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Bayesian Minimum Message Length87 with Parametric Heteroscedasticlinear Model

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Abstract

A Metropolis-Hasting algorithm was adapted to perform simulation on marginal posterior distribution of heteroscedastic linear model using Minimum Message Length87 which was conjugated with normal and inverted gamma priors to derive joint posterior distributions. The asymptotic behaviour was compared using absolute bias and mean square error criteria in order to ascertain consistency and efficiency of the estimator. The estimator is both asymptotically consistent and efficient. Results from this study would assist social and behavioural scientists if the methodology is adopted when presence of heteroscedasticity is established. This will enable them to have good precision of the inferences of the models parameters estimate. The estimator performed better when compare with conventional ordinary least square estimator. The algorithm runs faster in computation.

Keywords: Bayesian Inference, MML87, BMML87,Heteroscedastic, MCMC.
JEL: C52 & JEL C11.

1.0 Introduction

Minimum message length87 parameter estimation is similar to Bayesian inference because the duo incorporatespriors; the distinction is that mml87 uses negative log likelihood. Enes and Daniel [1,2] used mml87 to estimate efficient regression parameter estimates by adapting normal prior for parameterβ_i , uniform prior for P and discrete set gamma for σ^2. This works extends the work of Enes and Daniel by incorporating Harvey [3] heteroscedastic error structure into the mml87 likelihood. The study assumes unknown mean and precision thereby adapting normal gamma priors for parameter β_i and σ^2 respectively, the reference prior is placed on the parameter of heteroscedastic error structure λ_ii =0,..,2.

It is has been established in the literature that presence of heteroscedasticity in the data and or model often render the parameter estimates, standard error, test of hypothesis and confidence interval invalid Oloyede [4-6], Hadri and Guermat [7]**.**.This paper combines mml87 ols with heteroscedastic error structure to form mml87-het. Enes and Daniel [1] combined orthogonal least squares with mml criterion using polynomial regression to form mml-ols. This paper extends their work. Viswana than and Wallace[8] applied mml criterion to polynomial inference while Fitzigibbon *etal*[9] used mml criterion for Monte Carlo message length. In the Bayesian MML formulation, the sender and receiver agree on a prior distribution$h(\theta)$, and likelihood function $f(x/\theta)$ over the parameter space$\Theta$and data space X, which will then allow both to construct the codebook of minimum expected message length. Enes and Daniel[10] applied MML87 to logistic regression for both parameter estimation and model selection which are commonly estimated maximum likelihood approach and Akaike information criterion or Bayesian Information Criterion respectively. It was concluded that Minimum Message Length (MML) principle outperform maximum likelihood in model parameter estimation, and outperform both Akaike information criterion and Bayesian Information Criterion in model selection for both real and simulated data. Daniel and Enes [11] introduced minimum message length principle as model selection and ridge parameter estimation for generalized linear models and concluded that the MML87 criterion outperformed the corrected Akaike information criterion both in parameter estimation and model selection.

2.0 The Designs of Monte Carlo

Define our y as the combinations of X and u which is assumed to be *iid* as $u \sim \mu(0, \sigma_i^2)$ with unequal variances across the diagonals of $E(uu')$. We define our regressors to be matrix (x‘x) with error u incorporated as $(x'u^{-1}x)$. Thus, the

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parameters are defined as $\theta = (n, \sigma, \beta, \lambda)$. The task is to find a $\hat{\theta}$ that minimizes the length (MML87). Harvey [3] multiplicative heteroscedastic error structure considered and represented by w. Differentiating the full message length expression to obtain MML87 heteroscedastic estimator for continuous model parameters $(X'w^{-1}X + \sigma^2 diag(\gamma)^{-2})^{-1}(X'w^{-1}y)$.We derived our mml87 heteroscedasticity estimator through second derivative of sum of squares of mml87 heteroscedastic model. Six set of sample sizes were specified with seven scales of heteroscedasticity. For detail readers should see Oloyede[4-6]. In the Bayesian experiment, a Metropolis Hasting Algorithm was developed to simulate our MML87 heteroscedastic based models. The posterior simulation iteration was set to 10,000. The Burn-in is set at 1000 while thinning is set at 5.

The likelihood function of θ, where $\theta = (\beta_0, \beta_1, \beta_2, \sigma, \lambda)$ given the sample vector $X_1, X_2 = (1,2,\dots,n)'$ and $y = (y_1, y_2, \dots, y_n)'$The heteroscedastic mml87 is expressed as

$$MsgLen(\theta, y)_{het} = -log\left[h(\theta)\frac{[(2\pi\sigma^2)^{-n/2}\prod_{i=1}^{n} w^{-\lambda/2}\exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} w^{-\lambda}[y_i - x\beta]^2\}\epsilon^N}{\sqrt{\left(\frac{N}{\sigma^2}\right)\left(\frac{|X'X|}{\sigma^{2k}}\right)\left(\frac{1}{2\pi\exp(1)}\right)}}\right] + \frac{n}{2}(1 + \log k_n) - \log h(n) \quad (2.1)$$

