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Stochastic Analysis of Heterosexual Transmission of HIV/AIDS Epidemic in The Presence of Treatment

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Abstract

HIV/AIDS epidemic proves to be a deadly disease until recently when the treatment is being introduced. This paper addresses the scope of HIV/AIDS with respect to Heterosexual mode of transmission extensively with the introduction of treatment. The model was develop by solving the Chapman - Kolmogorov differential equation using birth – death process. Solving the model using probability Generating Functions (PGF) approach, the expectations and variance of probability distribution resulting from the model were obtained for Susceptible (S) persons, Infected (I) persons , Treatment (T) and AIDS (A) cases. Sensitivity analysis was carried out to investigate the influence of key parameters of the model on the spread of the disease.

Keywords: Probability Generating Function, HIV Transmission, Stochastic compartmental model, Heterosexual Transmission and Treatment.

1.0 Introduction

The human immunodeficiency virus (HIV) infection which often leads to acquired immunodeficiency syndrome (AIDS), has become a hazardous infectious disease in both the developed and developing nations. The disease break down the infected individual's immune system, leaving the victim vulnerable to a host of life threatening opportunistic infections, neurological disorders or unusual malignancies. It is a fatal disease and it has caused mortality of millions of people. Also, the threat of the disease has necessitated the expenditure of enormous amount of money in health care delivery and disease control [1].

Heterosexual is the group that is sexually mature and active, and therefore capable of reproduction, i.e the group that is responsible for the horizontal transmission of the HIV virus through sexual activities and for vertical transmission by infected mothers to their children [2]. By all indications the HIV/AIDS epidemic has continued to grow largely through Heterosexual unprotected sexual exposures [3].

Treatment is the process of offering the HIV positive individual with a life prolonging drugs/medicines known as antiretroviral (ARV) medicines or antiretroviral therapy (ART). ART drugs are the main types of treatment for HIV/AIDS. It is not a cure but it delays the onset of AIDS in the patients, thus enabling them to live longer than they would have done without the drugs. The therapy consists of drugs that have to be taken every day for the rest of patient's life. Treatment with anti-retroviral increases the life expectancy of people infected with HIV, even after HIV has progressed to diagnosable AIDS. The average survival time with antiretroviral therapy was estimated to be more than 5 years as at 2005 [4]

Olowofeso et al [5] studied MTCT transmission of virus in a varying population they developed SIA model, treatment not inclusive. In this research we adopt the models in [5] and incorporate treatment on the model for the mode of transmission. This is important because ARV treatment now plays a major role on the spread and progression of the disease.

2.0 The Model Formulation

We intend to propose a simple HIV/AIDS model with treatment class for Heterosexual mode of transmission. In this model, the sexually mature population is divided into four compartments: the Susceptible, the Infectives (also assumed to be Infectious), the Treatment and the AIDS population whose numbers are denoted by S,I,T and A respectively. The number of total population is denoted by N(t), at any time t i.e N=S+I+T+A. In the model, we assume that the Susceptibles become HIV

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infected via sexual contacts with infectives which may also lead to birth of infected children at the rate $w\delta c$, it is also assumed that population that are free of HIV from the Mother – to – child and the survivors of this subgroup over the development period (5,15) joing the susceptible group with the infant survival rate q and p the proportion of children that survive after age 15 years, we also assume that some of the infectives move to join treated class with the rate σ_2 , while others with serious infection directly join the AIDS class with a rate σ_1. The proposed schematic diagram is presented in Fig. 1.

