Journal of the Nigerian Association of Mathematical Physics
***Volume 31*, (July, *2015*), pp 57 – 60**

Comparison of Numerical Algorithms for Solving Non-Linear Equations

Abdullahi Salisu

**Department of Computer Science, Niger State Polytechnic,Zungeru,
Niger State, Nigeria.**

Abstract

There are no any general formulas for solving or finding the roots of Polynomial equations of higher degrees (higher than 4). Some polynomial equations can be solvable in some straight forward manner but most polynomials (in real life) aren't solvable. Instead we find the numerical approximation to some degree of accuracy. In this paper, we propose some new computational algorithms from the existing classical methods. The proposed method has been illustrated with some Numerical examples. The Numerical results obtained indicate that the new computational algorithms provide the good performance of iterations by reducing the number of iterations when compared with the Classical methods.

Keywords: Bisection Method, Regular Falsi Method, Nonlinear Equation, Numerical Examples.

1.0 Introduction

The use of simple operations to find the approximate solutions to complex problems constitutes the main goal of Numerical Analysis [1].
The problem of finding the approximation to the root of a non-linear equation can be found in many fields of science and Engineering [2-4]. A root finding Algorithm is a Numerical method or algorithm for finding a value x such that $f(x) = 0$ for a given function f such that x is called a root of the function$f$. The goal of this paper is to investigate the problem of finding the zero's of a non-linear equation

$f\,(x) = 0 \; . \; . \; . \; . (1)$

In this paper, we construct and propose new improve Bisection method (IBM)by introducing the equation of a straight line $y = mx + c$ into existing method of bisection,[2-4]. However, we also introduce the principle of Predictor-Corrector Method in modifying the Regular Falsi Method, such that the classic Regular Falsi Method (RFM) will be the predictor method and the New Modified Regular Falsi Method (MRFM) will be the corrector method. Many researchers have considered some predictor-corrector methods[5-6].

2.0 Review of the Classical Numerical Algorithms

2.1 Bisection Method (BM)

Let the function $y = f\,(x)$ be a continuous on an interval containing a and b, $a < b$, and such that $f(b)f(a) < 0$. Then $f\,(x) = 0$ has atleast one solution c, $a < c < b$. That is , we known from the intermediate value theorem, that a continues function has a root where $y$value changes sign from $a\ to\ b$; in particular, an odd-order polynomial always has atleast one real root [3].
We decide in which interval $[a, b]$ the root lies by computing f at the midpoint of interval, and picking that intermediate point where it differs in sign from either$a\ or\ b$.
After nth iterations, the root c′ is within the interval $[a_n, b_n]$ where the Error from the exact root c is

$$E = |c' - c| < \frac{b-a}{2^n} \quad . \quad . \quad . \quad . (2)$$

Corresponding author: Abdullahi Salisu, E-mail: saddullah2k3@yahoo.com, Tel.: +2348035925783

2.2 Regular Falsi Method (RFM)

Given the same condition as the bisection method above, if we connect the points $(a, f\ (a))\ and\ (b, f(b))$ by a straight line, then the x-intercept of line is a close approximation to the roots C of the equation $f\ (x) = 0, a < C < b$.
To find C

$$\frac{c-a}{f(a)} = \frac{b-c}{f(b)} \quad . \quad . \quad . \quad (3)$$

$$c = \frac{af(b) - bf(a)}{f(b) - f(a)} .. \quad . \quad . \quad .(4)$$

This process may be iterated by using "C" as the endpoint of a new interval, where a new line may be drawn and a new x-intercept found. Each new intercept usually add a new decimal place to the approximation.

2.3 Improved Bisection Method (IBM)

Let f be a continues function and define on $[a, b]$which$f\ (a)f(b) < 0$. Firstly, we set $a = a_1\ and\ b = b_1$ for an integer $k \geq 1$, by the bisection method, we have

$$C_k = \frac{a_{k+b_k}}{2}$$

Next, we consider a new interval $(a_k^*,\ b_k^*)$

$$(a_k^*, b_k^*) = (a_{k,} c_k\ )\ if\ \ f\ (a_{k,}) f(c_k) < 0 \ . \ . \ . \ . \ (5a)$$

$$(a_k^*, b_k^*) = (c_{k,} b_k\ )\ if\ \ f\ (c_{k,}) f(b_k) < 0 \ . \ . \ . \ . \ (5b)$$

Then, we can find the equation of a straight line from the points $((a_k^*, f(\ a_k^*))\ , ((b_k^*, f(\ b_k^*))$ as follows

$$y = mx + c \ . \ . \ . \ .(6)$$

Where

$$m = \frac{f(b_k^*) - f(a_k^*)}{b_k^* - a_k^*} . \ . \ . \ .(7)$$

And

$$C = f(b_k^*) - m. b_k^* \quad or\ \ C = \ f(a_k^*) - m. a_k^* \ . \ . \ . \ .(8)$$

