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A Tutorial Survey on the Fractional Order Derivatives of Exponential And Trigonometrical Functions

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Abstract

This paper reviewed the concept of fractional derivatives of exponential and Trigonometric functions. Some additional results on the Product and Quotient of both the fractional exponential and trigonometric functions were suggested. Proofs were presented with some examples.

1.0 Introduction

As it is well-known that Calculus provides the tools for describing motion quantitatively using the concept of differentiation and integration which are like inverse of each other as in the literature exampled in [1].

The answer to the question in differential and integral calculus that the derivative $\frac{d^n}{dx^n}$ of integer order n > 0, be extended to any fractional, irrational or complexorder is led to the development of a new theory which is called fractional calculus [1].The fractional calculus is a natural extension of the traditional calculus to real numbers.

As usual the derivative of a function $f(x)$ is given by

$$\frac{dy}{dx} = \lim_{\delta x \to o} \frac{f(x+\delta x)-f(x)}{\delta x} \qquad (1)$$

In the same way, if α is fraction$then$the α-derivative of $f(x)$ as defined in [2] is

$$f^{\alpha}(x) = \lim_{h\to 0} \frac{f^{\alpha}(x+h)-f^{\alpha}(x)}{(x+h)^{\alpha}-x^{\alpha}} \qquad (2)$$

This relation was also utilized in [2-6] among many others that were suggested in the literature. Similarly, utilizing L' Hospital Rule as in [2], the fractional derivative of $f(x)$ will be

$$f^{\alpha}(x) = \lim_{h\to 0} L\left(\frac{f^{\alpha}(x+h)-f^{\alpha}(x)}{(x+h)^{\alpha}-x^{\alpha}}\right)$$

2.0 Fractional Derivative of Exponential Function

Definition: Assume that $f(x): R \to Risafunction, \alpha \in R$ $(\alpha isfractionalnumber)$ then the fractional derivative of$f(x) = e^{x} is x^{1-\alpha} e^{\alpha x}$ as in [3]

1.1 Theorem: Fractional derivative of a function $f(x) = e^{nx} for n \in R is x^{1-\alpha} n e^{n\alpha x} where \alpha$ is fractional number.

Proof;

Using mathematical induction, we get following

For n =1

$$f(x) = e^{x}$$

From $f^{\alpha}(x) = \lim_{h\to 0} \frac{f^{\alpha}(x+h) f(x+h)^{\alpha-1}}{(x+h)^{\alpha-1}}$, where $\alpha isfractionalnumber$

$$f^{\alpha}(x) = \lim_{h\to o} \frac{\frac{d(e^{x+h})}{dh}(e^{x+h})^{\alpha-1}}{(x+h)^{\alpha-1}}$$

$$= \lim_{h\to 0} \frac{e^{x+h}(e^{x+h})^{\alpha-1}}{(x+h)^{\alpha-1}}$$

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$$=\frac{e^x(e^x)^{\alpha-1}}{x^{\alpha-1}}$$

$$f^\alpha(x)=x^{1-\alpha}e^{\alpha x}$$

Then assume it is true for n=k that is

$$iff(x)=e^{kx}then$$

$$f^\alpha(x)=x^{1-\alpha}ke^{k\alpha x}$$

For $n=k+1$

$$f^\alpha(x)=\lim_{h\to0}\frac{\frac{de^{(k+1)(x+h)}}{dh}\left(e^{(k+1)(x+h)}\right)^{\alpha-1}}{(x+h)^{\alpha-1}}$$

$$=\lim_{h\to0}\frac{(k+1)e^{(k+1)x+(k+1)h}\left(e^{(k+1)x+(k+1)h}\right)^{\alpha-1}}{(x+h)^{\alpha-1}}$$

$$\frac{(k+1)e^{(k+1)x}\left(e^{(k+1)x}\right)^{\alpha-1}}{x^{\alpha-1}}$$

$$x^{1-\alpha}(k+1)e^{(k+1)\alpha x} \qquad (4)$$

1.2 Theorem: Fractional derivative of a function $f(x)=\ e^{x^n}$ for n$>$ 0 is $f^\alpha(x)=\mathrm{n}x^{n-\alpha}e^{\alpha x^n}$, where αis fractional number.

