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Analysis of One-Dimensional Abrasive Wear Rate in Hot Forging Process

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Abstract

The analysis of one-dimensional abrasive wear rate in hot-forging process was computed using the Bubnov-Galerkin Finite Element Method in the present study. The weak form of the governing differential equation was obtained and nodal contact pressures for linear interpolation functions for different elements are calculated for Neumann boundary conditions. Time approximation was done with the aid of the Crank-Nicholson Finite difference scheme and time step $(\Delta t = 0.5)$ ***was used to obtain equation for the solution. Using a numerical example the results showed a maximum error of 0.5 percent for a number of linear elements. It is concluded that as the mesh is refined further progressively, the finite element solution approaches the exact solution which is an indication that the solutions are accurate and the method is robust.***

Keywords: Finite element method, Crank-Nicholson, Finite difference scheme, Bubnov-Galerkin, Time approximate.

1.0 Introduction

Wear is a damage to solid surface, generally involving progressive loss of material, due to relative motion between that surface and contacting substance or surface[1]. Wear is generally described as abrasive, adhesive or erosive[2]. Among these types, abrasive wear is the most important due to its destructive character[3]. Abrasive wear is the detachment of the material from surfaces in relative motion, caused by sliding of hard particles between the opposing surfaces, the hard particles normally slide on a softer surface and detach material from the latter [4]. The direct cost of wear failures increased work and time, loss of productivity as well as direct losses of energy and increased environmental burden are real problems in every day work.

A large number of researches on wear prediction exist in literature with mathematical models for the prediction of abrasive wear behavior in agricultural grade medium carbon steel [5], tool wear estimation using theoretical analysis and numerical stimulation technologies [6], development of a mathematical model for prediction of friction and wear when a soft surface slides against a harder rough surface [7], prevention of abrasive tool wear by optimizing the geometry of tool [8], experimental and theoretical investigation of ploughing, cutting and wedge[9], out a development of microcontact based modeling of abrasive [10].

Other studies are an analysis based on the estimation of interface temperature during contact sliding [11], and a theoretical estimation of abrasive wear resistance based on microscopic wear mechanism [12]. It can be seen from the literature that the potential of finite element method for addressing abrasive wear problems has not been given attention.

In this paper, we present the finite element analysis to solve the differential equation which governs the abrasive wear behaviour in hot forging process and compare the solution obtained with that of exact solution.

2.0 Governing Differential Equation

The governing differential equation for the abrasive wear behavior is given by:

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$$\frac{Sd^2p}{dv^2}+\frac{KVdp}{dv}+kp=\frac{-1}{v}\frac{dp}{dt} \tag{2.1}$$

$0<v<h$

Where,

S: sliding distance

K: dimensionless wear co-efficiencies

V: wear volume

P: contact pressure

The associated boundary conditions are given by

$p(o,t)=1$ and $\frac{dp}{dv}(o,t)=0$

And the initial condition is $p(v,o)=1$

2.1 Materials and Methods

The spatial domain of abrasive wear was divided into a number of uniform linear element with length ΔV .

Stiffness matrix, mass matrix and flux vector were generated for each element using Bubnov-Galerkin finite element method to get the contact pressure at nodal points. The stiffness matrix and mass matrix were assembled by enforcing continuity for the nodal degree of freedom to obtain the global system equations. The lagrange linear interpolation functions were used to obtain a solution.

A finite difference modeling was developed using the α - family of approximation in which a weighted average of time derived of the dependent variable p is approximated at two consecutive time steps by linear interpolation of the values of the variable at two steps. We then apply the Crank-Nicholson finite difference scheme by taking α =0.5 and a time step $(\Delta t=0.5)$ to obtain equation for the solution. A numerical analysis was done to compare the finite element results with the exact solution.

2.2 Weak Formulation

The weak form of equation (2.1) is obtained by multiplying the equation by a weight function W=w(t) and integrating it over the domain of the element and since it is time dependent and this becomes:

$$\int_o^h w(t)\frac{Sd^2pdvt}{dv^2}\int_o^h w(t)KV\frac{dpdv}{dv}+\int_o^h w(t)kpdv+\int_o^h w(t)\frac{1}{V}\frac{dp}{dt}dv=0 \tag{2.2}$$

That is,

$$\int_o^h\left(w(t)\frac{Sd^2p}{dv^2}+w(t)KV\frac{dp}{dv}dv+\int_o^h w(t)Kpdv+\int_o^h w(t)\frac{1}{V}\frac{dp}{dt}\right)dv=0 \tag{2.3}$$

