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Application of Multi-Set to loglinear models for arbitrary d-Dimensional Contingency Table and its Associated Closed-Form Formula for Parameter Estimations

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Abstract

The purpose of this research paper is to give a classical (combinatorial) proof for the closed-form formula that evaluates the parameter estimation for arbitrary d-dimensional Multi-index. Statistically, this can be referred to as d-dimensional contingency table, such that the index (running) variable $\bar{\imath}_d = i_1 i_2 \dots i_d or i_1, i_2, \dots, i_d$ is not necessarily a point (i), but rather a vector ($\bar{\imath}_d$)=($i_1, i_2, \dots, i_d$), where $i_r \epsilon [k_r]$, $k_r \epsilon \mathbb{N}$, $r \epsilon [d]$.

Keywords: Permutations, Combinations, Multi-Set, Multi-Index, Probability, contingency table.
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1.0 Introduction

The joint-distribution of several (categorical) variables defines an array of values that generate a array (matrice), which is statistically refers to as contingency table. research in the typical models for contingency tables are of growing interest due to the routine collection of data on moderate to large numbers of categorical variables. The statistical analysis of contingency table routinely relies on log-linear model, and it turns out that a direct route to a likelihood function for a log-linear model lead to multinomial (product-multinomial) distribution, that is to say that the underlying distribution for log-linear model will turn out to be multinomial (product-multinomial) distribution or the (product of) independent Poisson distribution. A standard approach to contingency table analysis parameterizes P as a log-linear model satisfying certain constraints. The log linear models express the logarithms of the joint probability mass function P of the variables as a linear function of parameters related to the index of each cell of the array or contingency table. Most of these parameters describe the interactions among the variables. Log-linear models for contingency table have many specialized applications in the social science, for example, the square-table, such as mobility table, where the variables in the table have the same categories. The development of appropriate models and test statistics are the major themes of several books **[1-14]** and others, have made important contributions to the contingency table literature. In the sequel,we shall state an explicit formula with proof for a closed form expression for:

(1). saturated log-linear models (with its subclass)

(2). their parameters estimate for arbitrary d-dimensionalarray (d-dimensional contingency table).

To achieve this, we give the following basic introduction, let α_{ij}be an observed frequency count in the (ij)th cell for a given $k \times m$contingency table for two (categorical) variables with k and m categories respectively, such that $\alpha_{i+} = \sum_{j=1}^{m} \alpha_{ij}$ is the marginal frequency count in the $(i)th$ row; $\alpha_{+j} = \sum_{i=1}^{k} \alpha_{ij}$ is the marginal frequency count in the $(j)th$ column and \$$n = \alpha_{++} = \sum_{i=1}^{k}.\sum_{j=1}^{m} \alpha_{ij}$ is the number of observation in the sample, with the assumption that the n observations are independently sampled from the population with proportion p_{ij}in cell {ij}, which correspond to the probability of sampling an individual observation in the cell. Consequently, the marginal probability distributions p_{i+} andp_{+j} is similarly defined as above. If the row and column variable are independent in the population, then $p_{ij} = p_{i+}p_{+j}$. Thus, if we let p_{ij} to be the expected probability in each cell, then the expected frequency μ_{ij} in cell $\{ij\}$ is define by $\mu_{ij} = n\pi_{ij}$; by independent of the variables we have ;

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$$\mu_{ij} = n\pi_{ij} = \frac{\mu_{i+}\mu_{+j}}{n} \quad (1.1)$$

Where $\mu_{i+} = \sum_{j=1}^{m} \pi_{ij}$ $and \mu_{+j} = \sum_{i=1}^{k} \pi_{ij}$

If we take natural log of (1.1), we have

$$\pi_{ij} = \log_e \mu_{i+} + \log_e \mu_{+j} - \log_e n \quad (1.2)$$

Where$\eta_{ij} = \log_e \mu_{ij}$.

