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Coefficient Estimates for New Subclasses of Ma-Minda Bi-starlike Functions

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Abstract

In this paper, we make use of the principle of subordination to define new subclasses of bi-univalent functions, the bounds of the coefficients of functions in these classes are obtained. Furthermore, Fekete-Szego functionals for the new classes involving sigmoid function are given.

Keywords: Bi-starlike function, Subordination, Sigmoid function, Fekete-Szego functional.
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1.0 Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1.1)$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$

We denote by S the subclass of A consisting of functions which are also univalent in U.

Let P denote the family of function $p(z)$, which are analytic in U such that $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0, z \in U$.

Zhigang and Han [1] investigated the bounds of the coefficients of several classes of bi-univalent functions. Srivastava and Bansal [2] considered a new subclass of the class consisting of analytic and bi-univalent functions in the open unit disk U. Furthermore, estimates on the first two Taylor-Maclaurin coefficients [a_2] and [a_3] were obtained. Bansal [3] studied bounds of second Hankel determinant $|a_2 a_4 - a_3^2|$ for functions belonging to a class of univalent functions using Toeplitz determinants. Andrei[4] introduced and studied certain subclasses of analytic functions in the open unit disk defined by differential operator. Tang et al [5] also introduced and investigated two new subclasses of Ma-Minda bi-univalent functions defined by subordination in the open unit disk. Estimates for initial coefficients $|a_2|$ and $|a_3|$ for these new subclasses were obtained. Fadipe-Joseph et al [6] investigated the properties of sigmoid function in relation to univalent functions theory.

In this work, new subclasses of bi-univalent functions are established. Furthermore, the coefficients $|a_2|$, $|a_3|$ were obtained. The Fekete-Szego functional for the new classes involving sigmoid function are given.

Definition 1.1: *(See[7]) An analytic function f is said to be subordinate to another analytic function g, written as* $f(z) \prec g(z)$ $(z \in U)$ if there exists a Schwarz function w, which is analytic in U with $w(0) = 0$ *and* $|w(z)| < 1$ such that $f(z) = g(w(z))$.

In particular, if the function g is univalent in U, then we have the following equivalence:

$f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ *and* $f(U) \subset g(U)$.

It is known that, if $f(z)$ is an analytic univalent function from a domain D_1 onto a domain D_2, then the inverse function g(z) defined by

$g(f(z)) = z, z \in D_1$

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Is an analytic and univalent from D_2 onto D_1.
The inverse of a function f(z) has a series expansion of the form:

$$f^{-1}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \ldots + \ldots \qquad (1.2)$$

which is convergent in some disk.
A function f(z) that is univalent in a neighborhood of the origin, and its inverse
$f^{-1}(w)$ satisfies the following conditions:

$$f(f^{-1}(w)) = w$$

Or, equivalently

$$w = f^{-1}(w) + a_2[f^{-1}(w)]^2 + a_3[f^{-1}(w)]^3 + \ldots \qquad (1.3)$$

If we use equation (1.1) and (1.2) in (1.3) we have

$$z + a_2 z^2 + a_3 z^3 + \ldots = z + (2a_2 + \gamma_2]z^2 + [a_3 + 2a_2(a_2 + \gamma_2) + (2a_2\gamma_2 + a_3 + \gamma_3)]z^3 + \ldots$$

By equating the corresponding coefficients, we have

$$a_2 = 2a_2 + \gamma_2$$

which yields $\gamma_2 = -a_2$
Also,

$$a_3 + 2a_2(a_2 + \gamma_2) + (2a_2\gamma_2 + a_3 + \gamma_3) = a_3$$

which yields $\gamma_3 = 2a_2^2 - a_3$
Substituting these back in equation (1.2), we have

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \ldots \qquad (1.4)$$

2.0 Preliminaries

Definition 2.1: A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U.
We denote by $\sum$ the class of all functions $f(z)$ which are bi-univalent in U and are given by (1.1).
(for more details, see also [1, 2])

Definition 2.2. Let ϕ be an analytic function with positive real part in U such that $\phi(0) = 1, \phi'(0) > 0$ and $\phi(U)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form:

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots \qquad (B_1 > 0) \qquad (2.1)$$

We now introduce a new subclass of bi-univalent functions.