The term $\frac{Sd^2p}{dv^2}$ was put in the weaker order $\frac{dp}{dv}$. Using integration by part principles

$$\int_o^h w(t)\left\{\frac{d}{dv}\left(\frac{sdp}{dv}\right)\right\}dv=\left[w(t)\frac{sdp}{dv}\right]_o^h-\int_o^h\frac{sdw}{dv}\frac{dp}{dv}dv \tag{2.4}$$

Substitute the weak form equation (2.4) into equation (2.3)

$$w(t)\frac{sdp}{dv}\int_o^h-S\int_o^h\frac{dp}{dv}\frac{dw}{dv}dv+KV\int_o^h w(t)\frac{dp}{dv}dv+K\int_o^h w(t)Pdv+\int_o^h w(t)\frac{1}{V}\frac{dp}{dv}dv=0 \tag{2.5}$$

Equation (2.5) becomes

$$S\int_o^h\frac{dp}{dv}\frac{dw}{dv}dv-KV\int_o^h w(t)\frac{dpdv}{dv}-k\int_o^h w(t)pdv-\int_o^h w(t)\frac{1}{V}\frac{dpdv}{dt}-S\left[w(t)\frac{dp}{dv}\right]_o^h=0 \tag{2.6}$$

Equation (2.6) is the weak form of equation (2.1)

2.3 Finite Element Modeling

Let the solution of equation (2.6) be of the separable variable form

$$p(v,t) \approx p^e(V,t) = \sum_{j=1}^{n} p_j(t)\psi_j^e(V) \qquad (2.7)$$

In finite element form, equation (2.7) becomes:

$$p(v,t) = \sum_{j=1}^{n} p_j^e(t)\psi_j^e(v) = \sum_{j=1}^{n} \left(p_j^s\right)\psi_j^e(v) \qquad (2.8)$$

Where ψ_j^e is lagrange interpolation function at the jth node and p_j^e the pressure at jth node of the element. Since Bubnov-Galerkin finite element is to be applied in the study, we assume that the weight function is equal to interpolation functions. That is:

$$w(t) = \psi_j(v) \qquad (2.9)$$

Substituting equation (2.8) and (2.9) into equation (2.6) we obtain:

$$S\int_o^h \frac{d\Sigma p_j^e(t)}{dv}\psi_j^e(v)\frac{d\psi_j^e}{dv}(v)dv - KV\int_o^h \psi_j^e(v)\frac{d\Sigma p_j^e(t)}{dv}\psi_j^e(v)$$

$$-k\int_o^h \psi_j^e(v)\sum p_j^e(t)\psi_j^e(v)dv - \frac{1}{V}\int_o^h \psi_j^e(v)\frac{d\Sigma p_j^e}{dv}(t)\psi_j^e(v) - S\left[\psi_j^e\frac{d\Sigma p_j^e}{dv}\right]_o^h = 0 \qquad (2.10)$$

Let

$$K_{ij} = S\int_o^h \frac{d\Sigma p_j^e}{dv}(t)\psi_i^e(v)\frac{d\psi_j^e}{dv}(v)dv - KV\int_o^h \psi_j^e(v)\frac{d\Sigma p_j^e}{dv}(t)\psi_j^e(v)dv$$

$$-K\int_o^h \psi_j^e(v)\sum p_j^e(t)\psi_j^e(v)dv$$

$$M_{ij} = \frac{1}{V}\int_o^h \psi_j^e(v)\frac{d\Sigma p_j^e}{dv}(t)\psi_j^e(v)$$

$$Q_i = S\psi_i^e(h_1)Q_1^e + S\psi_i^e(h_o)Q_o^e$$

That is,

$$\sum_{j=1}^{n}\left\{K_{ij}p_j^e + M_{ij}P_j^e\right\} - Q_i^e = 0 \qquad (2.11)$$

In matrix form, equation (2.11) becomes:

$$[K]\{p_j^e\} + [M]\{\dot{p}_j^e\} - \{Q_i^e\}$$

or

$$[M]\{\dot{p}_j^e\} + [K]\{p_j^e\} = \{Q_i^e\} \qquad (2.12)$$

Where

[M] = mass matrix

[K] = stiffness matrix

{Q} = Flux vector

Equation (2.12) is the finite element mode (FEM) for the analysis

The one-dimensional lagrange linear interpolation functions are:

$$\psi_1^e(v) = \frac{V_2 - V}{V_2 - V_1}$$

$$\psi_2^2(v) = \frac{V - V_1}{V_2 - V_1}$$

The Lagrange linear interpolation can be written compactly as $\psi_j(v) = \delta_{ji}$, where δ_{ji} is called kronecker delta and has the property.

$$\delta_{ji} = \begin{Bmatrix} 1 & if & j = l \\ 0 & if & j \neq l \end{Bmatrix}$$