Multivariate normal prior is considered for β, while inverse gamma is considered for σ and a uniform distribution is considered for λ which stand for $h(\theta)$:

$$\pi(\beta) \propto \frac{(\Psi)^{1/2}}{2\pi^{(k+1)/2}}\exp\left\{-\frac{1}{2}(\beta - \mu)\Psi(\beta - \mu)\right\}, \beta > 0; \quad (2.2)$$

$$\pi(\sigma^2) \propto (\sigma^2)^{-a_1+1}\exp(-b_1/\sigma^2), \sigma^2 > 0 \quad (2.3)$$

$$\pi(\lambda) \propto c\ isconstant (2.4)$$

The posterior distribution of$\theta = (\beta, \lambda, \sigma)$ .considering independence among the parameters is given by :

$$\pi(\beta, \lambda, \sigma|X, y) \propto -log\left[\frac{\frac{(\Psi)^{1/2}}{2\pi^{(k+1)/2}}(\sigma^2)^{-(a_1-1-n/2)}\exp\left\{-\frac{1}{2}(\beta-\mu)\Psi(\beta-\mu)\right\}\prod_{i=1}^{n} w^{-\lambda}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} w^{-\lambda}[y_i - x\beta]^2\right\}}{\sqrt{\left(\frac{N}{\sigma^2}\right)\left(\frac{|X'X|}{\sigma^{2k}}\right)\left(\frac{1}{2\pi\exp(1)}\right)}}\right] + \frac{n}{2}(1 + \log k_n) - \log h(n) \quad (2.5)$$

where a_1, b_1 are the hyper-parameters for the inverse-gamma distribution. Hyper-parameters are excluded for β-parameters since they would be estimated from the data and may be arbitrarily small leading to problems which may eventually affect the inferences. Integrating the posterior $\pi(\beta, \lambda, \sigma|X, y)$ with respect to β, thus we have joint posterior distribution for (β, λ)

$$\pi(\beta, \lambda, \sigma|X, y) \propto -log\left[\frac{(2\pi)^{-\frac{n}{2}}\exp\left\{-\frac{1}{2}(\beta-\mu)\Psi(\beta-\mu)\right\}\prod_{i=1}^{n} w^{-\lambda/2}\{-b_1 - \frac{1}{2}\sum_{i=1}^{n} w^{-\lambda}(y - X\hat{a})^2\}^{-(a_1-1-n/2)}}{\sqrt{\left(\frac{N}{\acute{o}^2}\right)\left(\frac{|X'X|}{\acute{o}^{2k}}\right)\left(\frac{1}{2\eth\exp(1)}\right)}}\right] + \frac{n}{2}(1 + \log k_n) - \log h(n) (2.6)$$

Metropolis Hasting Algorithm update is performed on the full conditional distribution of$\acute{o}^2 \propto IG(a_1 + \frac{n}{2}, b_1 + \frac{1}{2}\sum_{i=1}^{n} w^{-\lambda}(y - X\hat{a})^2)$

and$\beta \propto N\left((X'w^{-1}X + \sigma^2 diag(\gamma)^{-2})^{-1}(X'w^{-1}y), \sigma^2\frac{1}{2}\sum_{i=1}^{n} w^{-\lambda}(y - X\hat{a})^2\right)$ (2.7)

This yields the following full conditional density of the parameters â and ó:

In order to expunge the nuisance parameter Marginal posterior density is obtained by integrating the joint posterior density with respect to nuisance parameter. The study adapted distributional approach which seems to be simple to derive the marginal posterior density.

Recall

$$\int_0^\infty \frac{q^p}{\Gamma p}x^{-(p+1)}\exp\left(\frac{-q}{x}\right)dx = 1 \quad (2.8)$$

Relating the joint posterior density with inverted gamma above $\frac{(\Psi)^{1/2}}{2\pi^{(k+1)/2}}(\sigma^2)^{-(a_1-1-n/2)}\exp\left\{-\frac{1}{2}(\beta - \mu)\Psi(\beta - \mu)\right\}\prod_{i=1}^{n} w^{-\delta/2}\exp\{-\frac{1}{\sigma^4}(b_1 + \frac{1}{2}\sum_{i=1}^{n} w^{-\delta}(y - X\beta)^2\}$ (2.9)

By setting $p = a_1 - 1 - n/2$

$$q = \exp\left\{-\frac{1}{2}(\beta - \mu)\Psi(\beta - \mu)\right\}\prod_{i=1}^{n} w^{-\delta/2}\{-\frac{1}{\sigma^4}(b_1 + \frac{1}{2}\sum_{i=1}^{n} w^{-\delta}(y - X\beta)^2\} \quad (2.10)$$

$$x = \sigma^2$$

$$\int_0^\infty x^{-(p+1)}\exp\left(\frac{-q}{x}\right)dx = \frac{\Gamma p}{q^p} \quad (2.11)$$

Therefore

$$\int_0^\infty x^{-(p+1)}\exp\left(\frac{-q}{x}\right)dx \propto q^{-p} \quad (2.12)$$

Thus the marginal posterior distribution for β is obtained by substitution intoq^{-p}

$$\pi(\beta|\lambda, X, y) \propto -log\left[\frac{\exp\left\{-\frac{1}{2}(\beta-\mu)\Psi(\beta-\mu)\right\}\prod_{i=1}^{n} w^{-\lambda/2}\{-\frac{1}{2}\sum_{i=1}^{n} w^{-\lambda}(y-X\beta)^2\}^{-(a_1-1-n/2)}}{\sqrt{\left(\frac{N}{\sigma^2}\right)\left(\frac{|X'X|}{\sigma^{2k}}\right)\left(\frac{1}{2\pi\exp(1)}\right)}}\right] + \frac{n}{2}(1+\log k_n) - \log h(n) \quad (2.13)$$