Equation (1.2) is reminiscent of two-way ANOVA model. Hence by imposing the ANOVA-like sigma constraints on the model and then re-parameterize (1.2) so that

$$\eta_{ij} = \mu + \lambda_i - \gamma_j \quad (1.3)$$

Where $\lambda_+ = \sum_{i=1}^{k} \lambda_i = 0$ $and \gamma_+ = \sum_{j=1}^{m} \gamma_j = 0$ is the sigma constraints condition.

We call equation(1.3) the loglinear model for independency in the two-way (dimensional) table. Solving for the parameters in (1.3) we have that

$$\eta_{++} = \sum_{ij} \eta_{ij} = \sum_{ij} \mu + \sum_{ij} \lambda_i + \sum_{ij} \gamma_j ; \Longrightarrow \sum_{ij} \eta_{ij} = km\mu; \Longrightarrow \mu = \frac{\sum_{ij} \eta_{ij}}{km\mu}$$

$$\eta_{+} = \sum_{j} \eta_{ij} = \sum_{j} \mu + \sum_{j} \lambda_i + \sum_{j} \gamma_j ; \Longrightarrow \sum_{j} \eta_{ij} = m\mu + m\lambda_i; \Longrightarrow \mu = \frac{\sum_{ij} \eta_{ij}}{km\mu}$$

$$\eta_{+} = \sum_{i} \eta_{ij} = \sum_{i} \mu + \sum_{i} \lambda_i + \sum_{i} \gamma_j ; \Longrightarrow \sum_{ij} \eta_{ij} = k\mu + k\gamma_j; \Longrightarrow \mu = \frac{\sum_{ij} \eta_{ij}}{km\mu}$$

Where$\sum_{ij}. = \sum_{i=1}^{k}.\sum_{j=1}^{m}$.

Analogous to the two-way ANOVA model, we can add parameters to extend the loglinear model to data for which the row and column classifications are not independent in the population but are rather related in an arbitrary manner, as such we have.

$$\eta_{ij} = \mu + \lambda_i + \gamma_j + \delta_{ij} \quad (1.4)$$

Where $\mu = \lambda_+ = \gamma_+ = \delta_{+j} = \delta_{i+} = 0 \quad \forall\, i\, j$

Basically, the solution of (1.4) and (1.3) are the same for μ, λ_i, γ_j so that $\delta_{ij} = \eta_{ij} - \mu - \lambda_i - \gamma_j$is easily determined. δ_{ij} are referred as the Association Parameters since they represent deviation from independence. We call equation (1.4) the saturated (loglinear) model for the two-way contingency table.Similarly, a saturated loglinear model for a three-way contingency table of variables denoted by 1, 2, and 3 is given by

$$\eta_{ijk} = \mu + \mu_{1(i)} + \mu_{2(j)} + \mu_{3(k)} + \mu_{12(ij)} + \mu_{13(ik)} + \mu_{23(jk)} + \mu_{123(ijk)} ; \quad (1.5)$$

Using the sigma constraints on the parameter we can solve for the parameters as follows

$$\eta_{+++} = \sum_{ijk} \eta_{ijk} = \sum_{ijk} \mu + \sum_{ijk} (\mu_{1(i)} + \mu_{2(j)} + \mu_{3(k)} + \mu_{12(ij)} + \mu_{13(ik)} + \mu_{23(jk)} + \mu_{123(ijk)})$$

$$\Longrightarrow \sum_{ijk} \eta_{ijk} = k_1 k_2 k_3 \mu ; \Longrightarrow \mu = \frac{\sum_{ijk} \eta_{ijk}}{k_1 k_2 k_3}$$

$$\eta_{i++} = \sum_{jk} \eta_{ijk} = \sum_{jk} \mu + \sum_{jk} (\mu_{1(i)} + \mu_{2(j)} + \mu_{3(k)} + \mu_{12(ij)} + \mu_{13(ik)} + \mu_{23(jk)} + \mu_{123(ijk)})$$

$$\Longrightarrow \sum_{jk} \eta_{ijk} = k_2 k_3 \mu + k_2 k_3 \mu_{1(i)}; \Longrightarrow \mu_{1(i)} = \frac{\sum_{jk} \eta_{ijk}}{k_2 k_3} - \mu$$