3.0 Numerical Example

Use the finite element analysis to predict the abrasive wear rate in hot forging process. the governing differential equation is given:

$$\frac{Sd^2 p}{dv^2} + \frac{Kvdp}{dv} + KP = \frac{-1}{v}\frac{dp}{dt}$$

$$lm < V2m$$

Boundary condition

$p(o,t) = 1$ and $\frac{dp}{dv}(o,t) = 0$

Initial condition $p(v,o) = 1$

3.1 Solution

In solving the problem, we shall use linear interpolation functions for the solution. First, we will discretize the domain into eight linear elements which exposes nine nodes and observe the behaviour of the solution. First we will need to calculate the element mass matrix, stiffness matrix, noting that

$$[M_{ij}^e]\{\dot{p}_j\} + [k]\{p_j\} = \{Q_i^e\} \tag{3.1}$$

$$[M_{ij}^e] = \frac{1}{V}\int^e \frac{d\psi_i^e(v)}{dv}\frac{d\psi_j^e(v)}{dv} \tag{3.2}$$

$$[K_{ij}^e] = \left\{ S\int_o^e \frac{d\psi_j^e}{dv}\frac{d\psi_u^e}{dv} - KV\int_o^e \psi_i^e \frac{d\psi_i^e}{dv} - k\int_o^e \psi_i^e \psi_i^e \right\} dv \tag{3.3}$$

Where

$e = V_{i+1}$ = coordinate of the right end of element

$o = V_i$ = coordinate of the left end of element.

For a choice of linear interpolation lagrange functions, the element mass matrix and stiffness matrix will be in the form

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{Bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} \tag{3.4}$$

Where

$\dot{p}_1, \dot{p}_2, p_1, p_2$ = nodal degree of freedom Q_1, Q_2 = flux vector.

For a uniform increment, $V_{i+1} - V_i$ for N elements

That is,

$$V_{i+1} - V_i = \frac{2-1}{8} = 0.125$$

Thus,

$$M_{11} = \frac{1}{V}\int_{V_i}^{V_{i+1}} \frac{d\psi_1(v)}{dv} \times \frac{d\psi_1}{dv}(v)dV = 68$$

$$M_{12} = \frac{1}{V}\int_{V_i}^{V_{i+1}} \frac{d\psi_1(v)}{dv} \times \frac{d\psi_2}{dv}(v)dV = -68$$

$$M_{21} = \frac{1}{V}\int_{V_i}^{V_{i+1}} \frac{d\psi_2(v)}{dv} \times \frac{d\psi_1(v)}{dv} = -68$$

$$M_{22} = \frac{1}{V}\int_{V_i}^{V_{i+1}} \frac{d\psi_2}{dv}(v) \times \frac{d\psi_2(v)}{dv} = 68$$

In matrix form
Element 1
The element mass matrix is given as:

$$\begin{bmatrix} 68 & -68 \\ -68 & 68 \end{bmatrix} \begin{Bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix} \tag{3.5}$$

Element 2
The element mass matrix is given as:

$$\begin{bmatrix} 76 & -76 \\ -76 & 76 \end{bmatrix} \begin{Bmatrix} \dot{p}_2 \\ \dot{p}_3 \end{Bmatrix} = \begin{Bmatrix} Q_2^2 \\ Q_3^2 \end{Bmatrix} \tag{3.6}$$

Element 3
The element mass matrix is given as

$$\begin{bmatrix} 84 & -84 \\ -84 & 84 \end{bmatrix} \begin{Bmatrix} \dot{p}_3 \\ \dot{p}_4 \end{Bmatrix} = \begin{Bmatrix} Q_3^3 \\ Q_4^3 \end{Bmatrix} \tag{3.7}$$

Element 4
The element mass matrix is given as

$$\begin{bmatrix} 92 & -92 \\ -92 & 92 \end{bmatrix} \begin{Bmatrix} \dot{p}_4 \\ \dot{p}_5 \end{Bmatrix} = \begin{Bmatrix} Q_4^4 \\ Q_5^4 \end{Bmatrix} \tag{3.8}$$

Element 5
The element mass matrix is given as:

$$\begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} \dot{p}_5 \\ \dot{p}_6 \end{Bmatrix} = \begin{Bmatrix} Q_5^5 \\ Q_6^5 \end{Bmatrix} \tag{3.9}$$

Element 6
The element mass matrix is given as:

$$\begin{bmatrix} 108 & -108 \\ -108 & 108 \end{bmatrix} \begin{Bmatrix} \dot{p}_6 \\ \dot{p}_7 \end{Bmatrix} = \begin{Bmatrix} Q_6^6 \\ Q_7^6 \end{Bmatrix} \tag{3.10}$$

