Journal of the Nigerian Association of Mathematical Physics
Volume 31, **(July, *2015*), pp 255 – 262**

On Application of Differential Transform Method: A Modified Approach to Solution of Certain KdV Equations

M. I. Modebei[1], I. Otaide[2], O.O. Olaiya[3], E. J. Ejiofor[4]

[1,3,4]**Department of Mathematics Programme, National Mathematical Centre, Abuja, Nigeria.**
[2]**Department of Mathematics, Delta State University, Abraka.**

Abstract

In mathematics, the Korteweg–de Vries equation (KdV equation for short) is a mathematical model of waves on shallow water surfaces. It is particularly notable as the prototypical example of an exactly solvable model, that is, a non-linear partial differential equation whose solutions can be exactly and precisely specified. In this paper, we proposed the method of differential transform with a modified approach using the wave variable to obtain analytic solution of the KdV equation. This method helps to reduce minimally the enormous amount of mathematical computation in solving such kind of problem, and thus shows the efficiency of the method.

Keywords: Wave variables, KdV equation, Differential Transform Method (DTM), Tailor series, Differential Equations. .
2010 AMS Subject Classification; 34A34, 35A22, 34G20, 35L05, 41A58

1.0 Introduction

The Korteweg-de Vries equation (KdV in short), is fifth order nonlinear partial differential equation of the form:

$$\frac{\partial u}{\partial t}+Au^2\frac{\partial u}{\partial u}+B\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+Cu\frac{\partial^3 u}{\partial x^3}+D\frac{\partial^5 u}{\partial x^5}=0 \tag{1}$$

subject to the initial condition;

$$u(x,0)=f(x) \tag{2}$$

Where A, B, C, D are constants. This equation plays an important role in describing motion of long waves on shallow water under gravity; one dimensional nonlinear lattice, fluid mechanics, quantum mechanics, plasma physics, nonlinear optics and other areas of application, to mention but few. The KdV equation has several connections to physical problems. In addition to being the governing equation of the string in the Fermi–Pasta–Ulam problem in the continuum limit, it approximately describes the evolution of long, one-dimensional waves in many physical settings, including:

1. shallow-water waves with weakly non-linear restoring forces;
2. long internal waves in a density-stratified ocean;
3. ion acoustic waves in a plasma;
4. acoustic waves on a crystal lattice.

Complex as it may appear to be, solutions to some nonlinear partial differential equations sometimes possess exact solution and in most cases not at all, hence we seek numerical solutions. Even though many methods to nonlinear differential equations have been proposed and some found very proficient and efficient, among which are the Secant Method, Sine-Cosine Method more methods that are powerful are still under research [1,2].

Of late, are Fan Sub equation Method which is a unified algebraic method used to obtain many types of traveling wave solutions based on an auxiliary nonlinear ordinary differential equation with constant coefficients [3,4]; In recent times, the adomian decomposition method (ADM) was applied to the KdV equation [5]. Also a new modification of Laplace ADM [5,6,7] was implemented in the KdV equation [8].

Corresponding author: M. I. Modebei, E-mail: gmarc.ify@gmail.com, Tel.: +2348031576262 & 8172235091

Wazwaz [9,10], also derived a variety of traveling wave solutions of distinct physical structure [11]; Furihata [12] applied the finite different method to obtain a numerical solution of certain nonlinear PDE called the Cahn-Hilliard equation [13]. Research on solution of KdV-equation is becoming on the increase as new method and approaches are being developed in succession. Here, we propose a method first introduced by Zhou [14] with a modified approach. For other authors who have used the Differential Transform Method (DTM in short) interested readers should see [15-20], to mention but few.
The modified approach proposed in this paper uses the transformation of the nonlinear PDE into an ODE using a wave variable, hence applying the DTM to the transformed PDE to obtain an analytical series solution of the K-K equation.

2.0 Differential Transform Method [14,15,16].

Definition 2.1 *The differential transform of the Kth derivative of a function $f(x)$ is defined as:*

$$F(K) = \frac{1}{K!}\left[\frac{d^K f(x)}{dx^K}\right]_{x=x_0} \tag{3}$$

and the inverse differential transform of $F(K)$ is defined as:

$$f(x) = \sum_{K=0}^{\infty} F(K)(x-x_0)^K \tag{4}$$

In real application, the function $f(x)$ is expressed as approximation to a finite series and (4) can be written as

$$f(x) = \sum_{K=0}^{N} F(K)(x-x_0)^K \tag{5}$$

Combining equations (3) and (5) we obtain the Taylor's series expansion of a function $f(x)$ as

$$f(x) = \sum_{K=0}^{N} \frac{1}{K!}\left[\frac{d^K f(x)}{dx^K}\right]_{x=x_0} (x-x_0)^K \tag{6}$$

As a result of the equations (3) and (5), some important theorems can be deduced and these theorems are used to obtain some basic result and computation required in illustrating this method.