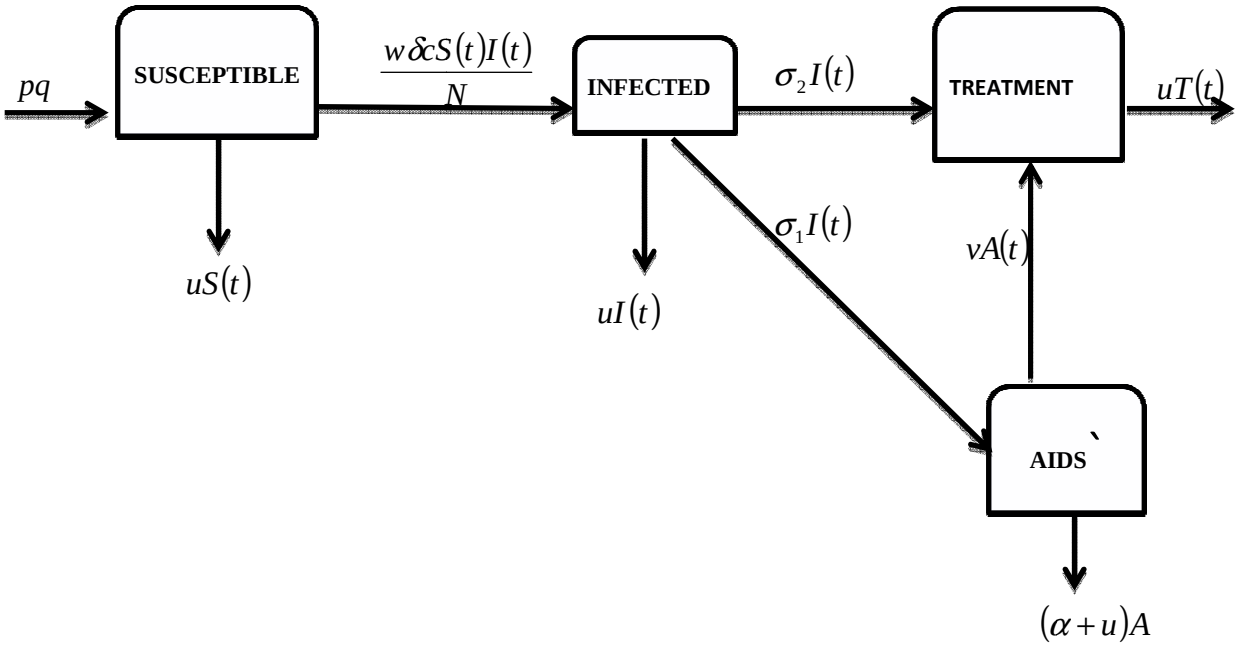

Fig.1: Proposed schematic diagram for Heterosexual mode of transmission of HIV/AIDS using Birth – Death process.

2.1 Notation for the Models

c – Average number of sexual partners per unit time.
α - Disease – induced death rate due to AIDS.
σ_1 - Rate of movement from infectious class to AIDS class.
σ_2 - Rate of movement from infectious class to treatment class.
u - Natural mortality rate.
v - Rate at which AIDS group get treatment
w - Sexual contact rate between a sexually mature S person and I person.
β - The contact rate of the epidemic.
p -The proportion of children that survive after age 15 years.
q – Infant Survival rate.

3.0 Method of Solution and Model Analysis

let
$p_n(t) =$ probability that exactly $"n"$ population are in the system at time (t)
$p_{n+1}(t) =$ probability that exactly $"n+1"$ population are in the system at time (t)
$p_{n-1}(t) =$ probability that exactly $"n-1"$ population are in the system at time (t)

$p_n(t+\Delta t)$ = probability that exactly "n" population are in the system at time $(t+\Delta t)$

Probability of no birth $= 1-(n\lambda\Delta t + o\Delta t)$

Probability of no death $= 1-(n\mu\Delta t + o\Delta t)$

Probability of birth $= n\lambda\Delta t + o\Delta t$

Probability of death $= n\mu\Delta t + o\Delta t$

Using the probability rule , we arrive at

$$p_n(t+\Delta t) = (1-n\lambda\Delta t + o\Delta t)(1-n\mu\Delta t + o\Delta t)p_n + p_{n+1}(1-n\lambda\Delta t + o\Delta t)(n\mu\Delta t + o\Delta t) + p_{n-1}(1-n\mu\Delta t + o\Delta t)(n\lambda\Delta t + o\Delta t) + p_n(n\mu\Delta t + o\Delta t)(n\lambda\Delta t + o\Delta t) \quad (3.1)$$

Simplifying and neglecting all $(o\Delta t)$, divide through by Δt and limit $\Delta t \to 0$, we obtained

$$p_n^{'} = -p_n(n\lambda + n\mu) + p_{n+1}(n\mu) + p_{n-1}(n\lambda)$$

Therefore ,

$$\frac{dp_n(t)}{dt} = -(n\lambda + n\mu)p_n(t) + (n\mu)p_{n+1}(t) + (n\lambda)p_{n-1}(t) \quad (3.2)$$