Definition 2.3 Let $0 \leq \lambda < 1$. A function $f \in \sum$ is said to be in the class $\sum(\lambda, \phi)$ if the following subordination conditions holds true:

$$\frac{1}{1-\lambda}\left[\frac{zf'(z)}{f(z)} - \lambda\right] \prec \phi(z) \qquad (z \in U) \qquad (2.2)$$

and

$$\frac{1}{1-\lambda}\left[\frac{wg'(w)}{g(w)} - \lambda\right] \prec \phi(w) \qquad (w \in U) \qquad (2.3)$$

(see [4])
To obtain the first two coefficients a_2, a_3 of the class $\sum(\lambda, \phi)$, we shall need the following lemma:

Lemma 2.0.1

If $p(z) + 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ is analytic on the unit disk U, and has a positive real part then,

$$|c_n| \leq 2 \qquad (n \in N)$$

This inequality is sharp for each $n \in N$ ([3,8]).

3.0 Main Results

In this section we prove the main results:

Theorem 3.1: Let $f(z) \in \sum(\lambda, \phi)$ be of the form (1.1). Then

$$|a_2| \le \left[\frac{B_1^{3/2}(1-\lambda)}{\sqrt{|B_1^2(1-\lambda)+(B_1-B_2)|}} \right]$$

$$|a_3| \le B_1(1-\lambda)[\tfrac{1}{2}+B_1(1-\lambda)]$$

where the coefficients B_1 and B_2 are given by (2.1)

Proof: Let $f(z)\in \sum(\lambda,\phi)$ and $g = f^{-1}$. Then there exists analytic function $u,v:U \to U$, with $u(0)=0, v(0)=0$ satisfying the following conditions

$$\frac{1}{1-\lambda}\left[\frac{zf'(z)}{f(z)}-\lambda\right] \prec \phi(u(z)) \qquad (z\in U) \qquad (3.1)$$

and

$$\frac{1}{1-\lambda}\left[\frac{wg'(w)}{g(w)}-\lambda\right] \prec \phi(v(w)) \qquad (w\in U) \qquad (3.2)$$

We define the function p_1 and p_2 in P by

$$p_1(z)=\frac{1+u(z)}{1-u(z)}=1+c_1z+c_2z^2+c_3z^3+\ldots \qquad (z\in U \qquad (3.3)$$

and

$$p_2(w)=\frac{1+u(w)}{1-u(w)}=1+b_1w+b_2w^2+b_3w^3+\ldots \qquad (w\in U) \qquad (3.4)$$

Then p_1 and p_2 are analytic in U with $p_1(0)=1=p_2(0),$ since $u,v:U\to U$, each of the function p_1 *and* p_2 has positive real part in U. Therefore, in view of the lemma (2.0.1), we have

$$|b_n| \le 2 \quad and \quad |c_n| \le 2 \quad (n\in N) \qquad (3.5)$$

Solving for u(z) and v(w), we get

$$u(z)=\frac{1}{2}\left[c_1z+(c_2-\frac{c_1^2}{2})z^2+\ldots.\right] \qquad (3.6)$$

and

$$u(w)=\frac{1}{2}\left[b_1w+(b_2-\frac{b_1^2}{2})w^2+\ldots.\right] \qquad (3.7)$$

Now substituting (3.6) into (3.1) we get

$$\frac{1}{1-\lambda}\left[\frac{zf'(z)}{f(z)}-\lambda\right]=\phi\left\{\frac{1}{2}\left|c_1z+(c_2-\frac{c_1^2}{2})z^2+\ldots]\right.\right\} \qquad (3.8)$$

But $\phi(z)=1+B_1z+B_2z^2+\ldots$

Therefore

$$\phi\left\{\frac{1}{2}[c_1z+(c_2-\frac{c_1^2}{2})z^2+\ldots]\right\}=1+(\frac{1}{2}B_1c_1)z+[\frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}B_2c_1^2]z^2+\ldots \qquad (3.9)$$

Also,

$$f(z)=z+\sum\nolimits_{k=2}^{\infty}a_kz^k$$

$$\Rightarrow f'(z)=1+\sum\nolimits_{k=2}^{\infty}ka_kz^{k-1}$$

and

$$zf'(z)=z+\sum\nolimits_{k=2}^{\infty}ka_kz^k$$

Therefore,

$$\frac{1}{1-\lambda}\left[\frac{zf'(z)}{f(z)}-\lambda\right]=1+\frac{a_2}{1-\lambda}z+\frac{2a_3-a_2^2}{1-\lambda}z^2+\ldots \qquad (3.10)$$

(3.8) thus gives

$$1+\frac{a_2}{1-\lambda}z+\left(\frac{2a_3-a_2^2}{1-\lambda}\right)z^2+\ldots=1+(\frac{1}{2}B_1c_1)z+[\frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}B_2c_1^2]z^2+\ldots$$