Hence, the x-intercept of the straight line is at point

$$x_k = -\frac{c}{m} . \ . \ . \ .(9)$$

$$x_k = b_k^* - f(b_k^*\ ) . \frac{b_k^* -\ a_k^*}{f(b_k^*) - f(a_k^*)} \ . \ . \ . \ .(10)$$

Or

$$x_k = a_k^* - f(a_k^*\ ) . \frac{b_k^* -\ a_k^*}{f(b_k^*) - f(a_k^*)} \ . \ . \ . \ .(11)$$

Finally, we choose the new the new subinterval for the next iteration as follows:

$$(a_{k+1}, b_{k+1}) = \begin{cases} a_k^*, x_k\ , & if\ \ f\ (a_k^*) f(x_k) < 0 \\ x_{k,}, b_k^*\ , & if\ \ f\ (x_{k,}) f(b_k^*) \geq 0 \end{cases} \ . \ . \ . \ . \ (12)$$

The process is continued until the interval is sufficiently small or the approximate solution is sufficiently close to the exact solution.

Theorem 2.1

Let f be a continuous function and define on [a,b] which $f(a)f(b) < 0$. The modified bisection method generates a sequence $\{x_n\}_{n=1}^{\infty}$ with $a_k < x_k < b_k\ ; for\ k\ \geq 1$

Proof: Since$f\ (a)f(b) < 0$, hence we separate to two cases

CASE 1: $f(a_k) < 0\ \ and\ f(b_k\ ) > 0$

Consider a subinterval (a_k^*, b_k^*) in equation

(i) $if\ f(a_k)f(c_k\ ) < 0$, then we have $a_k^* = a_k\ ,\ b_k^* = c_k\ and\ f(b_k^*) > 0$ so we have

$$f(b_k^*\ ) . \frac{b_k^* -\ a_k^*}{f(b_k^*) - f(a_k^*)} > 0 \ . \ . \ . \ . \ . \ (13)$$

Then

$$x_k = b_k^* - f(b_k^*\ ) . \frac{b_k^* -\ a_k^*}{f(b_k^*) - f(a_k^*)} < b_k^* < b_k \ . \ . \ . \ .(14)$$

Since

$$f(a_k^*\ ) . \frac{b_k^* -\ a_k^*}{f(b_k^*) - f(a_k^*)} < 0 \ . \ . \ . \ .(15)$$

Therefore

$$x_k = a_k^* - f(b_k^*\ ) .\frac{b_k^* - a_k^*}{f(b_k^*) - f(a_k^*)} > a_k^* = a_k \ . \ . \ . \ . (16)$$

Hence,

$$a_k < x_k < b_k$$

(ii) $if\ f(c_k)f(b_k\ ) < 0$, then we have $a_k^* = c_k\ ,\ b_k^* = b_k\ and\ f(a_k^*) < 0$, the proof is similarly.

CASE 2: $f(a_k) < 0\ and\ f(b_k\ ) > 0$

This proof is rather similar to the case 1 above.

2.4 Modified Regular Falsi Method (MRFM)

Suppose that $[a, b]$ is an interval that the equation $f(x) = 0$ has atleast one zero "r" in it. We will have two main cases:

Case 1:

$$c = \frac{af(b) - bf(a)}{f(b) - f(a)}. \quad . \quad . \quad . (17)$$

is the first approximation of the root of equation (1) with Regular falsi Method that is obtained from the linear interpolation

$$y - f(a) = \frac{f(b) - f(a)\ (x - a)}{b - a}. \quad . \quad . \quad . (18)$$

With setting $y = 0\ and\ x = c$. assume that $f(c\ )f(a) < 0$ and this linear interpolation intersects the line

$$y = \frac{-k\ f(b)\ (x - a)}{b - a}. \quad . \quad . \quad . (19)$$

Which is straight line that connect points $(a, 0\ )\ to\ (b, -kf(b))$

By considering equations (18) and (19) and eliminating y, we will have

$$x = \frac{(1 + k)bf(a) - f(b)}{(1 + k)f(a) - f(b)}. \quad . \quad . \quad . (20)$$

$if\ \ f\ (x)f(a) < 0$, then b is replace with x and $if\ \ f\ (x)f(a) > 0$, we replace b with c and a with x i.e b=c, a=x.