Proof;

Using mathematical induction, we have For n=k+1

$$f^\alpha(x)=\lim_{h\to0}\frac{\frac{d\left(e^{(x+h)^{k+1}}\right)}{dh}(e^{(x+h)^{k+1}})^{\alpha-1}}{(x+h)^{\alpha-1}}$$

$$=\lim_{h\to0}\frac{(k+1)(x+h)^k e^{(x+h)^{k+1}}(e^{(x+h)^{k+1}})^{\alpha-1}}{(x+h)^{\alpha-1}}$$

$$=\frac{(k+1)x^k e^{x^{(k+1)}}e^{\alpha x^{(k+1)}}e^{-x^{(k+1)}}}{x^\alpha x^{-1}}$$

$$=(k+1)x^{(k+1)-\alpha}e^{\alpha x^{(k+1)}}$$

Particular Examples

(1) $f(x)=e^x and\alpha=\frac{2}{3}$

The $\frac{2}{3}th-$ derivative of e^x is $x^{\frac{1}{3}}e^{\frac{2}{3}}$

(2) $f(x)=e^{-2x}$

The $\frac{2}{5}th-$ derivative of $e^{-2x}$is $-2x^{\frac{3}{5}}e^{-\frac{4}{5}x}$

3.0 Fractional Derivatives of Trigonometric Functions

3.1 Theorem: Fractional derivative of a function $f(x)=cos(nx)for n\in R is f^\alpha(x)=-\frac{ntanxcos^\alpha(nx)}{x^{\alpha-1}}$ where α is a fractional number

Proof;

If n = 1 $f(x)=cosx$

From $f^\alpha(x)=\lim_{h\to0}\frac{f'(x+h)f(x+h)^{\alpha-1}}{(x+h)^{\alpha-1}}$, where α is fractional number.

$$f^\alpha(x)=\lim_{h\to0}\frac{\frac{d(\cos(x+h))}{dh}(\cos(x+h))^{\alpha-1}}{(x+h)^{\alpha-1}}$$

$$=\lim_{h\to0}\frac{-sin(x+h)\left(cos(k+h)\right)^{\alpha-1}}{(x+h)^{\alpha-1}}$$

$$=\frac{-sinx(cosx)^{\alpha-1}}{x^{\alpha-1}}$$

$$=\frac{-tanxcos^\alpha x}{x^{\alpha-1}}$$

Then we now assume it is true for n=k that is $f^\alpha(x)=-\frac{ktanxcos^\alpha(kx)}{x^{\alpha-1}}$

For n= k+1

$$f^\alpha(x)=\lim_{h\to0}\frac{\frac{d(\cos(\mathrm{k}+1)(x+h))}{dh}(\cos(\mathrm{k}+1)\,(x+h))^{\alpha-1}}{(x+h)^{\alpha-1}}$$

$= \lim_{h\to 0} \frac{-(k+1)\sin(k+1)(x+h)(\cos(k+1)(x+h))^{\alpha-1}}{(x+h)^{\alpha-1}}$

$= -\frac{(k+1)sin(k+1)x(cos(k+1)x)^{\alpha-1}}{x^{\alpha-1}}$

$= \frac{-(k+1)\tan(k+1)xcos^{\alpha}(k+1)x}{x^{\alpha-1}}$

3.2 Theorem: Fractional derivative of a function$f(x) = sin(nx) for n \in R is f^{\alpha}(x) = \frac{ncotxsin^{\alpha}(nx)}{x^{\alpha-1}}$ where α is a fractional number

If $f(x): R \to R$is a function defined by $f(x) = \tan nx$and α a fractional number then the fractional derivative of $f(x) = tanx$ is $f^{\alpha}(x) = \frac{(1+tan^2x)tan^{\alpha-1}(x)}{x^{\alpha-1}}$

3.3 Theorem: A function $f(x) = tan(nx)$ has the fractional derivative as $f^{\alpha}(x) = \frac{(n+ntan^2nx)tan^{\alpha-1}(nx)}{x^{\alpha-1}}$ when α is a fractional number and $n \in R$.