$$\pi(\sigma|\theta. X, y) \propto -log\left[\frac{(\sigma^2)^{-(a_1-1-n/2)}\exp(-b_1/\sigma^2)\prod_{i=1}^{n} w^{-\lambda/2}\{-\frac{1}{\sigma^2}(b_4+\frac{1}{2}\sum_{i=1}^{n} w^{-\lambda}(y-X\beta)^2\}^{-(a_1+n/2)}}{\sqrt{\left(\frac{N}{\sigma^2}\right)\left(\frac{|X'X|}{\sigma^{2k}}\right)\left(\frac{1}{2\pi\exp(1)}\right)}}\right] + \frac{n}{2}(1+\log k_n) - \log h(n) \quad (2.14)$$

$$\pi(\lambda|\beta, X, y) \propto -log\left[\frac{\prod_{i=1}^{n} w^{-\delta/2}(b_1+\frac{1}{2}\sum_{i=1}^{n} w^{-\delta}(y-X\beta)^2\}^{-(a_4+n/2)}}{\sqrt{\left(\frac{N}{\sigma^2}\right)\left(\frac{|X'X|}{\sigma^{2k}}\right)\left(\frac{1}{2\pi\exp(1)}\right)}}\right] + \frac{n}{2}(1+\log k_n) - \log h(n) \quad (2.15)$$

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3.0 Results

In this study, we presented MML87 heteroscedastic contaminated linear model, using multiplicative heteroscedasticity structure. Parameters were obtained through the posterior location and spread estimates of Metropolis –Hasting Algorithm simulation, The level of convergence of the chains were monitored using the method proposed by Gelman and Rubin [12] and graphic analysis was carried out using coda package in R package. Multivariate normal and inverse gamma distributions were chosen as priors for parameter estimates and σ^2 respectively.

Table1: Absolute bias of Posterior Estimation of BMML87 Heteroscedastic Linear Model@0.0

Samples	Absolute bias		
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
25	0.06410	0.00663	0.00235
50	0.04157	0.00316	0.00010
100	0.00773	0.00083	0.00044
200	0.02629	0.00155	0.00141
500	0.01051	0.00109	0.00051
1000	0.00457	0.00066	0.00060

Table 2: Absolute bias of Posterior Estimation of BMML87 Heteroscedastic Linear Model@0.3

Samples	Absolute bias		
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
25	0.08924	0.00909	0.00307
50	0.05268	0.00390	5.87E-5
100	0.00185	0.00075	0.00120
200	0.03189	0.00187	0.00172
500	0.01106	0.00136	0.00097
1000	0.00545	0.00078	0.00063

Table 3: Absolute bias of Posterior Estimation of BMML87 Heteroscedastic Linear Model@0.5

Samples	Absolute bias		
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
25	0.10877	0.01104	0.00366
50	0.06367	0.00471	7.85E-5
100	0.00225	0.00092	0.00147
200	0.03870	0.00227	0.00209
500	0.01343	0.00165	0.00118
1000	0.00662	0.00095	0.00076

Table 4: Absolute bias of Posterior Estimation of BMML87 Heteroscedastic Linear Model @ 0.6

Samples	Absolute bias		
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
25	0.12007	0.01216	0.00399
50	0.06998	0.00518	9.08E-5
100	0.00249	0.00101	0.00162
200	0.04262	0.00251	0.00230
500	0.01479	0.00182	0.00131
1000	0.00729	0.00105	0.00084

Table 5: Absolute bias of Posterior Estimation of BMML87 Heteroscedastic Linear Model @ 0.9

Samples	Absolute bias		
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
25	0.16142	0.01626	0.00517
50	0.09288	0.00687	0.00014
100	0.00334	0.00135	0.00218
200	0.05691	0.00336	0.00309
500	0.01976	0.00243	0.00176
1000	0.00974	0.00140	0.00113

Table 6: Absolute bias of Posterior Estimation of BMML87 Heteroscedastic Linear Model@1

Samples	Absolute bias		
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
25	0.17812	0.01790	0.00563
50	0.10205	0.00755	0.00016
100	0.00369	0.00149	0.00241
200	0.06266	0.00371	0.00341
500	0.02176	0.00268	0.00194
1000	0.01072	0.00154	0.00124

Table 7: Absolute bias of Posterior Estimation of BMML87 Heteroscedastic Linear Model@2

Samples	Absolute bias		
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
25	0.47437	0.04673	0.01295
50	0.26030	0.01925	0.00067
100	0.00976	0.00391	0.00648
200	0.16308	0.00971	0.00912
500	0.05662	0.00699	0.00521
1000	0.02791	0.00402	0.00330

Performances of the BMML87 Heteroscedastic linear model on the basis of Absolute Bias criterion

Table 1-7 revealed the outcome of our estimation of MML87 heteroscedastic linear model. It shows that the bias for β_0 decreases algebraically as sample size increases while it increased at sample size 200 and thereafter decreases at higher sample sizes across all scale of heteroscedasticity considered in the study. The bias for β_1 is negative and absolutely decreases algebraically as sample size increases while it increased at sample size 200 and thereafter decreases at higher sample sizes across all scale of heteroscedasticity considered in the study. These depict consistency. The bias for β_2 is interchangeable, it increases and decreases algebraically as sample size increases which bring about inconsistency.Considering the degree of heteroscedasticity, we observed that the bias for β_0 increases algebraically as the scale of heteroscedasticity increases, the bias for β_1 is negative and absolutely increases algebraically as the scale of heteroscedasticity increases, and also the bias for β_2 absolutely increases algebraically as the scale of heteroscedasticity increases. Thus there exists consistency.