3.0 Basic Theorems

Let the differential transformation of the Kth derivative of the functions $f(x)$, $g(x)$, and $h(x)$ are respectively, $F(K)$, $G(K)$, and $H(K)$.

Theorem 3.1 *If* $f(x) = g(x) \pm h(x)$*, then* $F(k) = G(k) \pm H(k)$.

Theorem 3.2 *If* $f(x) = cg(x)$*, then* $F(k) = cG(k)$*, where* c *is a constant.*

Theorem 3.3 *If* $f(x) = \frac{d^n g(x)}{dx^n}$*, then* $F(K) = \frac{(K+n)!G(K+n)}{K!}$.

Theorem 3.4 *If* $f(x) = g(x)h(x)$ *, then* $F(K) = \sum_{K_1=0}^{K} G(K_1)H(K-K_1)$

Theorem 3.5 *If* $f(x) = g_1(x)g_2(x)g_3(x)$ *, then* $F(K) = \sum_{K_1=0}^{K}\sum_{j=0}^{i} G_1(j)G_2(i-j)G_3(k-i)$

Theorem 3.6 *If* $f(x) = x^n$*, then*

$$F(K) = \delta(K+n) = \begin{cases} 1 & if\ K = n \\ 0 & otherwise \end{cases}$$

4.0 Analysis of The Method

Given the general form of nonlinear partial differential equation

$$f(u, u_t, u_x, u_{xx}, u_{xt}, u_{xxx}, u_{txx}, u_{xtt}, \ldots) = 0 \tag{7}$$

where $u(x,t)$ is the unknown function. To find the solution $u(x,t)$ of (7), we introduce a traveling wave $\gamma = x - vt$, with the transformation

$$u(x,t)=u(\gamma) \tag{8}$$

where v is the wave speed given as $v=\sqrt{gh}$; $g=9.8ms^{-2}$ (gravitation constant) and h is the depth of the water. Then (7) can be transformed to the ordinary differential equation

$$f(u,vu',u',u'',vu'',u''',vu''',\ldots)=0 \tag{9}$$

and finally, we apply the differential transform method to (9), so that

$$u(\gamma)=\sum_{K=0}^{\infty}F(K)\gamma^K=\sum_{K=0}^{\infty}F(K)(x-vt)^K \tag{10}$$

where $F(K)$ is the differential transform of $u(\gamma)$

Hence the solution

$$u(\gamma)=u(x,t)=\sum_{K=0}^{\infty}F(K)(x-vt)^K \tag{11}$$

As mentioned earlier, in real application, we thus obtain an approximation of the form

$$u_*(x,t)=\sum_{K=0}^{n}F(K)(x-vt)^K \tag{12}$$

5.0 Main Result

In what follows we apply this approach to obtain a solution of KdV equation [1,2,6] as proposed. Consider the equation;

$$\frac{\partial u}{\partial t}+Au^2\frac{\partial u}{\partial u}+B\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+Cu\frac{\partial^3 u}{\partial x^3}+D\frac{\partial^5 u}{\partial x^5}=0 \tag{13}$$

subject to the initial condition;

$$u(x,0)=f(x) \tag{14}$$

Applying the transformation (8), to (13) we obtain the ordinary differential equation

$$vu'=-Du^{(v)}-Au^2u'-Bu'u''-Cuu''' \tag{15}$$

Thus applying the DTM, we obtain

$$T(k)DU(k+5)=-v\cdot(k+1)U(k+1)-A\sum_{i=0}^{k}A(i,k)-B\sum_{i=0}^{k}B(i,k)-C\sum_{i=0}^{k}\sum_{j=0}^{i}C(i,j,k)$$

where

$$T(K)=(k+1)(k+2)(k+3)(k+4)(k+5);$$
$$A(i,k)=(k-i+1)(k-i+2)(k-i+3)U(i)U(k-i+3);$$
$$B(i,k)=(i+1)(k-i+1)(k-i+2)U(i+1)U(k-i+2);$$
$$C(i,j,k)=(i+1)U(j)U(i-j)U(k-i-1).$$

and $U(k)$ is the differential transform of $u(\gamma)$. Thus the recursive relation