Proof;

By induction

For $n = k+1$

$f^{\alpha}(x) = \lim_{h\to 0} \frac{\frac{d(\tan(k+1)(x+h))}{dh}(\tan(k+1)(x+h))^{\alpha-1}}{(x+h)^{\alpha-1}}$

$= \lim_{h\to 0} \frac{(k+1)sec^2(k+1)(x+h)\left(tan(k+1)(x+h)\right)^{\alpha-1}}{(x+h)^{\alpha-1}}$

$= \frac{(k+1)(1+tan^2(k+1)x)(tan(k+1)x)^{\alpha-1}}{x^{\alpha-1}}$

$= \frac{\left((k+1)+(k+1)tan^2(k+1)x\right)(tan^{\alpha-1}(k+1)x)}{x^{\alpha-1}}$

Proved

3.4 Theorem: Fractional derivative of a function $f(x) = cos^n x$for n>0 $isf^{\alpha}(x) = \frac{-ntanx(cos^n x)^{\alpha}}{x^{\alpha-1}}$, where α is fractional number.

Proof,

To prove by induction

For n= k+1

$f^{\alpha}(x) = \lim_{h\to 0} \frac{\frac{d(\cos^{(k+1)}(x+h))}{dh}(\cos^{(k+1)}(x+h))^{\alpha-1}}{(x+h)^{\alpha-1}}$

$= \lim_{h\to 0} \frac{-(k+1)\sin(x)cos^k(x+h)(\cos^{(k+1)}(x+h))^{\alpha-1}}{(x+h)^{\alpha-1}}$

$= -\frac{(k+1)sinxcos^k x\left(cos^{(k+1)}x\right)^{\alpha-1}}{x^{\alpha-1}}$

$= \frac{-(k+1)tanx(cosx^{k+1}x)^{\alpha}}{x^{\alpha-1}}$

for the function $f(x) = sin^n x$has the fractional derivative when n>0 as$f^{\alpha}(x) = \frac{-ncotx(sin^n x)^{\alpha}}{x^{\alpha-1}}$ and the function $f(x) = tan^n x$ has the fractional derivative of$f^{\alpha}(x) = \frac{(n+ntan^2x)tan^{n\alpha-1}x}{x^{\alpha-1}}$, where α is fractional number.

Some Particular Examples

(1) The fractional derivative of the function $f(x) = cosx when \alpha = \frac{2}{3}$ is $tanx\sqrt[3]{xcos^2x}$

(2) The $\frac{2}{5} - derivative$of $sinx$is$cotx\sqrt[5]{xsin^2x}$

(3) The $f^{\frac{1}{3}}(x) of tanx$ is $\frac{(1+tan^2x)\sqrt[3]{x}}{\sqrt[3]{tanx}}$

In the same vein the product and quotient rules were established thus;

$If U(x) and V(x) are functions of x then the fractional derivative of thier product is given as$

$D^{\alpha}(UV) = U(x)U(x)^{\alpha-1}D^{\alpha}V(x) + V(x)V(x)^{\alpha-1}D^{\alpha}U(x)$.

Proof

From $f^{\alpha}(x) = \lim_{h\to 0} \frac{f'(x+h)f(x+h)^{\alpha-1}}{(x+h)^{\alpha-1}}$

$f(x) = U(x)V(x) then D^{\alpha}f(x) is$

$$D^{\alpha}f(x) = \lim_{h\to 0}\frac{\frac{d\left(U(x+h)V(x+h)\right)}{dh}\left(U(x+h)V(x+h)\right)^{\alpha-1}}{(x+h)^{\alpha-1}}$$
$$= \lim_{h\to 0}\frac{U(x+h)V'(x+h)(U(x+h)^{\alpha-1}V(x+h)^{\alpha-1}) + U'(x+h)V(x+h)(U(x+h)^{\alpha-1}V(x+h)^{\alpha-1})}{(x+h)^{\alpha-1}}$$
$$= U(x)U(x)^{\alpha-1}\lim_{h\to 0}\frac{V'(x+h)V(x+h)^{\alpha-1}}{(x+h)^{\alpha-1}} + V(x)V(x)^{\alpha-1}\lim_{h\to 0}\frac{U'(x+h)U(x+h)^{\alpha-1}}{(x+h)^{\alpha-1}}$$
$$D^{\alpha}f(x) = U(x)U(x)^{\alpha-1}D^{\alpha}V(x) + V(x)V(x)^{\alpha-1}D^{\alpha}U(x)$$