Similarly, as we varies the summation over ijk (in 2's) we have

$$\mu_{1(j)} = \frac{\sum_{jk} \eta_{ijk}}{k_1 k_3} - \mu$$

and

$$\mu_{1(k)} = \frac{\sum_{jk} \eta_{ijk}}{k_1 k_2} - \mu$$

$$\eta_{ij+} = \sum_{k} \eta_{ijk} = \sum_{k} \mu + \sum_{k} (\mu_{1(i)} + \mu_{2(j)} + \mu_{3(k)} + \mu_{12(ij)} + \mu_{13(ik)} + \mu_{23(jk)} + \mu_{123(ijk)})$$

$$\sum_{k} \eta_{ijk} = k_3 \mu + k_3 \mu_{1(i)} + k_3 \mu_{2(j)} + k_3 \mu_{12(ij)} ;$$

$$\Longrightarrow \mu_{12(ij)} = \frac{\sum_k \eta_{ijk}}{k_3} - \mu - \mu_{1(i)} - \mu_{2(j)}$$

Similarly, as we varies the summation over ijk we have

$$\mu_{13(ik)} = \frac{\sum_j \eta_{ijk}}{k_2} - \mu - \mu_{1(i)} - \mu_{3(k)}$$

and

$$\mu_{23(jk)} = \frac{\sum_i \eta_{ijk}}{k_1} - \mu - \mu_{2(j)} - \mu_{3(k)}$$

However, several other log linear models are nested within the saturated model and could be obtained by setting some of the parameters equal to zero. In specifying this restriction, we ensure that whenever a high order term is included in the model, it lower-order relatives are included as well. Loglinear models of this type are often called Hierarchical. Among all this authors mentioned above non have been able to formulate an explicit formula that is easy to work with and at the same time expressing the property of the interactions of the variables in a concise and simply form for loglinear model and there parameter estimates as we aimed to do for an arbitrarily d-dimensional contingency table. To achieve this, we shall introduce certain concept, notations and definitions in the section that follows.

The purpose of this research, is to consider the sets $X^{(d)}$and $M(\lambda, X^{(d)})$that is more general than the set Xsuch that the index (running) variable ($\bar{\imath}_d$)=($i_1, i_2, \dots, i_d$) is not necessarily a point, but rather a vector, where $i_r \epsilon [k_r],\ k_r \epsilon \mathbb{N},\ r \epsilon [d]$. Let

$$X^{(d)} = \{ x_{\bar{\imath}_d} : i_r \epsilon [k_r], k_r \epsilon \mathbb{N}, r \epsilon [d] \}$$

and then define

$$M(\lambda, X^{(d)}) = \{ {x_{\bar{\imath}_d}}^{\alpha_{\bar{\imath}_d}} : \alpha_{\bar{\imath}_d} \epsilon \alpha^{(d)} \}$$

to be the Multiset induced by $X^{(d)}$ due to the function $\lambda: X^{(d)} \to \mathbb{N}$such that $\lambda(x_{\bar{\imath}_d}) = \alpha_{\bar{\imath}_d}$. Where $\alpha^{(d)}$ is a multi-index. We then give a classical proof of the associated closed-form formula forestimating the associated parameters.

1.1 Multiset and Multinomial

Definition1.1.1 [15-16]: A finite multiset $M(\lambda, X)$ (orM) on a set X is a function $\lambda: X \to \mathbb{N}$such that