Table 8: Mean Squared Error criterion of Posterior Estimation MML87 Estimator@0.0

samples	Mean squared Errors		
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
25	6.48009	0.03813	0.03930
50	3.41774	0.01914	0.01456
100	1.66303	0.00831	0.00721
200	0.72911	0.00379	0.00391
500	0.32890	0.00163	0.00140
1000	0.15650	0.00077	0.00081

Table 9: Mean Squared Error criterion of Posterior Estimation MML87 Estimator@0.3

samples	Mean squared Errors		
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
25	11.7418	0.06844	0.07096
50	6.05299	0.03390	0.02620
100	2.99225	0.01507	0.01297
200	1.29466	0.00677	0.00707
500	0.58449	0.00292	0.00254
1000	0.28048	0.00138	0.00147

Table 10: Mean Squared Error criterion of Posterior Estimation MML87 Estimator @0.5

samples	Mean squared Errors		
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
25	17.4542	0.10112	0.10514
50	8.84275	0.04950	0.03883
100	4.41316	0.02236	0.01920
200	1.90625	0.01002	0.01047
500	0.86052	0.00432	0.00376
1000	0.41272	0.00204	0.00217

Table 11: Mean Squared Error criterion of Posterior Estimation MML87 Estimator@0.6

samples	Mean squared Errors		
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
25	21.2747	0.12286	0.12800
50	10.6849	0.05979	0.04728
100	5.35801	0.02723	0.02337
200	2.31238	0.01218	0.01274
500	1.04380	0.00525	0.00457
1000	0.50051	0.00247	0.00265

Table 12: Mean Squared Error criterion of Posterior Estimation MML87 Estimator@0.9

Samples	Mean squared Errors		
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
25	38.4844	0.22010	0.23103
50	18.8284	0.10526	0.08535
100	9.57812	0.04908	0.04211
200	4.12247	0.02186	0.02296
500	1.86048	0.00942	0.00824
1000	0.89159	0.00443	0.00474

Table 13: Mean Squared Error criterion of Posterior Estimation MML87 Estimator@1

Samples	Mean squared Errors		
	$\hat{\boldsymbol{\beta}}_0$	$\hat{\boldsymbol{\beta}}_1$	$\hat{\boldsymbol{\beta}}_2$
25	46.8747	0.26719	0.28133
50	22.7333	0.12705	0.10393
100	11.6201	0.05970	0.05126
200	4.99664	0.02655	0.02794
500	2.25482	0.01144	0.01002
1000	1.08038	0.00538	0.00577

Table 14: Mean Squared Error criterion of Posterior Estimation MML87 Estimator@2

Samples	Mean squared Errors		
	$\hat{\boldsymbol{\beta}}_0$	$\hat{\boldsymbol{\beta}}_1$	$\hat{\boldsymbol{\beta}}_2$
25	333.787	1.83311	2.02460
50	148.152	0.82395	0.74699
100	79.4487	0.41774	0.36643
200	33.8050	0.18288	0.19899
500	15.2393	0.07857	0.07152
1000	7.29500	0.03700	0.04094

Performances of the BMML87 Heteroscedastic Linear Model on the basis of Mean Squares Error Criterion

Tables 8-14revealed the mean squared error criterion, the mean squares error for β_0 decreases algebraically as the sample size increases irrespective of the scale of heteroscedasticity. Thus sample size 1000 has the least mean squares error, asymptotically, larger sample size bring about improvement in the estimation and reduce the effect of the error on the inferences. Moreover, the mean squares error for both β_1 and β_2 have asymptotic efficiency since the mse decreases as the sample size increases.

Considering the scale of heteroscedasticity, the study revealed that the mean squared error increases as the scale of heteroscedasticity increases for posterior mean of β_0, β_1 and β_2.

4.0 Conclusion

In this paper, we have presented a simple way of modeling and estimating heteroscedastic linear model under simulation approach (MCMC) by incorporating it into celebrated Minimum Message Length87. We observed that modeling heteroscedasticity in a full Bayesian improve the precision of the inferences of the estimates. We conclude that asymptotically there exist consistency and efficiency in the estimation. Results from this study would assist social and behavioural scientists if the methodology is adopted when presence of heteroscedasticity is established. This will enable them to have good precision of the inferences of the models parameters estimate. The algorithm is faster and efficient in computation. The performance of MML87 heteroscedastic prove robust compare to conventional ordinary least squares which had already been established in the literature that is inefficient in the face of heteroscedasticity Hadri and Gumert [7] Our approach can be applied to further studies in the area of simultaneous equation and other econometric models.

5.0 References

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APPENDIX

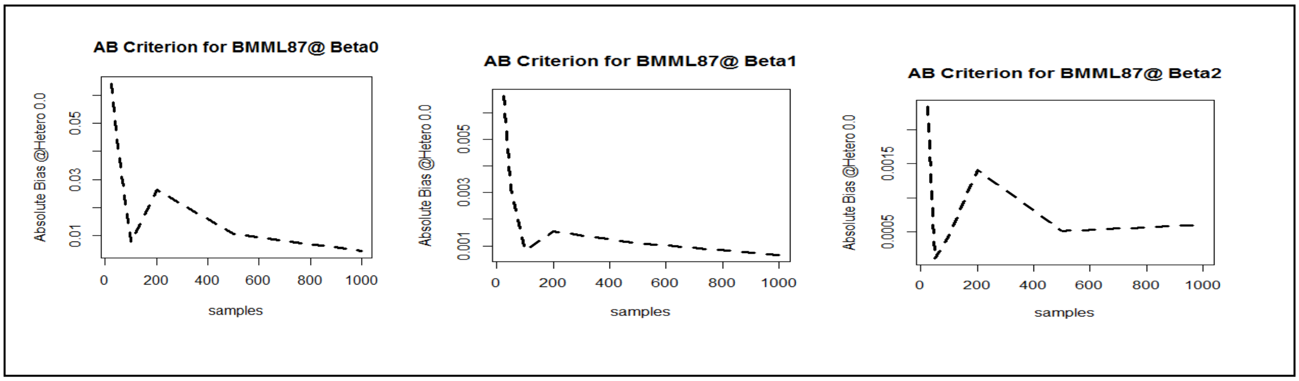


Figure 1: showing Absolute bias of Posterior Estimation of BMML87 Heteroscedastic Linear Model at beta 0, beta 1 and beta2 with 0 degree of heteroscedasticity

