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Paraletrix Linear Space

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Abstract

This paper established the concept of Paraletrix linear Space, an extension of rhotrix vector space to paraletrix and we present the concept of normed space on paraletrix. We extend by showing that the set of paraletrix over two binary operations addition and hearty multiplication of paraletrix forms a field. And state without proof that the norm linear space of paraletrix are Banach space.

1.0 Introduction

The concept of Rhotrix was introduced by Ajibade [1] as an extension of the initiative on matrix-tertions and matrix-noitrets suggested by Atanassov and Shannon [2]. The multiplication ofrhotrices defined by Ajibade [1] is as follows: let R and Q be two rhotrices such that;

$$R=\left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle, Q=\left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & k & \end{matrix} \right\rangle, \quad (1)$$

$$R\circ Q=\left\langle \begin{matrix} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{matrix} \right\rangle$$

and the addition of rhotrix R and Q defined by Ajibade [1] is

$$R+Q=\left\langle \begin{matrix} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{matrix} \right\rangle$$

while the multiplication of rhotrix R and Q defined by Sani [3] is

$$R\circ Q=\left\langle \begin{matrix} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bj+ek & \end{matrix} \right\rangle$$

The concept of Paraletrix was introduced by Aminu and Michael [4] as an extension of Rhotrix when the number of rows of the rhotrix is not equal to the number of columns. It is worth mentioning tosay that not all paraletrix has heart [4]. Let R and Q be two paraletrices such that;

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$$R=\left\langle\begin{array}{ccccc} & a_1 & & & \\ a_2 & a_3 & a_4 & & \\ & a_5 & a_6 & a_7 & \\ & & a_8 & a_9 & a_{10} \\ & & & a_{11} & \end{array}\right\rangle,\quad Q=\left\langle\begin{array}{ccccc} & b_1 & & & \\ b_2 & b_3 & b_4 & & \\ & b_5 & b_6 & b_7 & \\ & & b_8 & b_9 & b_{10} \\ & & & b_{11} & \end{array}\right\rangle, \qquad (2)$$

Multiplication of paraletrix R and Q using Ajibade's multiplication of rhotrix [1] is

$$R\circ Q=\left\langle\begin{array}{ccccc} & c_1 & & & \\ c_2 & c_3 & c_4 & & \\ & c_5 & c_6 & c_7 & \\ & & c_8 & c_9 & c_{10} \\ & & & c_{11} & \end{array}\right\rangle$$

where each $c_i = a_i h(Q) + b_i h(R) \ni i = 1,2,3,\cdots,11 \ni i \neq 6$ and $c_6 = h(Q)h(R)$ for the heart of the paraletrix $R\circ Q$, and $h(Q) = b_6$ and $h(R) = a_6$

Also, addition of paraletrix R and Q is given as

$$R+Q=\left\langle\begin{array}{ccccc} & d_1 & & & \\ d_2 & d_3 & d_4 & & \\ & d_5 & d_6 & d_7 & \\ & & d_8 & d_9 & d_{10} \\ & & & d_{11} & \end{array}\right\rangle$$ where each $d_i = a_i + b_i \ni i = 1,2,3,\cdots,11$

finally, the multiplication of paraletrix R and Q using Sani [3] rhotrix approach is only possible whenever the number of columns of R is equal to the number of rows of Q.

2 0 Definitions

2.1 *Group: let V be a non- empty set and $\circ$ be a binary operation. Then the ordered pair (V, $\circ$) is called a group if for any $x, y \in V$, thefollowing conditions are satisfied closure property, associativity, every non-zero element has inverse, identity element is contain in V. in addition, if commutativity is included, then we call it an abelian group.*

2.2 *Vector Space: let V be a non- empty set and$+$, $\circ$ be two binary operations (addition and scalar multiplication respectively). Then V is called a vector space over the field Kif;*

I. $(V,+)$ is an abelian group.

ii. V is closed under scalar multiplication i.e. for$x \in V, \alpha \in K, \alpha \circ x \in V$

iii. V has identity element under scalar multiplication

Iv. V is associative over scalar with respect to scalar multiplication

V. V is distributive over scalar and also distributive over vector. i.e. $\forall x, y \in V$ and $\alpha, \beta \in K$, $(\alpha+\beta)x = \alpha x + \beta x \in V$ *and* $\alpha(x+y) = \alpha x + \alpha y \in V$. *if all these conditions are satisfied, then $(V,\circ,+)$ is called a vector space or linear space.*