$$\sum \lambda\ (x) < \infty$$

If $\lambda(x) = n \forall x \in X$, then M is called an n-multiset, hence we write $n(M) = n$. Suppose $X = \{x_i : i = 1,2, \dots, k\}$ and $\lambda: X \to \mathbb{N}$ such that $\lambda\ (x_i) = \alpha_i$, we shall have $M = \{x_i^{\alpha_i} : i = 1,2, \dots, k\}$, where α_i is called the multiplicity of $x_i (inM)$ and $(\alpha_1, \alpha_2, \dots, \alpha_k)$ is called the (associated) multi-index (or weak composition), which is also a row matrix (vector). For simplicity we write $\alpha = (\alpha_{1i}, \alpha_{2i}, \dots, \alpha_k)$.We quickly remark that the function $\lambda: X^{(d)} \to \mathbb{N}$ is the so-called "random variable" as often used by statisticians. To see this, given any finite Multiset M, then there exist $\lambda: X^{(d)} \to \mathbb{N}$ such that $\lambda(x_i) = \alpha_i (i = 1,2, \dots, k)$ and $\sum \lambda\ (x_i) = n$ since M is finite. If we let $\lambda\ (x_i) = X_i$, then $X_i$is a random variable that count the occurences of outcome $x_i$in X (i.e. $X_i = \alpha_i;\ i = 1,2, \dots, k$).

Definition1.2.1 [17-19] By Multi-index, we mean a k-tuple vector (a row matrix or a column matrix) α_i, where each (α_i: $[k] = \{1,2, \dots, k\}$) is a non-negative interger. We define

The associated integer $|\alpha_i|$by

$$|\alpha| = \sum_{i=1}^{k} \alpha_i \qquad (1.6)$$

The associated monomial x^α by

$$x^\alpha = \prod_{i=1}^{k} x_i^{\alpha_i} \qquad (1.7)$$

The associated factorial $\alpha!$ by

$$\alpha! = \prod_{i=1}^{k} \alpha_i\,! \qquad (1.8)$$

Let $X = \{x_1, x_2, \dots, x_k\}$ be a distinct finite set of points. If we associate to each element $x_i \in X$ with the numberα_i in αthen certainly there exist a non-empty set $M(\alpha, X)$ induced by a non-negative integer $\lambda: X \to \mathbb{N}$such that x_i has multiplicity $\alpha_i$in $M(\alpha, X) or\ (M(\lambda, X))$, which is define by

$$M(\lambda, X) = \{x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_k^{\alpha_k}\} \qquad (1.9)$$

is the multiset associated with X with respect to the non-negative integer function on X. Now consider the expansion of $(x_1 + x_2 + \cdots + x_k)^n$, observe that if $k = 2,3$ then we have the binomial, trinomial expansion respectively. For arbitrary but fixed positive integer k the expansion of $(\sum_{i=1}^{k} x_i)^n$is a multinomial expansion of X in one running (index) variable i, which can be referred to as one category or class of data. Observe that each $x_i$has certain number of repeatition or multiplicity in the expansion of $(\sum_{i=1}^{k} x_i)^n$. There is no loss of generality if we assume that the multiplicity of x_i in the expansion of $(\sum_{i=1}^{k} x_i)^n$ is α_i; $i = 1,2,\ldots,k$ provided $\sum \alpha_i = n$. Thus, this will certainly induce a multiset representation due to the multinomial expansion, as such we have $\{x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_k^{\alpha_k}\}$as in (1.9). Furthermore, observe that each term (string) in this multinomial (k-nomial) expansion can be given in the general form

$$C(\alpha_1, \alpha_2, \ldots, \alpha_k) x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k} \quad (1.10)$$

Where $C(\alpha_1, \alpha_2, \ldots, \alpha_k)$is the associated k-nomial coefficient for each term. The following lemma gives the actual formular for $C(\alpha_1, \alpha_2, \ldots, \alpha_k)$ and corresponding probability mass function.

Now, we extend our description above to two-dimensional multiset and its associated two-dimensional multi-index $\alpha^{(2)} = (\alpha_{ij})$, by considering the expansion $\left(\sum_{i=1}^{k} . \sum_{j=1}^{m} x_i\right)^n$ where the multiplicity α_{ij} $(i \in [k],\ j \in [m],\ k, m \in \mathbb{N})$ for each term $x_{ij} \in X^{(2)} (i \in [k],\ j \in [m],\ k, m \in \mathbb{N})$ induces a $k \times m$ array (vector) where each α_{ij} $(i \in [k],\ j \in [m],\ k, m \in \mathbb{N})$ is a non-negative integer. Hence for the vector

$$\alpha^{(2)} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_{k1} & \cdots & \alpha_{km} \end{pmatrix}, or \alpha^{(2)} = (\alpha_{ij}): i \in [k], \qquad j \in [m]$$