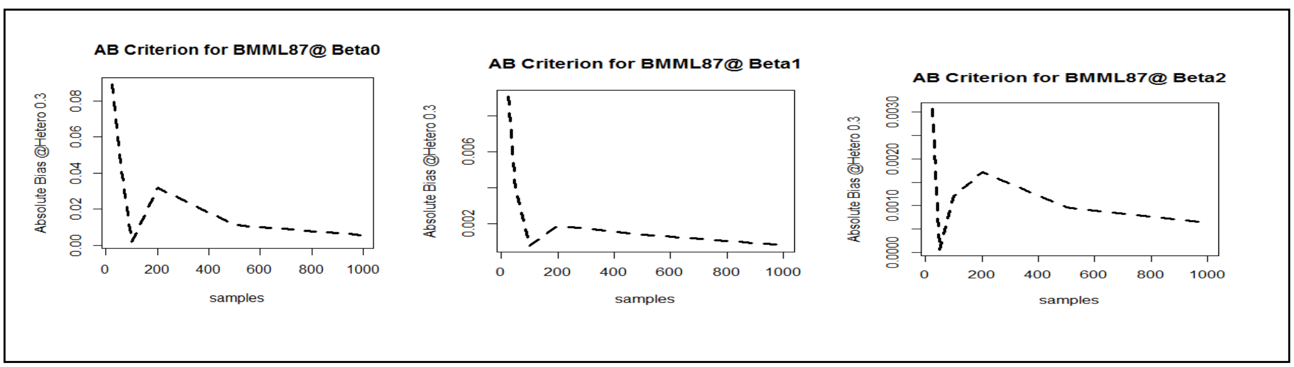


Figure 2: showing Absolute bias of Posterior Estimation of BMML87 Heteroscedastic Linear Model at beta 0, beta 1 and beta2 with 0.3 degree of heteroscedasticity

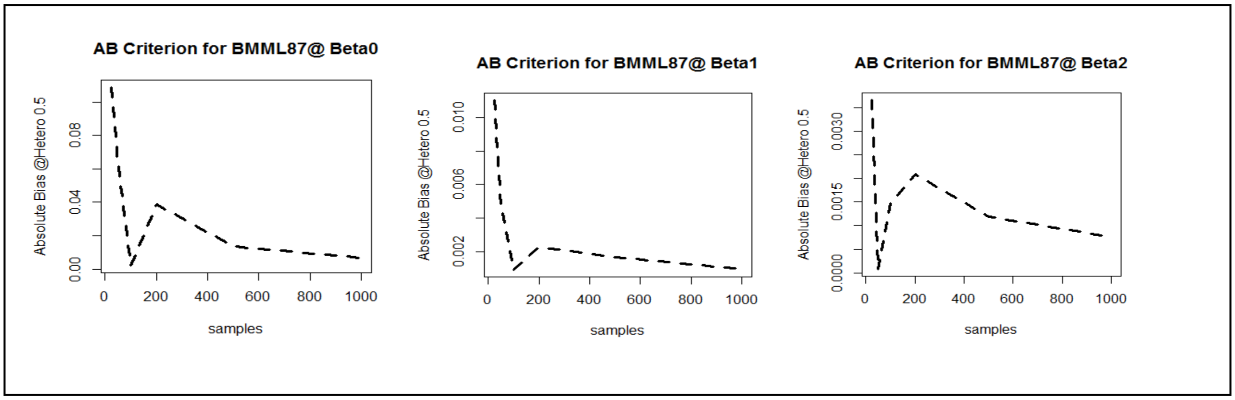


Figure 3: showing Absolute bias of Posterior Estimation of BMML87 Heteroscedastic Linear Model at beta 0, beta 1 and beta2 with 0.5 degree of heteroscedasticity

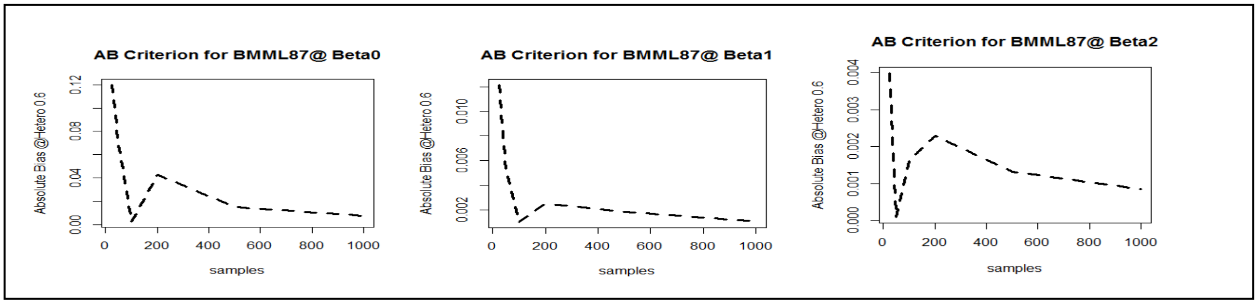


Figure 4: showing Absolute bias of Posterior Estimation of BMML87 Heteroscedastic Linear Model at beta 0, beta 1 and beta2 with 0.6 degree of heteroscedasticity

