = $\frac{1}{(n^2-n)(a+b)}$

$$= k\left[\frac{2b(n-1)n(n+1)}{2} - 2(b-a)\left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right)\right]$$

By simplifying the above equation and substituting for the value of k we obtain

$$\frac{(n+1)(b+2a)}{3(a+b)}$$

which is the required result.

ii. Finally, the Variance is given by $Var(x) = E(x^2) - [E(x)]^2$

$$E(x^2) = \sum_{x=1}^{n} x^2 f(x) = \sum_{x=1}^{n} \frac{2b(n-1)x^2-2(x^3-x^2)(b-a)}{(n^2-n)(a+b)}$$

$$= k[2b(n-1)\sum_{x=1}^{n} x^2 - 2(b-a)(\sum_{x=1}^{n} x^3 - \sum_{x=1}^{n} x^2)]$$

$$= k\left[2b(n-1)\frac{n(n+1)(2n+1)}{6} - 2(b-a)\left(\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right)\right]$$

where k = $\frac{1}{(n^2-n)(a+b)}$.

By simplifying the above equation we obtain

$$k(n+1)n\left[\frac{b(n-1)(2n+1)}{3} - \frac{(b-a)(3n+2)(n-1)}{6}\right]$$

On substituting the value of k with further simplification we obtain

$$E(x^2) = \frac{(n+1)[n(b+3a)+2a]}{6(a+b)}$$

Hence,

$$E(x^2) - [E(x)]^2 = \frac{(n+1)[n(b+3a)+2a]}{6(a+b)} - \left[\frac{(n+1)(b+2a)}{3(a+b)}\right]^2$$

$$= \frac{(n+1)[n(b+3a)+2a]}{6(a+b)} - \frac{(n+1)^2(b+2a)^2}{9(a+b)^2}$$

$$= (n+1)\left[\frac{n(a^2+4ab+b^2)-2(a^2+ab+b^2)}{18(a+b)^2}\right]$$

$$= \frac{(n+1)[(n-2)(a+b)^2 + 2(n+1)ab]}{18(a+b)^2}$$

which is the required result.

Theorem 2.4

If X is a random variable with arithmetical distribution

$$f(x) = P(X \le x) = \frac{2b(n-1)-2(x-1)(b-a)}{(n^2-n)(a+b)}; \quad x = 1,2,\dots,n.$$

on [a,b], then the moment generating function $M_x(t)$ is given by

$$M_x(t) = k\left[\frac{2(an-b)(e^{nt}-1)e^t}{e^t} + 2(b-a)\left(\frac{(e^t+n-1)e^{nt}-e^t}{e^t-1} - \frac{\left(e^{(n-1)t}-1\right)e^t}{(e^t-1)^2}\right)\right]$$

Where k = $\frac{1}{(n^2-n)(a+b)}$.

Proof:

$$M_x(t) = E(e^{xt}) = \sum_{x=1}^{n} e^{xt} f(x)$$

$$= \sum_{x=1}^{n} e^{xt} \frac{2b(n-1)-2(x-1)(b-a)}{(n^2-n)(a+b)}$$

$$= \frac{1}{(n^2-n)(a+b)}\left[2b(n-1)\sum_{x=1}^{n} e^{xt} + 2(b-a)\sum_{x=1}^{n} e^{xt} - 2(b-a)\sum_{x=1}^{n} xe^{xt}\right]$$

$$= \frac{1}{(n^2-n)(a+b)}\left[2(bn-a)\sum_{x=1}^{n} e^{xt} - 2(b-a)\sum_{x=1}^{n} xe^{xt}\right]$$

$$= \frac{1}{(n^2-n)(a+b)}\left[\frac{2(bn-a)(1-e^{nt})e^t}{e^t-1} - 2(b-a)\left(\frac{(e^t+n-1)e^{nt}-e^t}{e^t-1} - \frac{\left(e^{(n-1)t}-1\right)e^t}{(e^t-1)^2}\right)\right]$$

Corollary 2.4

If X is a random variable with arithmetical distribution and $h = 0 \; for \; all \; x = 1,2, \dots, n$ then,

$f(x) = \frac{1}{n} \quad \forall \; x = 1,2, \dots, n$

$F(x) = \frac{x}{n} \quad \forall \; x = 1,2, \dots, n$

This corollary is consequence of theorem 2.1, hence the proof is clear.**Corollary 2.5**

If X is a random variable with arithmetical distribution and $h = 0 \; for \; all \; x = 1,2, \dots, n$ then,

i. $E(x) = \frac{n+1}{2}$

ii. $Var(x) = \frac{n^2-1}{12}$

iii. $M_x(t) = \frac{(e^{nt}-1)e^t}{n(e^t-1)}$

iv. $Coefficient\ of\ variation = \left(\frac{n-1}{3(n+1)}\right)^{1/2}$

The proof of this corollary follows from theorem 2.3 and theorem 2.4.

Example 2.6

The table below shows the samples of bulbs manufactured by a company in her first 9 years of production. What is the probability that a customer will pick a bulb manufactured in the year 2004?

Table I (shows the number of bulbs produced for each year)

Year	2001	2002	2003	2004	2005	2006	2007	2008	2009
No. Of Bulbs	100	125	150	175	200	225	250	275	300

By the relative frequency definition of probability, we have that the probability of picking a bulb produced in 2004 is $\frac{175}{1800} = 0.0972.$

Solution

observe that

n = 9, x = 4, a = 100 , b = 300

Thus, $f(4) = \frac{2(100)(9-1) + 2(4-1)(300-100)}{\left(9^2-9\right)(300+100)} = \frac{2800}{28800} = \frac{7}{72} = 0.0972$

3.0 Conclusion

Observe that the results of corollary 2.4 and corollary 2.5 are for the Discrete Uniform Distribution Function, which are special cases of the results we obtained for the Arithmetic Probability Distribution Function in theorem 2.1 and theorem 2.2.

4.0 References

[1] R.R. Goldberg (1975); Methods of Real Analysis. New York, John Wiley and Sons.

[2] INC.K. Krishnamoorthy (2006); Handbook of Statistical Distributions with Application. New York, Taylor Francis Group, pg 29.

[3] R.S. Murry (2000); Probability and Statistics, Second Edition Singapore McGraw-Hill, INC.

[4] R.S. Murry (1999); Statistics Third Edition Singapore McGraw-Hill, INC.