By comparing coefficients, we have

$$\frac{a_2}{1-\lambda}=\frac{1}{2}B_1c_1 \tag{3.11}$$

$$\frac{2a_3-a_2^2}{1-\lambda}=\frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}B_2c_1^2 \tag{3.12}$$

Also by the subordination condition

$$\frac{1}{1-\lambda}\left[\frac{wg'(w)}{g(w)}-\lambda\right]=\phi(v(w)) \tag{3.13}$$

Similarly,

$$\frac{1}{1-\lambda}\left[\frac{wg'(w)}{g(w)}-\lambda\right]=1-\frac{a_2}{1-\lambda}w+\frac{(3a_2^2-2a_3)}{1-\lambda}w^2+\ldots \tag{3.14}$$

and

$$\phi(v(w))=1+\left(\frac{1}{2}B_1b_1\right)w+\left[\frac{1}{2}B_1(b_2-\frac{b_1^2}{2})+\frac{1}{4}B_2b_1^2\right]w^2+\ldots \tag{3.15}$$

Putting (3.15) and (3.14) into (3.13), we have

$$1-\frac{a_2}{1-\lambda}w+\left(\frac{3a_2^2-2a_3}{1-\lambda}\right)w^2+\ldots=1+(\frac{1}{2}B_1b_1)w+[\frac{1}{2}B_1(b_2-\frac{b_1^2}{2})+\frac{1}{4}B_2b_1^2]w^2+\ldots$$

Thus,

$$-\frac{a_2}{1-\lambda}=\frac{1}{2}B_1b_1 \tag{3.16}$$

$$\frac{3a_2^2-2a_3}{1-\lambda}=\frac{1}{2}B_1(b_2-\frac{b_1^2}{2})+\frac{1}{4}B_2b_1{}^2 \tag{3.17}$$

Now, we have the following system of equations

$$\frac{a_2}{1-\lambda}=\frac{1}{2}B_1c_1$$

$$\frac{2a_3-a_2^2}{1-\lambda}=\frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}B_2c_1^2$$

$$-\frac{-a_2}{1-\lambda}=\frac{1}{2}B_1b_1$$

$$\frac{3a_2^2-2a_3}{1-\lambda}=\frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2$$

From (3.11) and (3.16), we have that

$$\frac{1}{2}B_1c_1=-\frac{1}{2}B_1b_1$$

$$c_1=-b_1 \tag{3.18}$$

Adding equations (3.12) and (3.17), we have

$$\frac{2a_3-a_2^2}{1-\lambda}+\frac{3a_2^2-2a_3}{1-\lambda}=\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2+\frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2$$

$$\Rightarrow \frac{2a_2^2}{1-\lambda}=\frac{1}{2}B_1\left[(c_2+b_2)-\frac{1}{2}(c_1^2+b_1^2)\right]+\frac{1}{4}B_2(c_1^2+b_1^2)$$

$$\frac{2a_2^2}{1-\lambda}=\frac{1}{2}B_1(c_2+b_2)-\frac{1}{4}(B_1-B_2)(c_1^2+b_1^2)$$

Using the fact that $c_1 = -b_1$ from (3.18) and $c_1=\dfrac{2a_2}{(1-\lambda)B_1}$ from (3.11)

$$\frac{2a_2^2}{1-\lambda}=\frac{1}{2}B_1(c_2+b_2)-\frac{1}{4}(B_1-B_2)2c_1^2$$

$$=\frac{1}{2}B_1(c_2+b_2)-\frac{1}{2}(B_1-B_2)\left[\frac{2a_2}{(1-\lambda)B_1}\right]^2$$

$$=\frac{1}{2}B_1(c_2+b_2)-\frac{1}{2}(B_1-B_2)\frac{4a_2^2}{(1-\lambda)^2B_1^2}$$

$$\Rightarrow a_2^2\left[\frac{2}{1-\lambda}+\frac{2(B_1-B_2)}{(1-\lambda)^2B_1^2}\right]=\frac{B_1(c_2+b_2)}{2}$$

$$a_2\left[\frac{2(1-\lambda)B_1^2+2(B_1-B_2)}{(1-\lambda)^2B_1^2}\right]=\frac{B_1(c_2+b_2)}{2}$$

$$a_2^2\left[\frac{2(1-\lambda)B_1^2+2(B_1-B_2)}{(1-\lambda)^2B_1^2}\right]=\frac{B_1}{2}(c_2+b_2)$$