$$U(k+5)=\frac{-v\cdot(k+1)U(k+1)-A\sum_{i=0}^{k}A(i,k)-B\sum_{i=0}^{k}B(i,k)-C\sum_{i=0}^{k}\sum_{j=0}^{k-i}C(i,j,k)}{DT(k)} \tag{16}$$

and the initial condition;

$$U(0)=F(0) \tag{17}$$

6.0 Numerical Results

6.1 Ito Equation [7,10]

Consider the equation (13), where $A=2$, $B=6$, $C=3$, $d=1$ and $f(x)=-\frac{1}{40}\left(1-3\operatorname{sech}^2\left(\frac{x}{20}\right)\right)$ so that we have

$$u_t + 2u^2 u_x + 6u_x u_{xx} + 3uu_{xxx} + u_{xxxxx} = 0 \tag{18}$$

subject to

$$u(x,0) = -\frac{1}{40}\left(1 - 3\operatorname{sec}h^2\left(\frac{x}{20}\right)\right) \tag{19}$$

Hence equation (16) become

$$U(k+5) = \frac{-v\cdot(k+1)U(k+1) - 2\sum_{i=0}^{k} A(i,k) - 6\sum_{i=0}^{k} B(i,k) - 3\sum_{i=0}^{k}\sum_{j=0}^{k-i} C(i,j,k)}{T(k)} \tag{20}$$

and the initial condition;

$$U(0) = \frac{1}{20} \tag{21}$$

Here, from equation (21), we have;

$$U(1) = 0,;\; U(2) = -\frac{3}{8000},;\; U(3) = 0,\quad U(4) = \frac{3}{4000000}$$

and taking $h = 6.38m$. Hence we obtain from the recursive relation (16) the values of $U(k)$, $k = 0,1,2,\cdots$ To avoid complexity, we take $n = 10$ as in equation (12), and truncate the series at $O(x^9, t^9)$, hence factoring, we obtain the series;

$$\begin{aligned}
u_*(x,t) = &-0.1 + 0.0001875x^2 - 3.125*10^{-}7x^4 + 4.42708*10^{-}10x^6 - 5.76637E-13x^8 \\
&+ t(-9.375*E-8x + 3.125E-10x^3 - 6.64062E-13x^5 + 1.15327E-15x^7) \\
&+ t^2(1.17188E-11 - 1.17188E-13x^2 + 4.15039E-16x^4 - 1.00911E-18x^6 + 2.00835E-21x^8) \\
&+ t^3(1.95313E-17x - 1.38346E-19x^3 + 5.04557E-22x^5 - 1.3389E-24x^7) \\
&+ t^4(-1.2207E-21 + 2.59399E-23x^2 - 1.57674E-25x^4 + 5.85768E-28x^6 - 1.65334E-30x^8) \\
&+ t^5(-2.59399E-27x + 3.15348E-29x^3 - 1.7573E-31x^5 + 6.61335E-34x^7) \\
&+ t^6(1.08083E-31 - 3.94185E-33x^2 + 3.66105E-35x^4 - 1.92889E-37x^6 + 7.32892E-40x^8) \\
&+ t^7(2.81561E-37x - 5.23007E-39x^3 + 4.13335E-41x^5 - 2.09398 - 43x^7) \\
&+ t^8(-8.79878E-42 + 4.90319E-43x^2 - 6.45835E-45x^4 + 4.58057*E-47x^6 - 2.25424E-49x^8)
\end{aligned}$$

This approximate solution obtained can be compared with the exact solution u^{exact} as illustrated with the surface plot in Fig 1, where the exact solution is given as;

$$u(x,t) = -\frac{1}{40}\left(1 - 3\operatorname{sec}h^2\left(\frac{1}{20}\left(\frac{10^{-4}t}{4} + x\right)\right)\right) \tag{22}$$

for $-5 \le x \le 5$, $0 \le t \le 5$

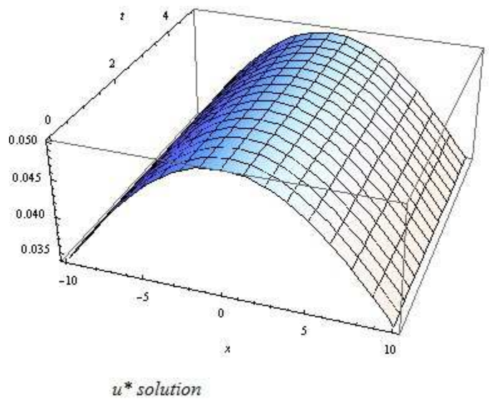

Fig (a)