SUSCEPTIBLE

The change in population size of this class during the time interval (t, t + Δt) is governed by the following conditional probabilities.

$$pr(S(t,t+\Delta t) = n+1/S(t)) = pq + o\Delta(t) \quad (3.2a)$$

$$pr(S(t,t+\Delta t) = n-1/S(t)) = (w\delta cI + u)S + o\Delta(t) \quad (3.2b)$$

$$pr(S(t,t+\Delta t) = n/S(t)) = 1 - pq - (w\delta cI + u)S - o\Delta(t) \quad (3.2c)$$

Substituting equation (3.2a), (3.2b) and (3.2c) into equation (3.2), we obtain Kolmogorov differential equation

$$\frac{dp_n(t)}{dt} = -(pq + (w\delta cI + u)S)p_n(t) - (n+1)(w\delta cI + u)Sp_{n+1}(t) + (n-1)pqp_{n-1}(t) \quad (3.3)$$

Solving (3.3) using probability generating function approach, we obtained

$$\frac{\partial G(z,t)}{\partial t} = -(pq + (w\delta cI + u)S)z\frac{\partial G(z,t)}{\partial z} + (w\delta cI + u)S\left[\frac{\partial G(z,t)}{\partial z} - p_1(t)\right] + pqz^n\frac{\partial G(z,t)}{\partial z} \quad (3.4)$$

Theorem 3.1.

For any initial state m at time $s, s < t$ *and* $|z| < 1$ then probability generating function $G(z,t)$ satisfies the relation

$\frac{\partial G(z,t)}{\partial t} = \sum z^n p(n,t|m,s)\{(z-1)\lambda_n(t) + (z^{-1}-1)\mu_n(t)\}$ (3.5)

Using theorem 3.1, equation (3.4) became,

$$\frac{\partial G(z,t)}{\partial t} = \frac{\partial G(z,t)}{\partial z}\{(z-1)[zpq + (w\delta cI + u)S]\} \quad (3.6)$$

Finding the general solution of the homogeneous partial differential equation we obtained

$$\frac{\partial G(z,t)}{\partial t} - \{(z-1)[zpq + (w\delta cI + u)S]\}\frac{\partial G(z,t)}{\partial z} = 0 \quad (3.7)$$

With initial condition

$$G(z,0) = z^m$$

The auxiliary equation for the equation (3.7) is

$$\frac{dt}{1} = \frac{\partial G(z,t)}{\{(1-z)[zpq + (w\delta cI + u)S]\}} = \frac{dG}{l} \quad (3.8)$$

The two solutions for problem in equation (3.8) is as follows;

$$\frac{dt}{1} = \frac{\partial G(z,t)}{\{(1-z)[zpq + (w\delta cI + u)S]\}} \quad (3.9)$$

And

$$\frac{dt}{1}=\frac{dG}{l} \tag{3.10}$$

Simplifying (3.10), letting $l=0$ then integrate

$$\int dG(z,t)=\int 0dt \tag{3.11}$$

$$G(z,t)=k$$

Also considering (3.9) ,Solving by separation of variables and simplifying, we obtained the distribution of the model as

$$G_s(z,t)=\left(\frac{(w\delta cI+u)S\left(e^{[pq-(w\delta cI+u)S]t}-1\right)+z\left(pq-(w\delta cI+u)Se^{[pq-(w\delta cI+u)S_1]t}\right)}{zpq\left(1-e^{[pq-(w\delta cI+u)S]t}\right)+pqe^{[pq-(w\delta cI+u)S]t}-(w\delta cI+u)S}\right)^m \tag{3.12}$$

Considering Bailey [6], differentiating (3.12) with respect to z and letting z=1, we obtain expectation as

$$E(G(z,t))=e^{[pq-(w\delta cI+u)S]t} \tag{3.13}$$

and Variance as

$$V(G(z,t))=\left(\frac{pq-(w\delta cI+u)S}{pq+(w\delta cI+u)S}\right)\ell^{(pq-(w\delta cI+u)S)t}\left(\ell^{(pq-(w\delta cI+u)S)t}-1\right) \tag{3.14}$$

INFECTED MODEL

The probability that there are n individuals in the infective population during the time interval $\{t,t+\Delta t\}$ is equal to probability

1 That there are " n" individuals by time t and nothing happens during the time interval $\{t,t+\Delta t\}$.

2 That there are "n – 1" individuals by time t and 1 is added by HIV transmission, immigration or Mother to child transmission during the time interval $\{t,t+\Delta t\}$.