$$a_2^2=\frac{B_1^3(1-\lambda)^2(b_2+c_2)}{4[1-\lambda)B_1^2+(B_1-B_2)}$$

$$\Rightarrow |a_2^2|=\left|\frac{B_1^3(1-\lambda)^2(b_2+c_2)}{4[1-\lambda)B_1^2+(B_1-B_2)}\right|$$

$$=B_1^3(1-\lambda)^2\left|\frac{b_2+c_2}{4[(1-\lambda)B_1^2+(B_1-B_2)]}\right|$$

$$\leq B_1^3(1-\lambda)^2\left|\frac{1}{(1-\lambda)B_1^2+(B_1-B_2)]}\right|$$

On taking the square root, we have

$$\Rightarrow |a_2|\leq\left[\frac{B_1^{3/2}(1-\lambda)}{\sqrt{|B_1^2(1-\lambda)+(B_1-B_2)|}}\right]$$

If we subtract (3.17) from (3.12), we have

$$\frac{2a_3-a_2^2}{1-\lambda}-\left(\frac{3a_2^2-2a_3}{1-\lambda}\right)=\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2-\left[\frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2\right]$$

$$\Rightarrow \frac{4a_3-4a_2^2}{1-\lambda}=\frac{1}{2}B_1\left[(c_2-b_2)-\frac{1}{2}(c_1^2-b_1^2)\right]+\frac{1}{4}B_2(c_1^2-b_1^2)$$

$$=\frac{1}{2}B_1(c_2-b_2)-\frac{1}{4}B_1(c_1^2-b_1^2)+\frac{1}{4}B_2(c_1^2-b_1^2)$$

$$=\frac{1}{2}B_1(c_2-b_2)-\frac{1}{4}(B_1-B_2)(c_1^2-b_1^2)$$

But $c_1=-b_1\Rightarrow c_1^2=b_1^2$

$$\Rightarrow \frac{4a_3 - 4a_2^2}{1-\lambda} = \frac{1}{2} B_1 (c_2 - b_2)$$

$$\Rightarrow \frac{4a_3}{1-\lambda} = \frac{1}{2} B_1 (c_2 - b_2) + \frac{4a_2^2}{1-\lambda}$$

By using $a_2 = \frac{(1-\lambda)B_1 c_1}{2}$ from (3.11)

$$\Rightarrow \frac{4a_3}{1-\lambda} = \frac{B_1}{2}(c_2 - b_2) + \frac{4}{1-\lambda}\left[\frac{(1-\lambda)B_1 c_1}{2}\right]^2$$

$$= \frac{B_1}{2}(c_2 - b_2) + (1-\lambda)B_1^2 c_1^2$$

$$\Rightarrow a_3 = \frac{(1-\lambda)}{4}\left[\frac{B_1}{2}(c_2 - b_2) + (1-\lambda)B_1^2 C_1^2\right]$$

$$= B_1 \frac{(1-\lambda)(c_2 - b_2)}{8} + \frac{(1-\lambda)^2 B_1^2 c_1^2}{4}$$

$$= a_3 = \frac{B_1(1-\lambda)(c_2 - b_2)}{8} + \frac{(1-\lambda)^2 B_1^2 b_1^2}{4}$$

$$\Rightarrow |a_3| = \left|\frac{B_1(1-\lambda)(c_2 - b_2)}{8} + \frac{(1-\lambda)^2 B_1^2 b_1^2}{4}\right|$$

$$\leq \left|\frac{B_1(1-\lambda)(c_2 - b_2)}{8}\right| + \left|\frac{(1-\lambda)^2 B_1^2 b_1^2}{4}\right|$$

$$\leq \frac{B_1(1-\lambda)}{2} + B_1^2(1-\lambda)^2$$

Therefore,

$$|a_3| \leq B_1(1-\lambda)\left[\frac{1}{2} + B_1(1-\lambda)\right]$$

This completes the prove.

Corollary 3.1: If λ =0 in Theorem 3.1, then $f \in \sum(0,\phi) \equiv \sum(\phi)$:

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \qquad (z \in U)$$

and

$$\frac{wg'(z)}{g(w)} \prec \phi(w) \qquad (w \in U)$$

If and only if

$$|a_2| \leq \frac{B_1^{3/2}}{\sqrt{|B_1^2 + B_1 - B_2|}}$$

$$|a_3| \leq B_1\left[\frac{1}{2} + B_1\right]$$

Remark 3.1: The class $\sum(0,\phi) \equiv \sum(\phi)$ is the class of Ma-Minda bi-starlike function

3.1 Sigmoid Function

The sigmoid function, g(z) is an analytic function of the form

$$g(z) - \frac{1}{1+e^{-z}}$$

The sigmoid function g(z) is an example of the activation function. There are three types of activation function, others being the threshold function and the piecewise linear function.