Case 2:

Consider $c = \frac{af(b)-bf(a)}{f(b)-f(a)}$

and (18) again. Now suppose that $f\ (x)f(a) > 0$ and the Regular Falsi Method (18) intersect with line

$$y = \frac{-k\ f(a)\ (x - a)}{b - a}. \quad . \quad . \quad . (21)$$

that is the straight line that lies between points $(b, 0\ )\ to\ (a, -kf(a))$

By considering equations (18) and (21), and eliminating y, we will have

$$x = \frac{(1 + k)af(b) - f(a)}{(1 + k)f(b) - f(a)}. \quad . \quad . \quad . (22)$$

$if\ \ f\ (x)f(a) < 0$ replace a with x and$if\ \ f\ (x)f(a) > 0$, replace a with c and b with x i.e a=c and b=x, the Algorithm is continued.

The parameter k≥ 0 is a weight relation with b-c and a-c, when $if\ \ f\ (c)f(a) < 0$ or $if\ \ f\ (c)f(a) > 0$ respectively.

We observe that when k=0, our method returns to Regular Falsi Method, so we will have

$$k = \begin{cases} \frac{\alpha\ f(c)}{b - a}, & if\ \ f\ (c)f(a) < 0 \\ \frac{\alpha\ f(c)}{a - c}, & if\ \ f\ (c)f(a) > 0 \end{cases} \quad . \quad . \quad . \quad . \quad . (23)$$

The parameter α is an arbitrary Number, and we set α = 1 in our example.

3.0 Numerical Examples

In this section we solved some problems with the new improved methods and the classical methods. Their results are tabulated. The solved equations are:

$f_{1(x)} = x^3 - 2x - 5,\ on\ [2,3]$

$f_{2(x)} = x^5 + x^3 - 3x - 2,\ on\ \ [1,2]$

Table 1: Numerical Results of BM,IBM,RFM and MRFM

PROBLEM	METHODS	INTERVAL	NO OF ITERATION	SOLUTION OBTAINED	ACTUAL SOLUTION
$f_{1(x)}$	BM	[2,3]	17	2.0945510864	2.0945515532
	IBM		13	2.0945514815	
	RFM		13	2.0945512550	
	MRFM		5	2.0945515791	
$f_{2(x)}$	BM	[1,2]	15	1.2993927020	1.299392094663
	IBM		6	1.2993209600	
	RFM		21	1.2993311460	
	MRFM		19	1.2993329034	

4.0 Conclusion and Remarks

Table 1 presents the Numerical results from Bisection Method, Improved Bisection Method, Regular Falsi Method and Modified Regular Falsi Method.

In problem 1, the new methods produce results accurate to 6 decimal place after 13 and 5 iterations for Improved Bisection Method and Modified Regular Falsi Method respectively when compared to results produce by classic methods accurate to 6 decimal place after 17 and 13 iterations for Bisection and Regular Falsi Methods respectively.

In problem 2, the new methods produce results accurate to 4 decimal place after 6 and 19 iterations for Improved Bisection Method and Modified Regular Falsi Method respectively when compared to results produce by classic methods accurate to 4 decimal place after 15 and 21 iterations for Bisection and Regular Falsi Methods respectively, the new methods can reduced the number of iterations less than the iteration number of the classical methods, which indicate that the new methods converge faster than the classic methods.

We can conclude, that the New Methods compare favorably with the well known classic methods. The New Methods are better method of approximation than the classic methods because they converge faster than the classic methods.

We are still in the process of exploring how we could make our root finding methods better and more efficient, it is an unending quest.

5.0 References

[1] Salisu,A and Adeboye,K.R : High order Quasi-Runge Kutta Methods By Refinement Process for the Solution of Initial Value Problems, J. NAMP Vol. 22,(2012) Pp 419-424

[2] Dalguist, G and Bjorck, A: Numerical Methods in Scientific Computing, SIAM, Philadelphia, PA, (2008).

[3] Mathews, J.H and Fink, F.F: Numerical Methods Using Matlab, 4th Edition, Prentice Hall, Upper Saddle River, NJ, (2004).

[4] Mathews, J.H: Modules for Numerical Methods and Numerical Analysis http:/maths.fullerton.edu/Mathews/n2003/NumericalUndergrad.html,September,2014.

[5] Noor, M.A, Ahmed, F.,Javeed, Sh: Two –Step iterative Methods for nonlinear equations, J. Appl. Math. Comput. 183 (2006) Pp 1068-1075.

[6] Noor, M.A, and Ahmed, F.: On a Predictor-Corrector Method for Solving nonlinear equations, J. Appl. Math. Comput. 183 (2006) Pp 128-133.

[7] Thukral,R: Introduction to fifth-Order Iteration Method for solving non-linear Equations. Inter. J. of Computational and Applied Mathematics,4 (2)(2009), 135-140