3 That there "n + 1" individuals by time t and 1 dies or converts to AIDS during the time interval $\{t,t+\Delta t\}$.

The change in population size during the time interval (t, t + Δt) is governed by the following conditional probabilities.

$$pr(I(t,t+\Delta t)=n+1/I(t))=w\delta cI+o\Delta(t) \tag{3.14a}$$

$$pr(I(t,t+\Delta t)=n-1/I(t))=(u+\sigma_2+\sigma_1)I+o\Delta(t) \tag{3.14b}$$

$$pr(I(t,t+\Delta t)=n/I(t))=1-w\delta cI-(u+\sigma_2+\sigma_1)I-o\Delta(t) \tag{3.14c}$$

Substituting equation (3.14a), (3.14b) and (3.14c) into equation (3.2), we obtain Kolmogorov differential equation

$$\frac{dp_n(t)}{\partial t}=-n(w\delta cI+(u+\sigma_2+\sigma_1)I)p_n(t)+n+1((u+\sigma_2+\sigma_1)I)p_{n+1}(t)+(n-1)[w\delta cI]p_{n-1}(t) \tag{3.15}$$

The resulting probability distribution of (3.15) using probability generating function approach is

$$G_I(z,t)=\left(\frac{zw\delta cI-(u+\sigma_2+\sigma_1)I+(u+\sigma_2+\sigma_1)I(1-z)e^{(w\delta cI-(u+\sigma_2+\sigma_1)I)t}}{zw\delta cI-(u+\sigma_2+\sigma_1)I+w\delta cI(1-z)e^{(w\delta cI-(u+\sigma_2+\sigma_1)I)t}}\right)^m \tag{3.16}$$

Considering Bailey [6], differentiating (3.16) with respect to z and letting z = 1, we obtained expectation as

$$E(G(z,t))=e^{[w\delta cI-(u+\sigma_2+\sigma_1)I]t} \tag{3.17}$$

and Variance as

$$V(G(z,t))=\left(\frac{w\delta cI-(u+\sigma_2+\sigma_1)I}{w\delta cI+(u+\sigma_2+\sigma_1)I}\right)\ell^{(w\delta cI-(u+\sigma_2+\sigma_1)I)t}\left(\ell^{(w\delta cI-(u+\sigma_2+\sigma_1)I)t}-1\right) \tag{3.18}$$

TREATMENT MODEL

$$pr(T(t,t+\Delta t)=n+1/T(t))=\sigma_2 I+\nu A+o\Delta(t) \tag{3.18a}$$

$$pr(T(t,t+\Delta t)=n-1/T(t))=(u)T+o\Delta(t) \tag{3.18b}$$

$$pr(T(t,t+\Delta t)=n/T(t))=1-(\sigma_2 I+\nu A)-(u)T-o\Delta(t) \tag{3.18c}$$

Substituting equation (3.18a), (3.18b) and (3.18c) into equation (3.2), we obtain Kolmogorov differential equation

$$\frac{\partial G(z,t)}{\partial t}=-n(\sigma_2 I+\nu A+(u)T)p_n(t)+n+1((u)T)p_{n+1}(t)+n-1(\sigma_2 I+\nu A)p_{n-1}(t) \tag{3.19}$$

The resulting probability distribution of (3.19) using probability generating function approach is.

$$G_T(z,t)=\left(\frac{z(\sigma_2 I+\nu A)-(u)T+(k+u)T(1-z)e^{((\sigma_2 I+\nu A)-(u)T)t}}{z(\sigma_2 I+\nu A)-(u)T+(\sigma_2 I+\nu A)(1-z)e^{((\sigma_2 I+\nu A)-(u)T)t}}\right)^m \tag{3.20}$$