Some important properties of sigmoid function are:
1. It outputs real numbers between 0 and 1.
2. It maps a very large input domain to a small range of outputs.
3. It never loses information because it is a one-to-one function.
4. It increases monotonically

The sigmoid function has S shape
(see [6]).

Definition 3.1: Let $S(z)=\frac{2}{1+e^{-z}}$ be an analytic function with positive real part in U such that S(0) =1, S'(0)= ½ and S(U) is symmetric with respect to the real axis. S(z) has a series expansions of the form.

$$S(z)=1+\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\frac{17}{40320}z^7+....$$

S(z) is called the modified sigmoid function.

Lemma 3.1: ([6]): Let g(z) be a sigmoid function and $S(z)=2g(z),\ then\ S(z)\in P, |z|<1.$

Definition 3.2: Let $0\le\lambda<1$. A function $f\in\sum$ is said to be in the class $\sum(\lambda,S)$ if the following subordination conditions holds true:

$$\frac{1}{1-\lambda}\left[\frac{zf'(z)}{f(z)}-\lambda\right]\prec S(z) \qquad (z\in U) \qquad (3.19)$$

and

$$\frac{1}{1-\lambda}\left[\frac{wg'(w)}{g(w)}-\lambda\right]\prec S(w) \qquad (w\in U) \qquad (3.20)$$

Theorem 3.2: If f(z) belongs to class $\sum(\lambda,S)$, then

$$|a_2|\le\frac{\sqrt{2}}{2}(1-\lambda)\left[\frac{1}{\sqrt{|3-\lambda|}}\right]$$

$$|a_3|\le\frac{1}{4}(1-\lambda)(2-\lambda)$$

Proof: The proof follows from Theorem 3.1 upon taking $B_1=\frac{1}{2}\ and\ B_2=0$

4.0 Fekete-Szego Functional For Class $\sum(\lambda,\phi)$ and $\sum(\lambda,S)$

The problem of finding the sharp bounds for the non-linear functional $|a_3-\alpha a_2^2|$; $0\le\alpha<1$, of any compact family of functions is popularly known as Fekete-Szego problem.

Here, sharp upper bound of the Fekete-Szego function $|a_3-\alpha a_2^2|$ for the class $\sum(\lambda,\phi)$ and $\sum(\lambda,S)$ are obtained.

Theorem 4.3: If f(z) given by (1.1) belongs to the class $\sum(\lambda,\phi)$, then for any real number α.

$$|a_3-\alpha a_2^2|\le\frac{B_1(1-\lambda)}{2}+(1-\lambda)^2B_1^2\frac{\alpha B_1^3(1-\lambda)^2}{|(1-\lambda)B_1^2+(B_1-B_2)|}$$

Proof: The proof follows upon putting

$$a_3=\frac{B_1(1-\lambda)(c_2-b_2)}{8}+\frac{(1-\lambda)^2B_1^2b_1^2}{4}$$

and

$$a_2^2=\frac{B_1^3(1-\lambda)^2(b_2+c_2)}{4[(1-\lambda)B_1^2+(B_1-B_2)}$$

Into the functional $|a_3-\alpha a_2^2|;0\le\alpha<1$

Theorem 4.4: If f(z) is given by (1.1) belongs to the class $\sum(\lambda,S)$, then for any real number α.

$$\left|a_3-\alpha a_2^2\right| \leq \frac{(1-\lambda)}{4}+\frac{(1-\lambda)^2}{4}+\frac{a(1-\lambda)^2}{2(3-\lambda)}$$

Proof: The proof follows upon putting

$$a_3=\frac{(1-\lambda)(c_2-b_2)}{16}+\frac{(1-\lambda)^2 b_1^2}{16}$$

$$a_2^2=\frac{(1-\lambda)^2(b_2+c_2)}{8(3-\lambda)}$$

into the functional $\left|a_3-\alpha a_2^2\right|; 0 \leq \alpha<1$

5.0 Conclusion

The first two coefficients of some subclasses of Ma-Minda Bi-Starlike functions were obtained. Also, the Fekete-Szego coefficients functional for the classes were established.

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