For example $f(x) = e^{x}sinx then D^{\alpha}f(x) is$

$let U = e^{x} and V = sinx$

$from D^{\alpha}f(x) = UU^{\alpha-1}D^{\alpha}V + VV^{\alpha-1}D^{\alpha}U$

$$= e^{x}e^{\alpha x-x}\frac{(cosxsin^{\alpha-1}x)}{x^{\alpha-1}} + sinxsin^{\alpha-1}x\frac{(e^{x}e^{\alpha x-x})}{x^{\alpha-1}}$$
$$= \frac{e^{\alpha x}sin^{\alpha}x(1+cosxsin^{-1}x)}{x^{\alpha-1}}$$
$$= \frac{e^{\alpha x}sin^{\alpha}x(1+tanx)}{x^{\alpha-1}}$$

Similarly, the fractional derivative of their quotient will be given as follows:

$$D^{\alpha}f(x) = \frac{1}{V(x)^{\alpha}}D^{\alpha}U(x) - \frac{U(x)^{\alpha}}{V(x)^{2\alpha}}D^{\alpha}V(x)$$

Proof

From $f^{\alpha}(x) = \lim_{h\to 0}\frac{f'(x+h)f(x+h)^{\alpha-1}}{(x+h)^{\alpha-1}}$

$$D^{\alpha}f(x) = \lim_{h\to 0}\frac{\frac{d\left(\frac{U(x+h)}{V(x+h)}\right)}{dh}\left(\frac{U(x+h)}{V(x+h)}\right)^{\alpha-1}}{(x+h)^{\alpha-1}} = \lim_{h\to 0}\frac{\frac{V(x+h)U'(x+h)-U(x+h)V'(x+h)}{\left(V(x+h)\right)^{2}}\left(\frac{U(x+h)^{\alpha-1}}{V(x+h)^{\alpha-1}}\right)}{(x+h)^{\alpha-1}}$$
$$= \lim_{h\to 0}\left(\frac{\frac{V(x+h)U'(x+h)U(x+h)^{\alpha-1}}{U(x+h)^{\alpha+1\alpha}} - \frac{U(x+h)V'(x+h)U(x+h)^{\alpha-1}}{V(x+h)^{\alpha+1}}}{(x+h)^{\alpha-1}}\right).\frac{V(x+h)^{\alpha-1}}{V(x+h)^{\alpha-1}}$$
$$= \lim_{h\to 0}\frac{U'(x+h)U(x+h)^{\alpha-1}}{(x+h)^{\alpha-1}} - \frac{U(x)^{\alpha}}{V(x)^{2\alpha}}\lim_{h\to 0}\frac{V'(x+h)V(x+h)^{\alpha-1}}{(x+h)^{\alpha-1}}$$

$$D^{\alpha}f(x) = \frac{1}{V(x)^{\alpha}}D^{\alpha}U(x) - \frac{U(x)^{\alpha}}{V(x)^{2\alpha}}D^{\alpha}V(x)$$

For example$f(x) = \frac{1}{sinx} then D^{\alpha}f is D^{\alpha}f(x) = \frac{1}{sin^{\alpha}x}.0 - \frac{1}{sin^{2\alpha}x}.\frac{cosxsin^{\alpha-1}x}{x^{\alpha-1}} = \frac{-cosxsin^{-(1+\alpha)}x}{x^{\alpha-1}}$

4.0 Conclusion

Fractional derivativesof exponential and trigonometric functions were surveyed. Some theorems are proved and some numerical examples were discussed based on the proposed product and quotient of the functions. It is hope that these will be utilized as tools in solving complex problems in applications..

5.0 References

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