We define

The associated integer $|\alpha^{(2)}|$ by

$$\alpha^{(2)} = \sum_{i=1}^{k} . \sum_{j=1}^{m} \alpha_{ij} \ ; where \left|\alpha_i^{(2)}\right| = \sum_{j=1}^{m} \alpha_{ij} \ ; \ i \in [k] \quad (1.11)$$

The associated monomial x^{α} by

$$x^{\alpha^{(2)}} = \prod_{i=1}^{k} . \prod_{j=1}^{m} x_{ij}^{\alpha_{ij}} \ ; \text{where } x_i^{\alpha^{(2)}} = \prod_{j=1}^{m} x_{ij}^{\alpha_{ij}} \quad (1.12)$$

The associated factorial $\alpha!$ by

$$\alpha^{(2)}! = \prod_{i=1}^{k} . \prod_{j=1}^{m} \alpha_{ij}!; \text{ where } \alpha_i^{(2)}! = \prod_{j=1}^{m} x_{ij}^{\alpha_{ij}}! \quad (1.13)$$

It is important to remark that in lemma 1.2.1, given the set $X^{(2)} = \{x_{ij} : i \in [k], j \in [m]\}$and $\lambda: X^{(2)} \to \mathbb{N}$**,** then that there exist a two-dimensional finite multiset $M_{(2)}(\alpha^{(2)}, X^{(2)}) = \{x_{ij}^{\alpha_{ij}} : i \in [k], j \in [m]\}$ on $X^{(2)}$ with it corresponding multi-index $\alpha^{(2)} = (\alpha_{ij}) i \in [k], j \in [m]$. By lemma 1.2.1, we wish to generalise the result for an arbitrary d-dimensional finite multiset $M_{(d)}(\alpha^{(d)}, X^{(d)})$ with the corresponding d-dimensional multi-index $\alpha^{(d)} = \alpha_{i_1, i_2, \ldots, i_d}$, by considering the expansion $\left(\sum_{i_1=1}^{k_1} \ldots \sum_{i_d=1}^{k_d} x_{i_1, i_2, \ldots, i_d}\right)^n$ where the multiplicity $\alpha_{i_1, i_2, \ldots, i_d}$ $(i_r \epsilon [k_r], k_r \epsilon \mathbb{N}, r \epsilon [d])$ for each term $x_{i_1, i_2, \ldots, i_d} \in X^{(d)} (i_r \epsilon [k_r], k_r \epsilon \mathbb{N}, r \epsilon [d])$ induces a $k_1 \times k_2 \times \cdots \times k_d$ vector (array) where each $\alpha_{i_1, i_2, \ldots, i_d}$ $(i_r \epsilon [k_r], k_r \epsilon \mathbb{N}, r \epsilon [d])$ is a non-negative integer. Similarly for the vector $\alpha^{(d)}$. we define

The associated integer $|\alpha^{(2)}|$ by

$$\alpha^{(d)} = \sum_{i_1=1}^{k_1} \ldots \sum_{i_d=1}^{k_d} \alpha_{i_1, i_2, \ldots, i_d} \quad (1.14)$$

$$where \ \left|\alpha_{i_1, i_2, \ldots, i_u}^{(d)}\right| = \sum_{i_{u+1}=1}^{k_{u+1}} \ldots \sum_{i_d=1}^{k_d} \alpha_{i_1, i_2, \ldots, i_u i_{u+1}, \ldots, i_d} \ ; 1 \le u < d$$