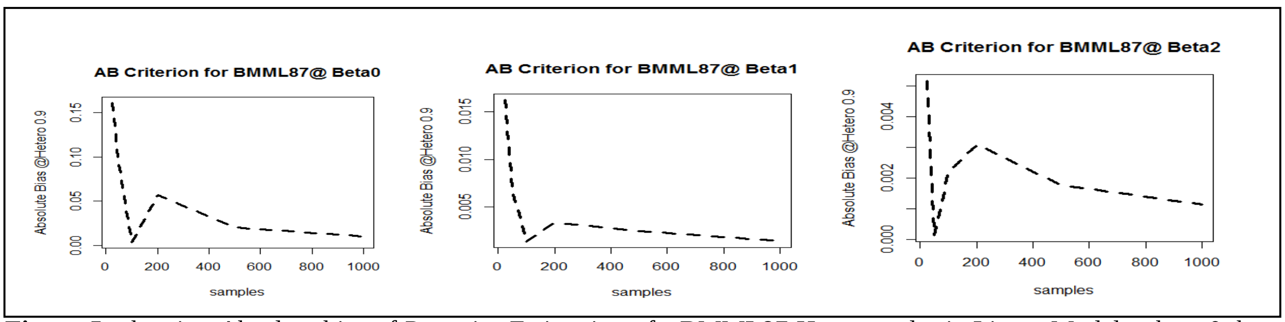


Figure 5: showing Absolute bias of Posterior Estimation of BMML87 Heteroscedastic Linear Model at beta 0, beta 1 and beta2 with 0.9 degree of heteroscedasticity

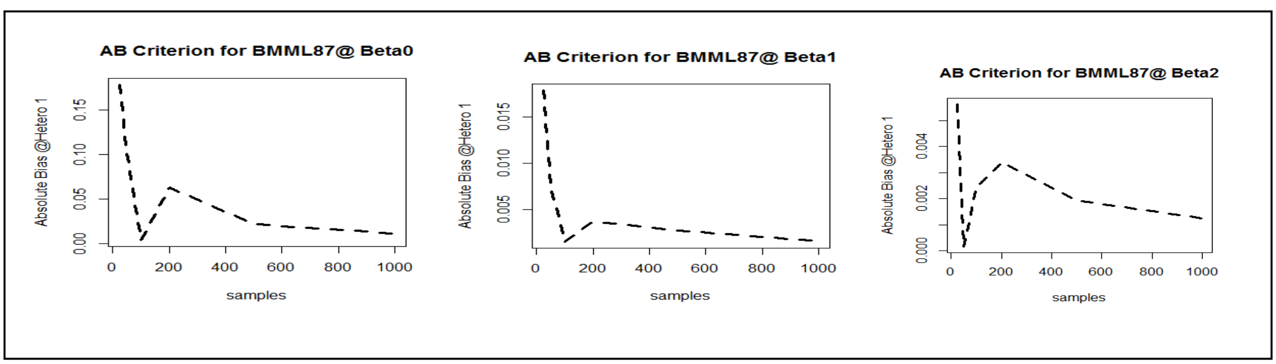


Figure 6: showing Absolute bias of Posterior Estimation of BMML87 Heteroscedastic Linear Model at beta 0, beta 1 and beta2 with 1 degree of heteroscedasticity

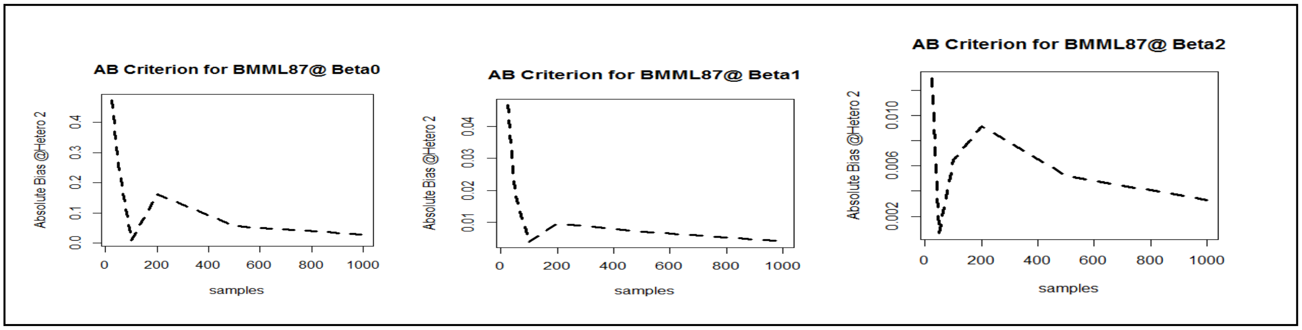


Figure 7: showing Absolute bias of Posterior Estimation of BMML87 Heteroscedastic Linear Model at beta 0, beta 1 and beta2 with 2 degree of heteroscedasticity