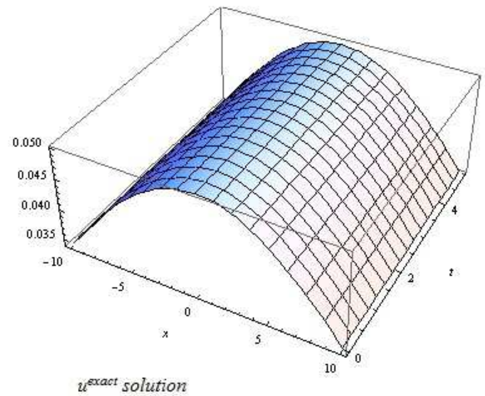

Fig (b)

Figure 1: Surface plot of the approximate solution Fig (a) and the exact solution Fig (b) respectively.

Table 1: Values of the approximate and exact solutions at some selected points. Here, error = $|Appr. - Exact|$

	$x = -0.5$			$x = 1.0$			$x = 2.5$		
$t \times 10^4$	u^*	u^{exact}	$error$	u^*	u^{exact}	$error$	u^*	u^{exact}	$error$
0	*0.0499531*	0.0499531	0	0.0498127	0.0498128	1E-07	0.0488401	0.0488402	E-07
1.0	*0.0483371*	0.0483375	4E-07	0.0495792	0.0495797	5E-07	*0.049*	*0.050*	0.001
2.0	*0.0446019*	0.0446023	4E-07	0.0470780	0.0470782	2E-07	0.0488401	0.0488402	1E-07
3.0	*0.0391722*	0.0391729	7E-07	0.0426040	0.0426043	3E-07	0.0455001	0.0455011	1E-06
4.0	*0.0326080*	0.0326082	2E-07	0.0366501	0.0366501	0	0.0403684	0.0403685	1E-07
5.0	*0.0254891*	0.0254895	4E-07	0.0297961	0.0297962	1E-07	0.0339833	0.0339836	1E-07

The graphs in Figure 1 and the values in Table 1 illustrate the exact and approximate solutions. Clearly it can be seen that the derived apptoximate solution with this method has very small and insignificant diffeence from the exaction solution.

6.2 Caudrey-Dodd-Gibbon Equation (CDG Equation) [7,10].

Consider the equation (13), where $A = 180$, $B = 30$, $C = 30$, $d = 1$ and $f(x) = -\frac{1}{1200}\left(1 - 3\sec h^2\left(\frac{x}{20}\right)\right)$ so that we have

$$u_t = 180u^2u_x + 30u_xu_{xx} + 30uu_{xxx} + u_{xxxxx} \tag{23}$$

subject to

$$u(x,0) = -\frac{1}{1200}\left(1 - 3\sec h^2\left(\frac{x}{20}\right)\right) \tag{24}$$

Hence equation (16) becomes

$$U(k+5) = \frac{v\cdot(k+1)U(k+1) - 180\sum_{i=0}^{k}A(i,k) - 30\sum_{i=0}^{k}B(i,k) - 30\sum_{i=0}^{k}\sum_{j=0}^{k-i}C(i,j,k)}{T(k)} \tag{25}$$

and the initial condition;

$$U(0) = \frac{1}{600} \tag{26}$$

From equation (26), we have;

$$U(1) = 0,; \; U(2) = -\frac{1}{80000},; \; U(3) = 0, \;\; U(4) = \frac{1}{4000000}$$

Hence we obtain from the recursive relation (16) the values of $U(k)$, $k = 0,1,2,\cdots$. To avoid complexity, we take $n = 10$ as in equation (12), and truncate the series at $O(x^9, t^9)$, hence factoring, we obtain the series;

$$
\begin{aligned}
u_*(x,t) = {} & 0.00166667 - 6.25E-6x^2 + 1.04167E-8x^4 - 1.46484E-11x^6 + 1.88192E-14x^8 \\
& + t^3(6.51042E-22x - 4.61155E-24x^3 + 1.68186E-26x^5 - 4.45876E-29x^7) \\
& + t^7(9.38537E-46x - 1.74336E-47x^3 + 1.37778E-49x^5 - 6.97931E^{-}52x^7) \\
& + t^5(-8.64665E-34x + 1.05116E-35x^3 - 5.85768E-38x^5 + 2.20389E-40x^7) \\
& + t(-3.125E-10x + 1.04167E-12x^3 - 2.21354E-15x^5 + 3.8239E-18x^7) \\
& + t^2(-3.90625E-15 + 3.90625E-17x^2 - 1.38346E-19x^4 + 3.36202E-22x^6 - 6.66278E-25x^8) \\
& + t^6(-3.60277E-39 + 1.31395E-40x^2 - 1.22035E-42x^4 + 6.42949E-45x^6 - 2.44185E-47x^8) \\
& + t^8(2.93293E-51 - 1.6344E-52x^2 + 2.15278E-54x^4 - 1.52685E-56x^6 + 7.51275E-59x^8) \\
& + t^4(4.06901E-27 - 8.64665E-29x^2 + 5.25581E-31x^4 - 1.95238E-33x^6 + 5.50385E-36x^8)
\end{aligned}
$$