The associated monomial x^{α} by

$$x^{\alpha^{(d)}} = \prod_{i_1=1}^{k_1} \ldots \prod_{j_d=1}^{k_d} x_{i_1, i_2, \ldots, i_d}^{\alpha_{i_1, i_2, \ldots, i_d}} \quad (1.15)$$

$$\text{where } x_{i_1, i_2, \ldots, i_u}^{\alpha^{(2)}} = \prod_{i_{u+1}=1}^{k_{u+1}} \ldots \prod_{i_d=1}^{k_d} x_{i_1, i_2, \ldots, i_u i_{u+1}, \ldots, i_d}^{\alpha_{i_1, i_2, \ldots, i_u i_{u+1}, \ldots, i_d}} \ ; 1 \le u < d$$

The associated factorial $\alpha^{(d)}!$ by

$$\alpha^{(d)}! = \prod_{i_1=1}^{k_1} \cdots \prod_{j_d=1}^{k_d} \alpha_{i_1,i_2,\dots,i_d}! = |\alpha_{\bar{\imath}_d}|! \quad (1.16)$$

Where

$$\alpha^{(d)}_{i_1,i_2,\dots,i_u}! = \prod_{i_{u+1}=1}^{k_{u+1}} \cdots \prod_{j_d=1}^{k_d} \alpha_{i_1,i_2,\dots,i_u i_{u+1},\dots,i_d}! \, and \, |\alpha_{\bar{\imath}_u}|! = |\alpha_{i_{u-1}1}|!\,|\alpha_{i_{u-1}2}|! \cdots |\alpha_{i_{u-1}k_u}|!$$

2.0 Preliminary

For arbitrary but fixed $d \in \mathbb{N}, let\ [d] = \{1,2,\dots,d\}$ denote the set of d categorical variables. Let $\alpha^{(d)}$ be a d-dimensional array; that is $k_1 \times k_2 \times \cdots \times k_d$ contingency table with cell counts (frequencies) $\alpha_{i_1,i_2,\dots,i_d}$. For any $r \in \mathbb{N}$ such that $1 < r < d$, then the subset variable $[r] \subset [d]$ generate $\alpha^{(d)} r$-dimensional sub array with cell counts (frequencies) $\alpha_{i_1,i_2,\dots,i_r} = \alpha_{\bar{\imath}_r}$ $(r \le d)$. Let $\mu_{[r](\bar{\imath}_r)}$ denote the interaction among the variables in the index subset $[r]$ of the r-dimensional sub table of $\alpha^{(d)}$ that correspond to $\bar{\imath}_r$ cell. we shall assume that $[r] = \emptyset\ if\ r = 0$, so that we define $\mu_{[\emptyset](\bar{\imath}_\emptyset)} = \mu$.

Definition 2.1.1([20])

A loglinear model is said to be hierarchical if for every $r \in \mathbb{N}$ such that $1 < r < d$ $([r] \subset [d])$ for which $\mu_{[r](\bar{\imath}_r)} = 0$, then we have $\mu_{[s](\bar{\imath}_s)} = 0$ for all $s \ge r$ $([r] \subset [s])$

Furthermore, let $S^c_{d,r}$ be the set of strings of r-combinations (in increasing order) of elements of $[d]$ and $P([d]) = 2^{[d]}$denote the power set of $[d]$. Thus, for any $r \in \mathbb{N}$ such that $1 < r < d$ $([r] \subset [d])$ we define

$$P([d]: 0 \le n(\{\bar{\jmath}\}) \le r) := \left\{\{\bar{\jmath}\} \in P([d]): 0 \le n(\{\bar{\jmath}\}) \le r\right\} \quad (2.1)$$

Where $\bar{\jmath} \in S^c_{d,r}$for $r = 0,1,2,\dots,d$ $(d \le r)$with $\{\bar{\jmath}\} = \emptyset$ if $r = 0$ and $n(\{\bar{\jmath}\})$ denote the lenght of the string $\bar{\jmath} \in S^c_{d,r}$ or the cardinality of the set $\{\bar{\jmath}\} \in P([d])$. Observe that $P([d]: 0 \le n(\{\bar{\jmath}\}) \le r)$ is simply a subclass of $P([d])$, however, $P([d]: 0 \le n(\{\bar{\jmath}\}) \le r)$ is equal to the power set $P([d])$ if $d = r$. furthermore, by this, is easy to see that

$$P([d]: 0 \le n(\{\bar{\jmath}\}) \le d) := \bigcup_{r=0}^{d} P([d]: n(\{\bar{\jmath}\}) = r) = \bigcup_{r=0}^{d} \{\bar{\jmath}\}_r \quad (2.2)$$