Differentiating (3.20) with respect to z and lettingz=1, and simplifying we obtain expectation as

$$E(G(z,t))=e^{[(\sigma_2 I+\nu A)-(u)T]t} \tag{3.21}$$

and Variance as

$$V(G(z,t))=\left(\frac{(\sigma_2 I+\nu A)-(u)T}{(\sigma_2 I+\nu A)+(u)T}\right)\ell^{((\sigma_2 I+\nu A)-(u)T)t}\left(\ell^{((\sigma_2 I+\nu A)-(u)T)t}-1\right) \tag{3.22}$$

AIDS MODEL

The change in population size of this class during the time interval (t, t + Δt) is governed by the following conditional probabilities.

$$pr(A(t,t+\Delta t)=n+1/A(t))=\sigma_1 I+o\Delta(t) \tag{3.22a}$$

$$pr(A(t,t+\Delta t)=n-1/A(t))=(\alpha+\nu+u)A+o\Delta(t) \tag{3.22b}$$

$$pr(A(t,t+\Delta t)=n/A(t))=1-(\sigma_1 I)-(\alpha+\nu+u)A-o\Delta(t) \tag{3.22c}$$

Substituting equation (3.22a), (3.22b) and (3.22c) into equation (3.2), we obtain Kolmogorov differential equation

$$\frac{dp_n(t)}{dt}=-n(\sigma_1 I+(\alpha+\nu+u)A)p_n(t)+n+1((\alpha+\nu+u)A)p_{n+1}(t)+n-1(\sigma_1 I)p_{n-1}(t) \tag{3.23}$$

The resulting probability distribution of (3.23) using probability generating function approach is.

$$G_A(z,t)=\left(\frac{z(\sigma_1 I)-(\alpha+\nu+u)A+(\alpha+\nu+u)A(1-z)e^{((\sigma_1 I)-(\alpha+\nu+u)A)t}}{z(\sigma_1 I)-(\alpha+\nu+u)A+(\sigma_1 I)(1-z)e^{((\sigma_1 I)-(\alpha+\nu+u)A)t}}\right)^m \tag{3.24}$$

Differentiating (3.24) with respect to z and letting z= 1, and simplifying, we obtain expectation as

$$E(G(z,t))=e^{[(\sigma_1 I)-(\alpha+\nu+u)A]t} \tag{3.25}$$

and Variance as

$$V(G(z,t))=\left(\frac{(\sigma_1 I)-(\alpha+\nu+u)A}{(\sigma_1 I)+(\alpha+\nu+u)A}\right)\ell^{((\sigma_1 I)-(\alpha+\nu+u)A)t}\left(\ell^{((\sigma_1 I)-(\alpha+\nu+u)A)t}-1\right) \tag{3.26}$$

4.0 Result and Discussion

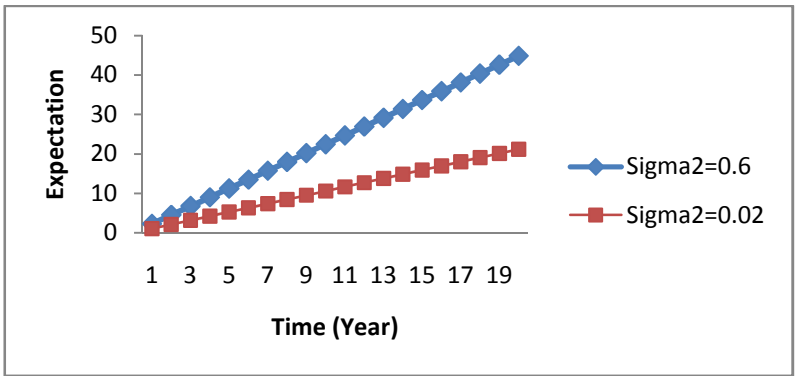

Fig.4.1: Expectation plot of Treatment with varying value of σ_2 *when* $I(0)=1.793,\ \sigma_1=0.004, v=0.3, u=0.03$