2.3 *Field: a non-empty set V, together with two binary operations addition (+) and multiplication($\circ$) is called a field if $(V,+)$and $(V,\circ)$are both Abelian group and the distributive law is satisfied.*

2.4. *Normed Linear Space:let V be a linear space. A mapping $\|\ \|: V \to \mathbb{R}$ is called a norm on V if the following conditions are satisfied;*

$\prod_1 := \|x\| \geq 0 \quad \forall x \in V$

$\prod_2 := \|x\| = 0 \quad iff \quad x = 0$

$\prod_3 := \|\alpha x\| = |\alpha|\|x\| \quad \forall \alpha \in K$ and $\forall x \in V$

$\prod_1 := \|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$

2.5. *Complete Normed Linear Space: A normed linear space X is called complete if every Cauchy sequence in X converges.*

2.6. *Banach Space: A complete metric space in a normed linear space is called a banach space.*

3.0 The concept of paraletrix linear space

Introduction: Paraletrix linear space considers a set of certain dimension paraletrix with two binary operations; addition (+) and scalar multiplication (∘), and we discover that it formed a linear space (just like matrix linear space). We establish the following theorems to justify this hypothesis.Throughout this chapter, we shall use the hearty multiplication of paraletrix.

Theorem 3.1; let P *be the set of all* $m\times n$ *dimensional paraletrix over the field* K ($\mathbb{R}$ or $\mathbb{C}$) *i.e* $P=\{R_i:$ each R_i is an $n\times m$ dimensional paraletrix, $i=1,2,3,...\}$ *.let* $\circ,+$ *be two binary operations, addition of paraletrix and scalar multiplication respectively. Then* $(P,+,\circ)$ *is a linear space.*

Proof:

It is sufficient to show that P *satisfies all the conditions in Definition 2.2. Since each element in the set P is a paraletrix, then* $P\neq\phi$. *Let* $\psi,\chi\in P$, *then;*

i. $(P,+)$ *is an Abelian group. It is easy to show that for any elements* $\psi,\chi,\varphi\in P$, *they are closed, associative, there exist* κ which is a (null) zero paraletrix, $k\in P\ \forall\psi\in P,\ni\psi+k=\psi=k+\psi$, *also every element of* P has inverse i.e. $\forall\psi\in P\ \exists(-\psi)\in P\ni\psi+(-\psi)=\kappa$ *and finally, since paraletrix addition is commutative, hence the proof.*

ii.There exist $e\in P\ \forall\psi\in P\ \ni e\circ\psi=\psi=\psi\circ e$ *this* e *is called the identity paraletrix [4]*

iii. $\forall\psi\in P$ and $a,b\in\mathbb{R}$ *then* $(a+b)\psi=a\psi+b\psi$ which implies distributive law is satisfied, since for any paraletrix, the product with a scalar is also a paraletrix. And by closure property of addition of paraletrix, the result follows immediately.

iv. $\psi,\chi\in P$, *and* $a\in\mathbb{R}$, $a(\psi+\chi)=a\psi+a\chi\in P$, *this result follows directly from elementary matrix theorem.*

v. $\forall\, a,b\in\mathbb{R}$ and $\psi\in P$, $(a\circ b)\circ\psi=a\circ(b\circ\psi)$ which implies associativity with scalar multiplication, this is true because the product of any real number is a real number.

Since all the above proof has been verified, it means $(P,+,\circ)$ *is a linear space.*

Theorem 3.2: For any paraletrix, P; it is invertible if and only if $h(P)$ *exist and* $h(P)\neq0$, *where* $h(P)$ *is the heart of the paraletrix.*

Proof: the proof follows directly from Ajibade's definition of inverse of a rhotrix[1].

Theorem 3.2; Let P *be the set of paraletrix of the same order, if* $h(P)$ *exist and* $h(P)\neq0$, *then* $(P,+\,*)$ *is a field.Where + and * are paraletrix addition and hearty multiplication respectively.*

Proof: the proof follows directly from the definition of field in 2.3 and also from theorem 3.2 as it is easy to see that $(P,+)$ *and* $(P,*)$ *are Abelian groups and also distributive. We know that paraletrix hearty multiplication is commutative, hence the proof.*

4.0 Paraletrix Norm Space

Let P be an $m\times n$ *paraletrix linear space. A map* $\|\ \|:P\to R$ *is called a norm on P if Definition 2.4 is satisfied.*

Theorem 4.1: Consider the space of $P_{3\times7}$ *dimensional paraletrix and define a map* $\|\ \|:P_{3\times7}\to R$ *by*

$$\|B\|=\max_{i\le11}|a_i|\text{ where }B=\left\langle\begin{array}{ccccc} & a_1 & & & \\ a_2 & a_3 & a_4 & & \\ & a_5 & a_6 & a_7 & \\ & & a_8 & a_9 & a_{10} \\ & & & a_{11} & \end{array}\right\rangle\in P_{3\times7},\text{ then } B \text{ is a norm on } P_{3\times7} \text{ and } (P_{3\times7},\|B\|) \text{ is a}$$

normed space.