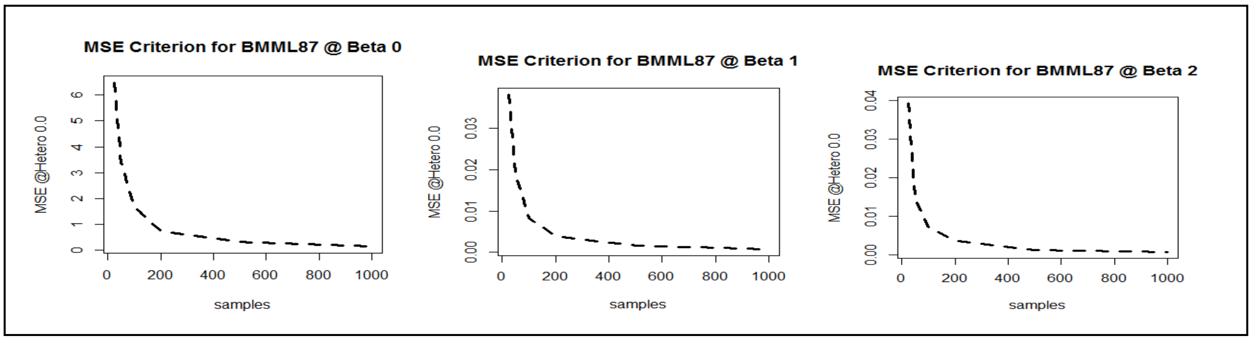
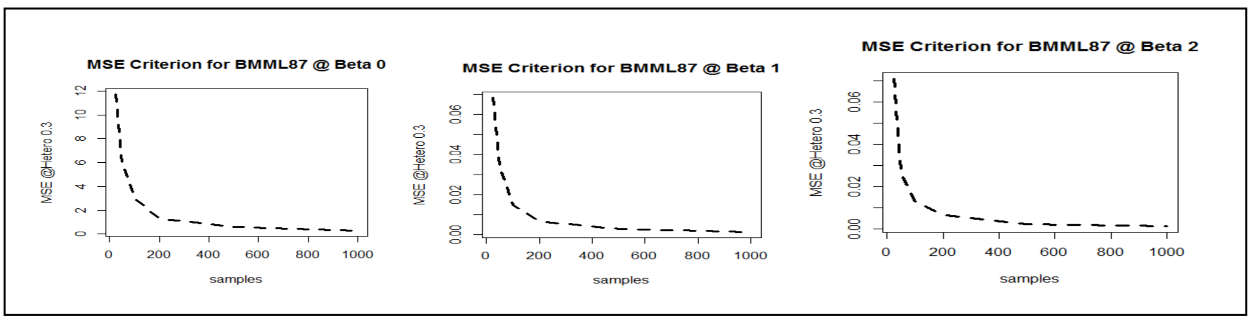


Figure 8: showing Mean Squared Error criterion of Posterior Estimation MML87 Estimator at beta 0, beta 1 and beta2 with 0 degree of heteroscedasticity



Mean Squared Error criterion of Posterior Estimation MML87 Estimator @0.5

Figure 9: showing Mean Squared Error criterion of Posterior Estimation MML87 Estimator at beta 0, beta 1 and beta2 with 0.3 degree of heteroscedasticity

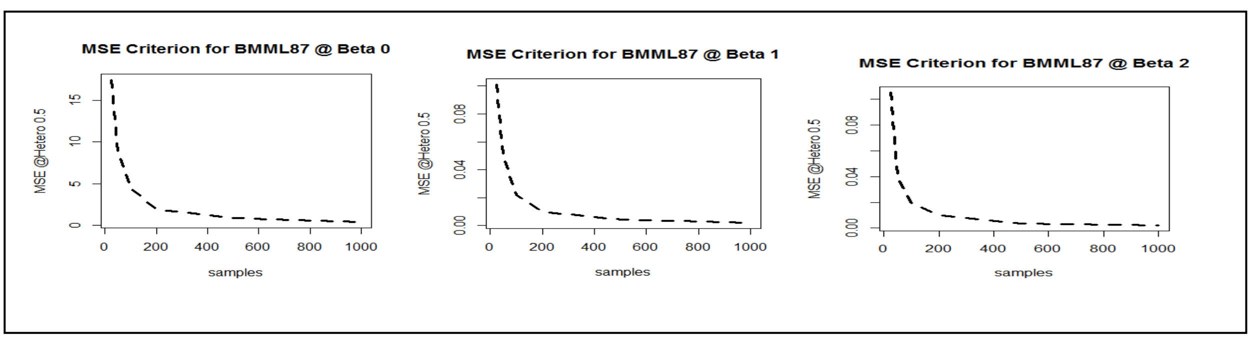


Figure 10: showing Mean Squared Error criterion of Posterior Estimation MML87 Estimator at beta 0, beta 1 and beta2 with 0.5 degree of heteroscedasticity

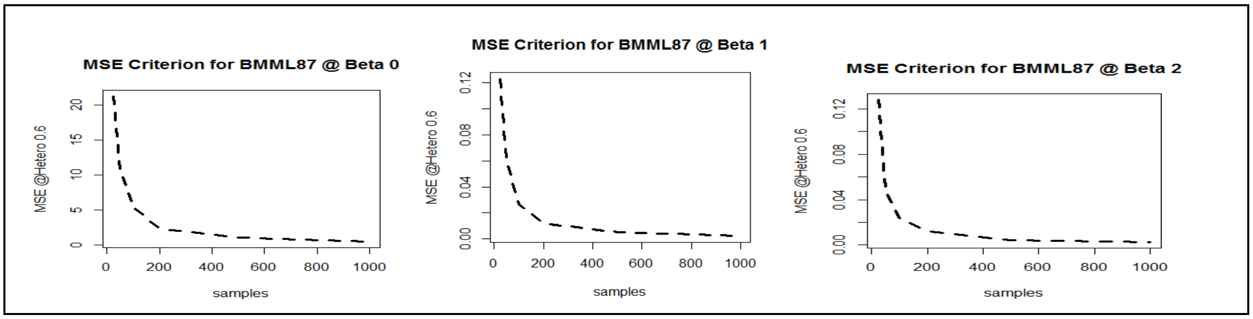


Figure 11: showing Mean Squared Error criterion of Posterior Estimation MML87 Estimator at beta 0, beta 1 and beta2 with 0.6 degree of heteroscedasticity