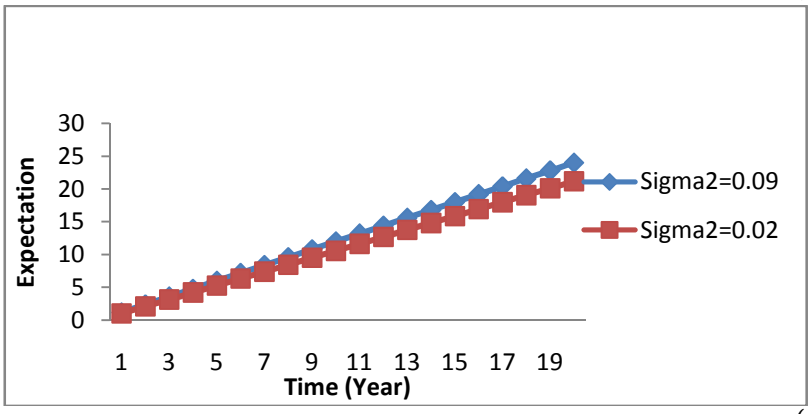

Fig.4.2: Expectation plot of Treatment with varying value of σ_2 *when* $I(0)=1.793,\ \sigma_1=0.004, v=0.3, u=0.03$

Figure 4.1 and 4.2 explained the expected population in treated class. From Figure 4.1, when σ_2 is 0.6 at year 13, the expected number treatment population is 29.15 while the expected number treated population is 13.77 when σ_2 is 0.09, the expected population is 15.61 at the same point. With varying value of σ_2, it was observed that as σ_2 increases the expected population of treated class is also increasing, which likely result to decrease in expected population in the infected class. This is as a result of making ARVs more available to the society. So, this can serve as a control measure for Heterosexual transmission.

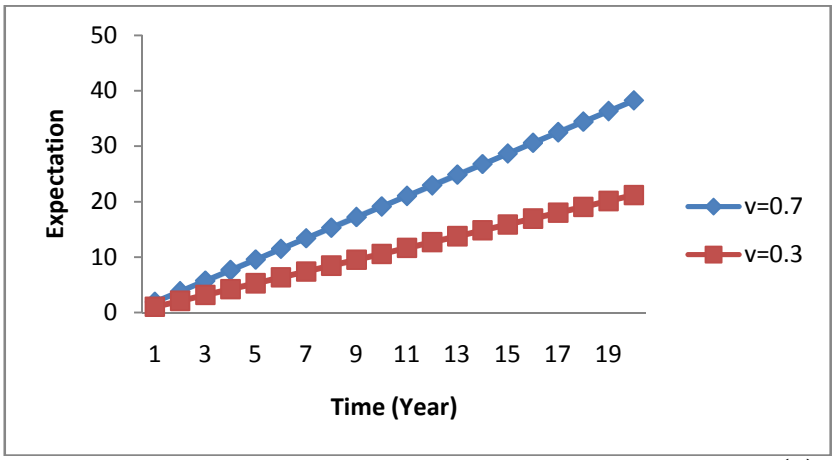

Fig.4.3: Expectation plot of Treatment with varying value of v *when* $I(0)=1.793,\ \sigma_1=0.004,\ u=0.03$

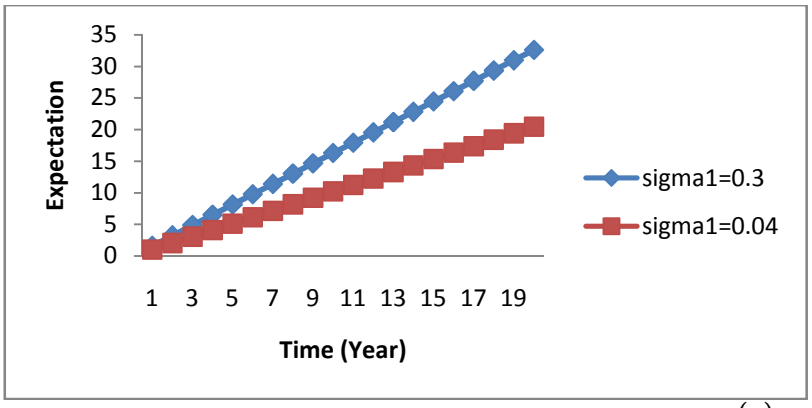

Fig.4.4: Expectation plot ofAIDS with varying value of σ_1 *when* $I(0)=1.793, \alpha=0.1, v=0.3, u=0.03$

From Fig.4.3 and Fig 4.4, we discovered that as the rate at which in the AIDS class get treatment increases, the expected population of treated class increases in Fig.4.3 which is likely to be as a result of increase in public awareness of HIV/AIDS and treatment . Also from Fig. 4.4 we can see that as σ_1 is increases the expectation of population of AIDS class is increase, when $\sigma_1 = 0.3$ the expected population is 16.29 and also at 0.04 the expected population is 10.23, this may be as a result of nonchalant attitude or fear of stigmatization that make the infected people not to go for HIV screening and treatment on time before it get to chronic stage.