Element 7
The element mass matrix is given as:

$$\begin{bmatrix} 116 & -116 \\ -116 & 116 \end{bmatrix} \begin{Bmatrix} \dot{p}_7 \\ \dot{p}_8 \end{Bmatrix} = \begin{Bmatrix} Q_7^7 \\ Q_8^7 \end{Bmatrix} \tag{3.11}$$

For the 8^{th} elements, we have nodes 8 and 9. Superimposing element 8 on element 1, we have node 8 becomes node 1 and node 9 becomes node 2. $V_i = 1.875$ and $V_{i+1} = 2.0$. In matrix form, element (8) mass matrix becomes:

$$\begin{bmatrix} 124 & -124 \\ -124 & 124 \end{bmatrix} \begin{Bmatrix} p_8 \\ p_9 \end{Bmatrix} = \begin{Bmatrix} Q_8^8 \\ Q_9^8 \end{Bmatrix} \tag{3.12}$$

The next step is to assemble the element mass matrices for all the eight elements. We obtain the system mass matrix below:

$$\begin{bmatrix} 68 & -68 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -68 & 144 & -76 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -76 & 160 & -84 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -84 & 176 & -92 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -92 & 192 & -100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -100 & 208 & -108 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -108 & 224 & -166 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -116 & 240 & -124 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -124 & 124 \end{bmatrix} \begin{Bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \\ \dot{p}_5 \\ \dot{p}_6 \\ \dot{p}_7 \\ \dot{p}_8 \\ p_9 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_2^2 \\ Q_3^2 + Q_3^3 \\ Q_4^3 + Q_4^4 \\ Q_5^4 + Q_5^5 \\ Q_6^5 + Q_6^6 \\ Q_7^6 + Q_7^7 \\ Q_8^7 + Q_8^8 \\ Q_9^8 \end{Bmatrix} \tag{3.13}$$

Similarly, we evaluated for stiffness matrix to obtain

$$\left[k^{e}\right]=\begin{bmatrix}\dfrac{6s+kh_e^2}{6h_e} & \dfrac{-6s+kh_e^2}{6h_e}\\ \dfrac{-6s+kh_e^2}{6h_e} & \dfrac{6s-5kh_e^2}{6h_e}\end{bmatrix}\begin{Bmatrix}p_1\\ p_2\end{Bmatrix}=\begin{Bmatrix}Q_1^1\\ Q_2^1\end{Bmatrix} \qquad (3.14)$$

Where

K= the dimensionless wear coefficient (10^{-2})

S = sliding distance

h_e = wear depth

Element 1

The element stiffness matrix is given as

$$\begin{bmatrix}1.002 & -1.002\\ -1.002 & 0.999\end{bmatrix}\begin{Bmatrix}p_1\\ p_2\end{Bmatrix}=\begin{Bmatrix}Q_1^1\\ Q_2^1\end{Bmatrix} \qquad (3.15)$$

Element 2

The element stiffness matrix is given as:

$$\begin{bmatrix}2.002 & -2.002\\ -2.002 & 1.999\end{bmatrix}\begin{Bmatrix}p_2\\ p_3\end{Bmatrix}=\begin{Bmatrix}Q_2^2\\ Q_3^2\end{Bmatrix} \qquad (3.16)$$

Element 3

The element stiffness matrix is given as:

$$\begin{bmatrix}3.002 & -3.002\\ -3.002 & 2.999\end{bmatrix}\begin{Bmatrix}p_3\\ p_4\end{Bmatrix}=\begin{Bmatrix}Q_3^3\\ Q_4^3\end{Bmatrix} \qquad (3.17)$$

Element 4

The element stiffness matrix is given as:

$$\begin{bmatrix}4.002 & -4.002\\ -4.002 & 3.999\end{bmatrix}\begin{Bmatrix}p_4\\ p_5\end{Bmatrix}=\begin{Bmatrix}Q_4^4\\ Q_5^4\end{Bmatrix} \qquad (3.18)$$

Element 5

The element stiffness matrix is given as:

$$\begin{bmatrix}5.002 & -5.002\\ -5.002 & 4.999\end{bmatrix}\begin{Bmatrix}p_5\\ p_6\end{Bmatrix}=\begin{Bmatrix}Q_5^5\\ Q_6^5\end{Bmatrix} \qquad (3.19)$$

Element 6

The element stiffness matrix is given as:

$$\begin{bmatrix}6.002 & -6.002\\ -6.002 & 5.999\end{bmatrix}\begin{Bmatrix}p_6\\ p_7\end{Bmatrix}=\begin{Bmatrix}Q_6^6\\ Q_7^6\end{Bmatrix} \qquad (3.20)$$

Element 7

The element stiffness matrix is given as:

$$\begin{bmatrix} 7.002 & -7.002 \\ -7.002 & 6.999 \end{bmatrix} \begin{Bmatrix} p_7 \\ p_8 \end{Bmatrix} = \begin{Bmatrix} Q_7^7 \\ Q_8^7 \end{Bmatrix} \tag{3.21}$$

Element 8

For element 8, we have nodes 8 and 9, $p_8, p_9, S = 20$ and $k = 10^{-2}$. We then used superimposition of element 8 on element 1, node 8 becomes node 1 and node 9 becomes node 2.