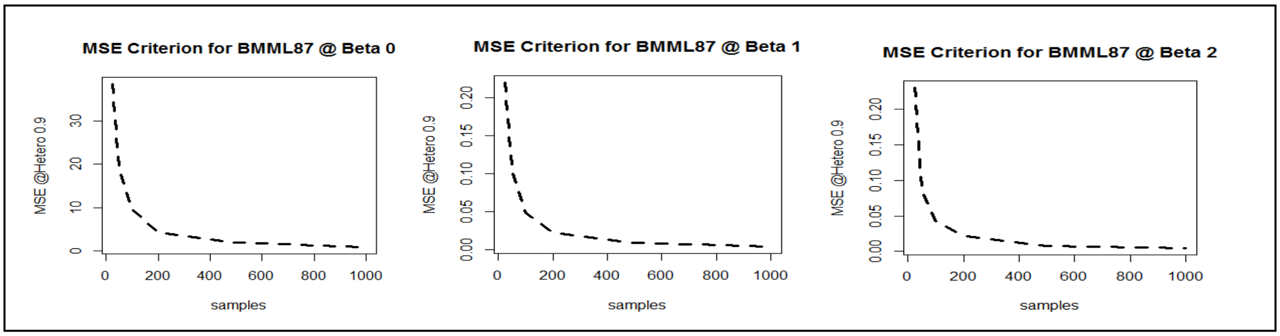
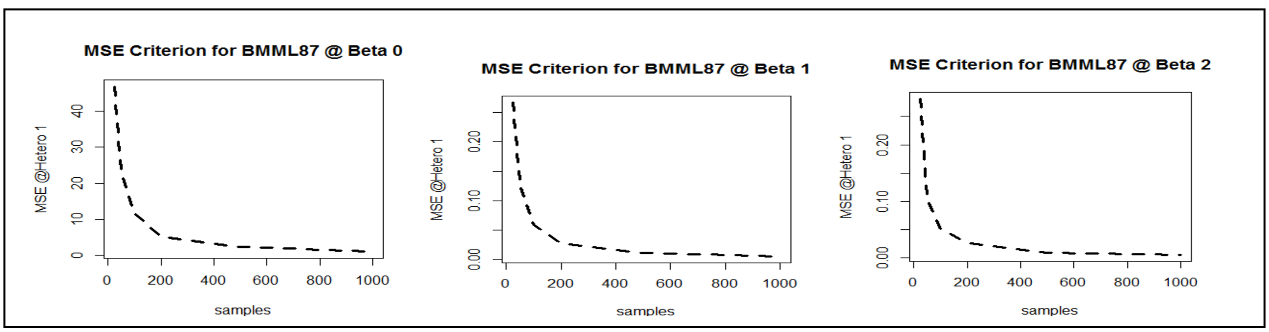


Figure 12: showing Mean Squared Error criterion of Posterior Estimation MML87 Estimator at beta 0, beta 1 and beta2 with 0.9 degree of heteroscedasticity



Mean Squared Error criterion of Posterior Estimation MML87 Estimator@2

Figure 13: showing Mean Squared Error criterion of Posterior Estimation MML87 Estimator at beta 0, beta 1 and beta2 with 1 degree of heteroscedasticity

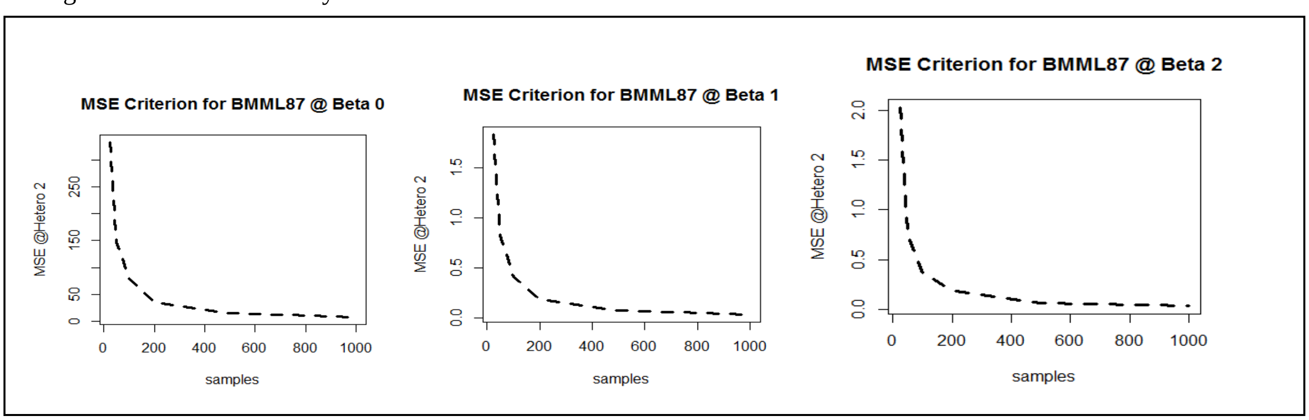


Figure 14: showing Mean Squared Error criterion of Posterior Estimation MML87 Estimator at beta 0, beta 1 and beta2 with 2 degree of heteroscedasticity