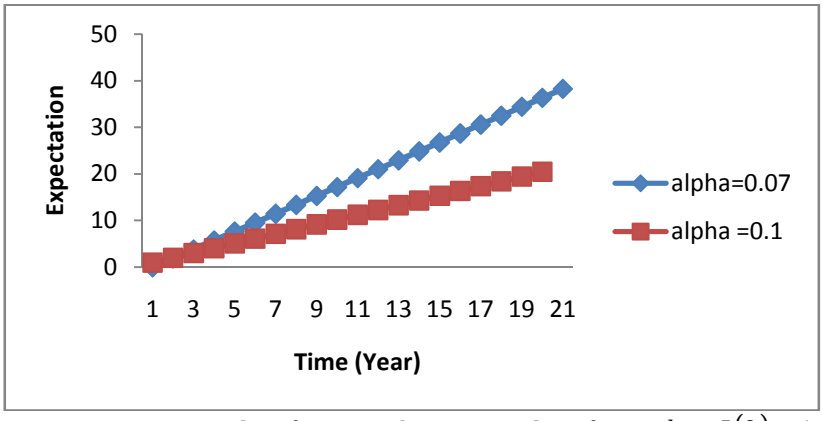

Fig.4.5: Expectation plot ofAIDS with varying value of α, *when* $I(0)=1.793, \alpha=0.1, v=0.3, u=0.03, \sigma_1=0.004$

We can see from Fig. 4.5 that as the value of disease induce death (α) increasing the population of AIDS class is decreasing, from the graph it was observed that at year 2 there is equilibrium point for the two value of α but at the later years the population was decreasing as the α is increasing.

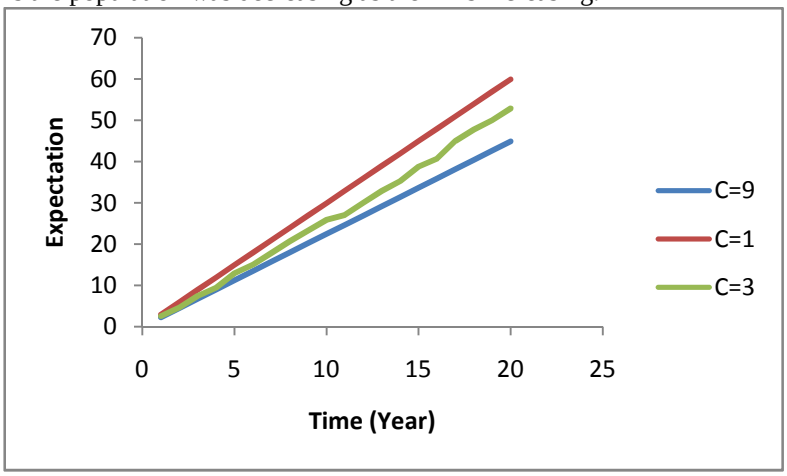

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Fig.4.6: Expectation plot of suceptible with varying value of C when $T(0)=1.7449, v=0.3, u=0.03, \sigma_1=0.004, q=0.9, \sigma_2=0.04$

It can be seen from Fig. 4.6 that as Sexual contact increases, the Susceptible decreases which could be resulted to increase in the Infected and Treated class as a result of loss in immunity. It can be concluded that as C increases the infective also increase. Thus it can be concluded that, in order to reduce the spread of the disease, the number of sexual partners as well as unsafe sexual interaction with an infective should be restricted.

5.0 Conclusion

In this research, stochastic model for the HIV/AIDS was presented compartmentally for Heterosexual transmission. The expectation and the variance of the models were then obtained. Sensitivity analysis was carried out to investigate the potential impact of treatment on disease progression. The analyses show that an increase in the treatment rate results in an expected decline of new disease incidences.

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