The element stiffness matrix obtained is:

$$\begin{bmatrix} 16.002 & -16.002 \\ -16.002 & 15.999 \end{bmatrix} \begin{Bmatrix} p_8 \\ p_9 \end{Bmatrix} = \begin{Bmatrix} Q_8^8 \\ Q_9^8 \end{Bmatrix} \tag{3.22}$$

Using continuity for node 8, which is common to element 7 and element 8. The matrix for all the element (1,2,3,4,5,6,7,8 and 9) assembled to becomes

$$\begin{bmatrix} 1.002 & -1.002 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.002 & 3.002 & -2.002 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.002 & 5.001 & -3.002 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3.002 & 7.001 & -4.002 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4.002 & 9.001 & -5.002 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5.002 & 11.001 & -6.002 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6.002 & 13.001 & -7.002 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7.002 & 23.001 & -16.002 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16.002 & 15.999 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_2^2 \\ Q_3^2 + Q_3^3 \\ Q_4^3 + Q_4^4 \\ Q_5^4 + Q_5^5 \\ Q_6^5 + Q_6^6 \\ Q_7^6 + Q_7^7 \\ Q_8^7 + Q_8^8 \\ Q_9^8 \end{Bmatrix} \tag{3.23}$$

The global assembled equation for mass matrix $\left[M^e\right]$, stiffness matrix $\left[K^e\right]$ and flux vector $\left\{Q^e\right\}$ becomes:

$$\begin{bmatrix} 68 & -68 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -68 & 144 & -76 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -76 & 160 & -84 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -84 & 176 & -92 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -92 & 192 & -100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -100 & 208 & -108 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -108 & 224 & -116 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -116 & 240 & -124 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -124 & 124 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \end{Bmatrix} +$$

$$\begin{bmatrix} 1.002 & -1.002 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.002 & 3.002 & -2.002 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.002 & 5.001 & -3.002 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3.002 & 7.001 & -4.002 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4.002 & 9.001 & -5.002 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5.002 & 11.001 & -6.002 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6.002 & 13.001 & -7.002 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7.002 & 23.001 & -16.002 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16.002 & 15.999 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_9 \end{Bmatrix} \tag{3.24}$$

Due to balance of internal fluxes, it follows that $Q_3^2 + Q_3^3 = Q_5^4 + Q_5^5 = Q_7^6 + Q_7^7 = 0$ and $Q_2^2 = Q_4^4 = Q_6^6 = Q_8^8 = 0$
Equation (3.24) becomes:

$$\begin{bmatrix} 68 & -68 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -68 & 144 & -76 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -76 & 160 & -84 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -84 & 176 & -92 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -92 & 192 & -100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -100 & 208 & -108 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -108 & 224 & -116 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -116 & 240 & -124 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -124 & 124 \end{bmatrix} \begin{Bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \\ \dot{p}_5 \\ \dot{p}_6 \\ \dot{p}_7 \\ \dot{p}_8 \\ \dot{p}_9 \end{Bmatrix} +$$

$$\begin{bmatrix} 1.002 & -1.002 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.002 & 3.002 & -2.002 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.002 & 5.001 & -3.002 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3.002 & 7.001 & -4.002 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4.002 & 9.001 & -5.002 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5.002 & 11.001 & -6.002 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6.002 & 13.001 & -7.002 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7.002 & 23.001 & -16.002 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16.002 & 15.999 \end{bmatrix} \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_9 \end{Bmatrix} \qquad (3.25)$$

We consider the boundary condition $p(o,t)=0$ which implies that $\frac{dp}{dv}(1,t)=0$ initial condition $p(v,o)=1.0=p_1$ which implies that $Q_1=0$

$$\begin{bmatrix} 68 & -68 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -68 & 144 & -76 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -76 & 160 & -84 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -84 & 176 & -92 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -92 & 192 & -100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -100 & 208 & -108 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -108 & 224 & -116 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -116 & 240 & -124 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -124 & 124 \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \\ \dot{p}_5 \\ \dot{p}_6 \\ \dot{p}_7 \\ \dot{p}_8 \\ \dot{p}_9 \end{Bmatrix} +$$

$$\begin{bmatrix} 1.002 & -1.002 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.002 & 3.002 & -2.002 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.002 & 5.001 & -3.002 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3.002 & 7.001 & -4.002 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4.002 & 9.001 & -5.002 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5.002 & 11.001 & -6.002 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6.002 & 13.001 & -7.002 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7.002 & 23.001 & -16.002 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -16.002 & 15.999 \end{bmatrix} \begin{Bmatrix} 0 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_9 \end{Bmatrix} \qquad (3.26)$$