From (1.3), notice that $P([d]: n(\{\bar{\jmath}\}) = r)$ is structurally equal to the (set) collections of elements of $S^c_{d,r}$. As a consequence of above concept and definitions, we shall rather replace the notation $\mu_{[s](\bar{\imath}_s)}$ by $\mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})}$ such that $n(\{\bar{\jmath}\}) = r$. However, these notations could be used interchangeable if need be in the course of this work, also if$n(\{\bar{\jmath}\}) = r$, then $\{\bar{\jmath}\} \in \{\bar{\jmath}\}_r$. The follwing lemma shall be useful in the sequel.

3.0 Main Results

Lemma 3.1

For arbitrary but fixed $d \in \mathbb{N}$ and let $\mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})}$ be as define above such that $\bar{\jmath} \in S^c_{d,r}$ $(\{\bar{\jmath}\} \in P([d]: n(\{\bar{\jmath}\}) = p)$ then $\{\bar{\jmath}\} \in$ P$([d]: 0 \le n(\{\bar{\jmath}\}) \le d)$

$$\sum_{q=0}^{d} \cdot \sum_{1 \le j_1 < j_2 < \cdots < j_u \le d} \mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})} = \sum_{\{\bar{\jmath}\} \in P([d]: 0 \le n(\{\bar{\jmath}\}) \le d)} \mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})}$$

Proof

$$\sum_{q=0}^{d} \cdot \sum_{1 \le j_1 < j_2 < \cdots < j_u \le d} \mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})} = \sum_{q=0}^{d} \cdot \sum_{\bar{\jmath} \in S^c_{d,r}} \mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})} = \sum_{q=0}^{d} \cdot \sum_{\{\bar{\jmath}\} \in \{\bar{\jmath}\}_q} \mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})}$$

$$\sum_{q=0}^{d} \cdot \sum_{\{\bar{\jmath}\} \in P([d]: n(\{\bar{\jmath}\}) = q)} \mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})} = \sum_{\{\bar{\jmath}\} \in \bigcup_{q=0}^{d} P([d]:\, n(\{\bar{\jmath}\}) = q)} \mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})} = \sum_{\{\bar{\jmath}\} \in P([d]: 0 \le n(\{\bar{\jmath}\}) \le d)} \mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})}$$

Theorem 3.2

Let $\alpha^{(d)}$ be a d-dimensional $k_1 \times k_2 \times \cdots \times k_d$ contingency table with cell counts (frequencies)$\alpha_{i_1,i_2,\dots,i_d}$ $(i_r \in [k_r], k_r \in \mathbb{N}, r \in [\, d\,])$, then the saturated loglinear model for the d-way table of variables indexed in $[d]$ is given by;

$$\eta_{\bar{\jmath}_d} = \sum_{\{\bar{\jmath}\}\in \mathrm{P}([d]:0\le n(\{\bar{\jmath}\})\le d)} \mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})}$$

Where $\log_e \mu_{\bar{\jmath}_d} = \eta_{\bar{\jmath}_d}$

Proof

It suffices to show that, there are 2^d terms of the parameters involving μ in the model. Since $\{\bar{\jmath}\}$ is a (set) runing variable over $\mathrm{P}([d]: 0 \le n(\{\bar{\jmath}\}) \le d)$, then the number of terms of μ involving in the summation is determined by the cardinality of $\mathrm{P}([d]: 0 \le n(\{\bar{\jmath}\}) \le d)$. Thus

$$\mathrm{n}\big(\mathrm{P}([d]: 0 \le n(\{\bar{\jmath}\}) \le d)\big) = \mathrm{n}\left(\bigcup_{q=0}^{d} P([d]: n(\{\bar{\jmath}\}) = q)\right)$$

$$\sum_{q=0}^{d} \mathrm{n}\big(P([d]: n(\{\bar{\jmath}\}) = q)\big) = \sum_{q=0}^{d} \mathrm{n}\big(S^c_{d,q}\big) = \sum_{q=0}^{d} \binom{d}{q} = 2^d$$

This completes the proof.