Proof: we shall use the definitions in 2.4 to show that the above map is a norm on $P_{3\times7}$. It is good to mention that this same technique can be used to generalise for any $m\times n$ dimensional paraletrix.

i. $\forall a_i\in P_{3\times7},|a_i|\geq0$

$$\Rightarrow\max_{i\le11}|a_i|\geq0,$$

$\Rightarrow \|B\| \geq 0$ Satisfying the first condition.

ii. If $a_i = 0 \ \forall i = 1, 2, 3, ..., 11$

Then $\Rightarrow |a_i| = 0 \ \forall i$

$$\Rightarrow \max_{i \leq 11} |a_i| = 0, \Rightarrow \|B\| = 0. \qquad (4.1)$$

Conversely, if $\|B\| = 0$

$$\Rightarrow \max_{i \leq 11} |a_i| = 0, \forall i$$

$$\Rightarrow |a_i| = 0, \quad \Rightarrow a_i = 0, \ \forall \ i \qquad (4.2)$$

From (4.1) and (4.2), condition two of definition 2.4 is satisfied.

iii. Let $\alpha \in \mathbb{R} \ \ B \in P_{3\times 7}$

$$\|\alpha B\| = \max_{i \leq 11} |\alpha a_i|,$$

$$= \max_{i \leq 11} |\alpha||a_i|,$$

$$= |\alpha| \max_{i \leq 11} |a_i|,$$

$= |\alpha|\|B\|$ Satisfying condition three of definition 2.4.

iv. Let $B, C \in P_{3\times 7}$, then;

Consider $\|B + C\| = \max_{i \leq 11} |a_i + d_i|, d_i \in C$

$$\|B + C\| = \max_{i \leq 11} [|a_i| + |d_i|], d_i \in C$$

$$\Rightarrow \|B + C\| \leq \max_{i \leq 11} |a_i| + \max_{i \leq 11} |d_i|$$

$$\Rightarrow \|B + C\| \leq \|B\| + \|C\|$$

Since (i),(ii),(iii) and iv are satisfied, the $\|B\| = \max_{i \leq 11} |a_i|$ *is a norm on* $P_{3\times 7}$ *and* $(P_{3\times 7}, \|B\|)$ *is a normed linear space.*

Theorem 4.2:*Consider the space of* $P_{m\times n}$ *dimensional paraletrix whose heart exists, a map* $\| \ \| : P_{m\times n} \to R$ *by* $\|A\| = |a_h|$ *is a normed linear space. Where* a_h *is the heart of the paraletrix A.*

Proof:

Clearly, if $a_h = 0$*, the result follows immediately.*

But if $a_h \neq 0$*, the result follows from the definition of norm space and from theorem 4.1.*

5.0 Some Special Cases of Paraletrix

It is good to mention that if we see our paraletrix as in the sani's [3] row-column forms of rhotrix, the following are normed space;

i. $\|A\|_p = \left[\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2\right]^{1/2}$ *and* $\|B\|_p = \left[\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^2\right]^{1/2}$

ii. $\|A\|_1 = \max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|$

iii. $\|A\|_z = \left[\sum_{i=1}^{m}\sum_{j=1}^{n} (a_{ij}+c_{ij})^2\right]^{1/2}$

Finally, since we know that every rhotrix is a paraletrix whenever n=m, which implies that all the discussed theorem are also trivially true for a rhotrix as we can see in the norm define below;

$\|R\| = \det(R)$ *Called the determinant of the rhotrix R. It is also a normed space on the set of n dimensional rhotrix.*

Theorem 4.3; The Normed linear space of paraletrix are Complete linear space and hence Banach space

Proof; this proof follows from the application of a theory in functional analysis [6], that every finite dimensional normed linear space is complete, since paraletrix are of finite dimensions, hence the result.

6.0 Conclusions

We have presented a concept of linear space (vector space) in paraletrix, introduced the concept of group, field and norm on a paraletrix. We also state and proof a necessary and sufficient condition for a paraletrix to be called a field. And finally define some norm on the set of Paraletrix Linear space.

7.0 References

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