This approximate solution obtained can be compared with the exact solution u^{exact} as illustrated with the surface plot in Fig 2, where the exact solution is given as;

$$
u(x,t) = -\frac{1}{1200}\left(1 - 3\sec h^2\left(\frac{1}{20}\left(\frac{10^{-4}t}{4} + x\right)\right)\right) \qquad (27)
$$

for $-5 \le x \le 5$, $0 \le t \le 5$

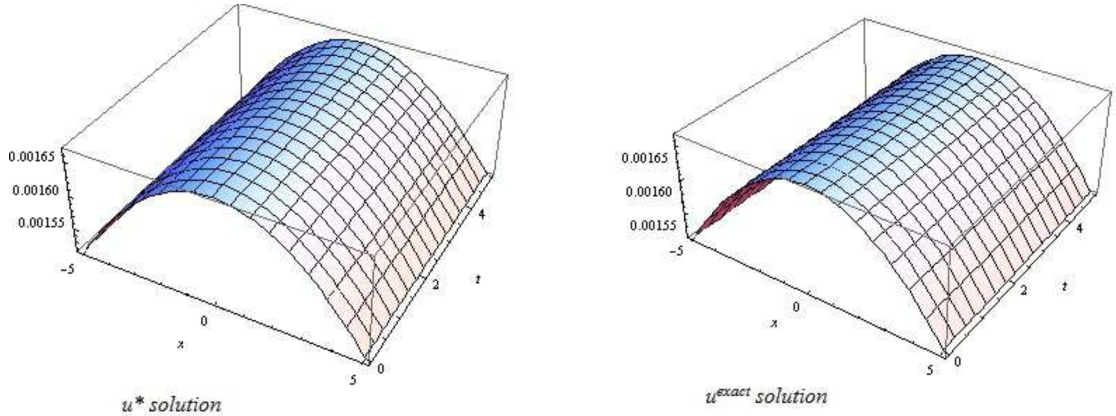

Fig (a) Fig (b)

Figure 2: Surface plot of the approximate solution Fig (a) and the exact solution Fig (b) respectively.

Table 2: Values of the approximate and exact solutions at some selected points. Here, error = $|Appr. - Exact|$

	$x = -0.5$			$x = 1.0$			$x = 2.5$		
$t \times 10^4$	u^*	u^{exact}	$error$	u^*	u^{exact}	$error$	u^*	u^{exact}	$error$
0	0.0016650	0.0016651	1E-07	0.00166041	0.00166043	2E-08	0.0016280	0.00162801	1E-08
1.0	0.00166625	0.00166628	3E-08	0.00165691	0.00165693	2E-08	0.00161997	0.00161999	2E-08
2.0	0.00166662	0.00166667	5E-08	0.00165259	0.00165266	7E-08	0.00161122	0.00161125	3E-08
3.0	0.00166623	0.00166628	5E-08	0.00164761	0.00164762	1E-08	0.0016017	0.0016018	1E-07
4.0	0.0016650	0.0016651	1E-07	0.00164182	0.00164183	1E-08	0.00159161	0.00159164	3E-08
5.0	0.00166312	0.00166315	3E-08	0.00163527	0.00163529	2E-08	0.0015804	0.0015808	4E-07

Similarly, the graphs in Figure 2 and the values in Table 2 illustrate the exact and approximate solutions. Clearly it can be seen that the derived apptoximate solution with this method has very small and insignificant diffeence from the exaction solution.

7.0 Conclusion

In this paper, the transformation of PDE into ODE using the wave variable has shown the effectiveness of the concept of differential transform method and its suitability for solving both linear and nonlinear differential equation. This method as applied has shown its proficiency, effectiveness and very high accuracy and is a very good tool for solving even higher order differential equations. It can be concluded that the use of wave variable as illustrated in this paper and the application of Differential Transform Method (DTM) is very powerful, efficient and less time consuming in finding the analytic series solutions for a wide class of (higher order) differential equations. The method gives more realistic series solutions that converge very rapidly in physical problems.

8.0 References

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