Theorem 3.3

Let $m, d \in \mathbb{N}$ and the saturated loglinear model for the d-dimensional contingency table $\alpha^{(d)}$ associated with $M_{(d)}\big(p\,, \alpha^{(d)},\ X^{(d)}\big)$given by

$$\eta_{\bar{\jmath}_d} = \sum_{r=0}^{d}\cdot \sum_{\bar{\jmath}\in S^c_{d,r}} \mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})}$$

Then the parameter estimators is

$$\mu_{\{\bar{\jmath}_m\}(i_{\bar{\jmath}m})} = \begin{cases} \dfrac{\sum_{\bar{\imath}_d} \alpha_{\bar{\imath}_d}}{I_{\bar{\imath}_d}}\ ;\ if\ m = 0 \\ \dfrac{\alpha_{(i_{\bar{\jmath}_r},+)}}{I_{i_{\bar{\jmath}_d}\backslash i_{\bar{\jmath}m}}} - \sum_{r=0}^{m-1}\cdot \sum_{\bar{\jmath}\in S^c_{d,r}} \mu_{\{\bar{\jmath}_q\}(i_{\bar{\jmath}_q})}\,; \\ \forall\ \bar{\jmath}_m \in S^c_{d,m}\ ;\ m = 1,2,\dots,d \end{cases}$$

Where $\alpha_{(i_{\bar{\jmath}_r},+)} = \sum_{i_{\bar{\jmath}_d}\backslash i_{\bar{\jmath}_r}} \alpha_{\bar{\imath}_d}$

Proof

Let $\bar{\jmath} \in S^c_{d,r}$ such that$n(\{\bar{\jmath}\}) = m\ (0 < m \le r \le d)$. Thus using the sigma constrain condition on $\eta_{\bar{\jmath}_d}$ we have

$$\sum_{i_{\bar{\jmath}_d}\backslash i_{\bar{\jmath}m}} \eta_{\bar{\jmath}_d} = \sum_{i_{\bar{\jmath}_d}\backslash i_{\bar{\jmath}m}} \cdot \sum_{r=0}^{d}\cdot \sum_{\bar{\jmath}\in S^c_{d,r}} \mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})}$$

$$= \sum_{i_{\bar{\jmath}_d}\backslash i_{\bar{\jmath}m}} \left(\sum_{r=0}^{m}\cdot \sum_{\bar{\jmath}\in S^c_{d,r}} \mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})} + \sum_{r=m+1}^{d}\cdot \sum_{\bar{\jmath}\in S^c_{d,r}} \mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})} \right)$$

$$= \sum_{i_{\bar{\jmath}_d}\backslash i_{\bar{\jmath}m}} \cdot \sum_{r=0}^{d}\cdot \sum_{\bar{\jmath}\in S^c_{d,r}} \mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})}$$

$$\mu_{\{\bar{\jmath}_m\}(i_{\bar{\jmath}m})} = \frac{\alpha_{(i_{\bar{\jmath}_r},+)}}{I_{i_{\bar{\jmath}_d}\backslash i_{\bar{\jmath}m}}} - \sum_{r=0}^{m-1}\cdot \sum_{\bar{\jmath}\in S^c_{d,r}} \mu_{\{\bar{\jmath}\}(i_{\bar{\jmath}})}$$

4.0 Conclusion

We observed that the results obtained in this paper solve certain parameter estimation problem with sigma constrain condition in the generalised sense.It is however interesting to note that this solutionhave pave way forresearchersin multivariate analysis to analyse data for any arbitrary d-dimensional $k_1 \times k_2 \times \cdots \times k_d$ contingency table.We are not aware of the existence of the results obtained in this paper in literature.

5.0 References

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