The condensed equations are:

$$\begin{bmatrix} 144 & -76 & 0 & 0 & 0 & 0 & 0 & 0 \\ -76 & 160 & -84 & 0 & 0 & 0 & 0 & 0 \\ 0 & -84 & 176 & -92 & 0 & 0 & 0 & 0 \\ 0 & 0 & -92 & 192 & -100 & 0 & 0 & 0 \\ 0 & 0 & 0 & -100 & 208 & -108 & 0 & 0 \\ 0 & 0 & 0 & 0 & -108 & 224 & -116 & 0 \\ 0 & 0 & 0 & 0 & 0 & 116 & 240 & -124 \\ 0 & 0 & 0 & 0 & 0 & 0 & -124 & 124 \end{bmatrix} \begin{Bmatrix} \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \\ \dot{p}_5 \\ \dot{p}_6 \\ \dot{p}_7 \\ \dot{p}_8 \\ \dot{p}_9 \end{Bmatrix} +$$

$$\begin{bmatrix} 3.002 & -2.002 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.002 & 5.001 & 3.002 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3.002 & 7.001 & -4.002 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4.002 & 9.001 & -5.002 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5.002 & 11.001 & -6.002 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6.002 & 13.001 & -7.002 & 0 \\ 0 & 0 & 0 & 0 & 0 & -7.002 & 23.001 & -16.002 \\ 0 & 0 & 0 & 0 & 0 & 0 & -16.002 & 15.999 \end{bmatrix} \begin{Bmatrix} p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \end{Bmatrix} = \begin{Bmatrix} -1.002 \\ 1.002 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (3.27)$$

Recall equation (2.12) that the finite element model (FEM) was the form:
Thus,

$$[M]=\begin{bmatrix} 144 & -76 & 0 & 0 & 0 & 0 & 0 & 0 \\ -76 & 160 & -84 & 0 & 0 & 0 & 0 & 0 \\ 0 & -84 & 176 & -92 & 0 & 0 & 0 & 0 \\ 0 & 0 & -92 & 192 & -100 & 0 & 0 & 0 \\ 0 & 0 & 0 & -100 & 208 & -108 & 0 & 0 \\ 0 & 0 & 0 & 0 & -108 & 224 & -116 & 0 \\ 0 & 0 & 0 & 0 & 0 & 116 & 240 & -124 \\ 0 & 0 & 0 & 0 & 0 & 0 & -124 & 124 \end{bmatrix} \quad (3.28)$$

$$[K]=\begin{bmatrix} 3.002 & -2.002 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2.002 & 5.001 & 3.002 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3.002 & 7.001 & -4.002 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4.002 & 9.001 & -5.002 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5.002 & 11.001 & -6.002 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6.002 & 13.001 & -7.002 & 0 \\ 0 & 0 & 0 & 0 & 0 & -7.002 & 23.001 & -16.002 \\ 0 & 0 & 0 & 0 & 0 & 0 & -16.002 & 15.999 \end{bmatrix} \quad (3.29)$$

$$\{Q\}=\begin{Bmatrix} -1.002 \\ 1.002 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (3.30)$$

4.0 Finite Difference Modeling

In this study, we used the α -family of approximation in which weighted average of time derived of dependent variable P is approximated at two consecutive time steps by linear interpolation of the values of the variable at two steps.

$$(1-\alpha)\{\dot{p}\}_s + \alpha\{\dot{p}\}_{s+1} = \frac{\{p\}_{s+1} - \{p\}_s}{\Delta t_{s+1}} \text{ for } 0 \le \alpha \le 1 \quad (4.1)$$

Where $\{\ \}_s$ refers to the value of the enclosed quantity at time $t = t_s = \sum_{i=1}^{\infty} \Delta t_1$ since the finite element model in valid for any t>0, it is valid for $t=t_s$ and $t=t_{s+1}$

$$[M]\{\dot{p}\}_s + [K]\{p\}_s = \{Q\}_s \quad (4.2)$$

$$[M]\{\dot{p}\}_{s+1} + [K]\{\ \}_{s+1} = \{Q\}_{s+1} \quad (4.3)$$

We multiply both sides of equation (4.1) by $\Delta t_{s+1}[M]$ to get:

$$\Delta t_{s+1}\alpha[M]\{\dot{p}\}_{s+1} + \Delta t_{s+1}(1-\alpha)[M]\{\dot{p}\}_s = [M](\{p\}_{s+1} - \{p\}_s) \quad (4.4)$$

We substitute for $[M]\{\dot{p}\}_{s+1}$ and $[M]\{\dot{p}\}_s$ from equations (4.2) and (4.3) respectively.

$$\Delta t_{s+1}\alpha(\{Q\}_{s+1}-\{p\}_s)$$

Rearranging the terms into known and unknown, we get

$$([M]+\Delta t_{s+1}\alpha[k])\{p\}_{s+1}=([M]-\Delta t_{s+1}(1-\alpha)[K])\{p\}_s+\Delta t_{s+1}(\alpha\{Q\}_{s+1}+(1-\alpha)\{Q\}_s)$$

But

$$\{Q\}_{s+1}=\{Q\}_s=\{Q\}$$

Therefore writing

$$\Delta t_{s+1}=\Delta t$$

$$([M]+\Delta t\alpha[K])\{p\}_{s+1}=([M]-\Delta t(1-\alpha)[K])\{p\}_s+\Delta t\{Q\} \tag{4.4}$$

We apply the crank-Nicholson finite difference scheme i.e. we take α =0.5 equation (4.4) becomes:

$$\left([M]+\frac{\Delta t[k]}{2}\right)\{p\}_{s+1}=\left([M]-\frac{\Delta t[K]}{2}\right)\{p\}_s+\Delta t\{Q\}$$

$$\{p\}_{s+1}=\left[\left[[M]+\frac{\Delta t[K]}{2}\right]^{-1}\left[[M]-\frac{\Delta t[K]}{2}\right]\right]\{p\}_s+\left[[M]+\frac{\Delta t[K]}{2}\right]^{-1}[\Delta t\{Q\}] \tag{4.5}$$

For one-element mesh, we have:

$$\begin{bmatrix} M_{11}^1+\alpha\Delta tk_{11}^1 & M_{12}^1-\alpha\Delta tk_{12}^1 \\ M_{21}^1-\alpha\Delta tk_{21}^1 & M_{22}^1+\alpha\Delta tk_{22}^1 \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}_{s+1}=\begin{bmatrix} M_{11}^1-(1-\alpha)\Delta tk_{11}^1 & M_{12}^1+(1-\alpha)\Delta tk_{12}^1 \\ M_{21}^1+(1-\alpha)\Delta tk_{21}^1 & M_{22}^1-(1-\alpha)\Delta tk_{22}^1 \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}_s+\alpha\Delta t\begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix} \tag{4.6}$$

Using the boundary conditions, we recast for the one-element model as:

$$\{p_2\}_{s+1}=\left[\frac{M_{22}^1-(1-\alpha)\Delta tk_{22}^1}{M_{22}^1+\alpha\Delta tk_{22}^1}\right]\{p_2\}_s+\left[\frac{\Delta tQ_2^1}{M_{22}^1+\alpha\Delta tk_{22}^1}\right]\{p_2\}_s \tag{4.7}$$

The solutions are then obtained by substituting into equation (4.7) value of M_{22}^1, K_{22}^1 and taking a time step $\Delta t = 0.5$. We then solved repeatedly for P_2 at difference times, S=0,1,2,3,…30.

5.0 Exact Solution

$$\frac{Sd^3p}{dv^2}+KV\frac{dp}{dv}+Kp+\frac{1}{V}\frac{dp}{dt}=0$$

We assume that $p=V(v)T(t)$

$$V\left(\frac{sd^2p}{dv^2}+KV\frac{dp}{dv}+Kp\right)=\frac{-dp}{dt} \tag{5.1}$$

We solved equation (5.1) by separation of variables using

$$p(v,t)=V(v)T(t)$$

Thus, equation (5.1) becomes

$$\frac{s}{KV}\frac{d^2V}{dv^2}+\frac{dV}{dV}+1=\frac{-1}{KT}\frac{dT}{dt}=\frac{1}{\lambda^2} \tag{5.2}$$

Where $\frac{1}{\lambda^2}$ (Constant of separation)

We assumed the L.H.S. of equation (5.2) is independent of (t) and R.H.S. independent of (V), then each side of equation (5.2) can be equated to the constant of separation. We solve for both L.H.S and R.H.S. to obtain a generally solution for any n.

$$p_n(V,t)=\sum_{n=1}^{\infty}\beta_n Sin\left(\frac{n\pi v}{h}\right)\ell^{-k\left(\frac{n\pi}{h}\right)^2 t} \tag{5.3}$$

We apply the initial condition

$$p(V,o)=\sum_{n=1}^{\infty}\beta_n Sin\left(\frac{n\pi V}{h}\right) \tag{5.4}$$

We multiply both sides of equation (5.4) by $Sin\left(\frac{m\pi v}{h}\right)$ and integrating from 0 to h gives

$$\int_o^h Sin\left(\frac{m\pi v}{h}\right)p(V,o)dv=\int_o^h\sum_{n=1}^{\infty}\beta_n Sin\left(\frac{m\pi v}{h}\right)Sin\left(\frac{n\pi v}{h}\right)dV \tag{5.5}$$

Using the orthogonality of $Sin(m\pi v)$ and $Sin(n\pi v)$ we get

$$\int_o^h Sin\left(\frac{n\pi v}{h}\right)p(v,o)dv=\frac{h}{2}\beta_n \tag{5.6}$$

Where

$$\beta_n=\frac{2}{h}\int_o^h Sin\left(\frac{n\pi v}{h}\right)p(v,o)dv \tag{5.7}$$

$(n=1,2,3,4,..)$

Applying the initial condition $p(v,o)=p(v)$ at t=0, we have

$$p(v)=\sum_{n=1}^{\infty}\beta_n Sin\left(\frac{n\pi v}{h}\right) \text{ for } 0<V<h$$

So that

$$\beta_n=\frac{2}{h}\int_o^h p(v)Sin\frac{n\pi V}{h}dv$$

$$=\frac{2}{h}\int_o^h V\ Sin\frac{n\pi v}{h} \tag{5.8}$$

6.0 Results and Discussion

The abrasive wear rates at the nodes for different meshes using linear interpolation functions are shown in Table 6.1. The abrasive wear rates at points between nodes are also shown in Table 6.1. The numerical value of the calculated nodal degree of freedom shows progressive improvement of abrasive wear rates with convergence characteristic. The absolute point wise error is not greater than 0.5 percent for all points considered along the domain showing an admirable rate of convergence to the exact solution. Successive decrease in the length of the elements produces solutions which approach the exact solution which is an indication that the solutions are accurate and the method very robust.

Table 6.1: A comparison of the finite element solutions obtained for linear element using time approximate scheme ($(\Delta t = 0.5)$ with exact solution.

t(s)	1L	2L	4L	5L	Exact
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
0.5	0.9926	0.9869	0.9785	0.9753	0.9772
1.0	0.9854	0.9740	0.9574	0.9512	0.9502
1.5	0.9782	0.9613	0.9368	0.9277	0.9240
2.0	0.9710	0.9487	0.9167	0.9048	0.9146
2.5	0.9639	0.9363	0.8970	0.8825	0.8739
3.0	0.9568	0.9241	0.8777	0.8607	0.8613
3.5	0.9498	0.9120	0.8588	0.8394	0.8300
4.0	0.9429	0.9001	0.8404	0.8187	0.8172
4.5	0.9360	0.8883	0.8223	0.8185	0.8013
5.0	0.9291	0.8767	0.8223	0.7985	0.7952
5.5	0.9223	0.8653	0.8046	0.7788	0.7629
6.0	0.9156	0.8540	0.7873	0.7596	0.7478
6.5	0.9089	0.8428	0.7704	0.7408	0.7218

Continuation of Table 6.1					
7.0	0.9022	0.8318	0.7538	0.7225	0.7054
7.5	0.8956	0.8209	0.7376	0.7047	0.6884
8.0	0.8891	0.8102	0.7217	0.6873	0.6612
8.5	0.8826	0.7996	0.7062	0.6703	0.6408
9.0	0.8761	0.7892	0.6910	0.6538	0.6204
9.5	0.8697	0.7789	0.6762	0.6376	0.6151
10.0	0.8633	0.7687	0.6616	0.6219	0.5903
10.5	0.8570	0.7586	0.6335	0.5918	0.5800
11.0	0.8507	0.7487	0.6199	0.5772	0.5601
11.5	0.8445	0.7389	0.6065	0.5629	0.5503
12.0	0.8322	0.7293	0.5935	0.5491	0.5353
12.5	0.8261	0.7198	0.5807	0.5355	0.5206
13.0	0.8200	0.7103	0.5682	0.5223	0.5011
13.5	0.8140	0.7011	0.5560	0.5094	0.4912
14.0	0.8081	0.6919	0.5441	0.4968	0.4800
14.5	0.8022	0.6829	0.5324	0.4845	0.4624
15.0	0.7963	0.6739	0.5209	0.47261	0.4594

7.0 Conclusion

Finite element analysis of abrasive wear rate in hot forging process has been presented. It has been shown that the present method can be used to predict the abrasive wear rate behavior accurately with successive mesh refinement. The potential of the finite element method has been successfully demonstrated.

8.0 References

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